124j Stalactite Growth as a Free-Boundary Problem

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The astonishing variety and beauty of structures found in limestone caves, from stalactites and stalagmites to soda straws, draperies, and helictites, have been the subject of human wonder for hundreds if not thousands of years. There is little debate about the fundamental chemical processes responsible for their development. Water enters the cave from the overlying environment with significant concentrations of dissolved carbon dioxide and calcium. As the partial pressure of carbon dioxide in the cave is lower than that in the overlying rock, carbon dioxide outgases from the water. This raises the pH and leads to supersaturation and then precipitation of calcium carbonate. Yet, this chemical picture is only part of the story, for it does not in any direct way answer the most obvious morphological question: Why are stalactites long and slender, often roughly conical (resembling icicles)? While some studies address the dynamics of speleothem morphology, none quantitatively explains this most basic fact. In the work to be presented, we show that the combination of thin film fluid dynamics, calcium carbonate chemistry, and carbon dioxide diffusion and outgassing leads to a local geometric growth law for the surface evolution that quantitatively explains the shapes of natural stalactites. Here we provide details of this free-boundary calculation, exploiting a strong separation of time scales between that for diffusion within the layer, the time during which a fluid parcel is in contact with the growing surface, and the time scale of growth. When the flow rate, the scale of the stalactite, and the chemistry are in the ranges typically found in nature, the local growth rate is proportional to the local thickness of the fluid layer, itself determined by Stokes flow over the surface. Analytical and numerical studies of this law establish that a broad class of initial conditions is attracted to an ideal universal shape. Found under a set of limiting assumptions, this may be thought of as the Platonic ideal of speleothem growth. While real stalactites have more complex shapes due to instabilities and cave inhomogeneities, statistical analysis of stalactite shapes from Kartchner Caverns (Benson, AZ) yields excellent agreement between the average shape of natural stalactites and the ideal shape. This work serves to emphasize a broad class of problems that demands considerable attention--free-boundary dynamics in precipitative pattern formation. Beyond speleothems, these include structures as diverse as hydrothermal vents, chemical gardens, mollusc shells, and tubes whose growth is templated by bubbles.