

Quantitative Feedback Design for Systems with Probabilistic Parameterisations

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Abstract

This paper examines the feedback design possibilities for uncertain plants where the underlying parameterisation is described by a probabilistic rather than a deterministic set membership. The design perspective is Horowitz' quantitative feedback theory.

1. Introduction

Horowitz' quantitative feedback theory (QFT) is a well-established engineering design philosophy for uncertain feedback problems (see Horowitz, 1991 and references therein). In its usual form, the plant is a member of an uncertain set and at each design frequency, ω_i , this set results in a *template* on the arithmetic, or more usually, the log-polar complex plane (Nichols chart). In some practical applications the plant knowledge is probabilistic and in this case the QFT templates do not have crisp edges. It may appear that the notion of a quantitative, yet probabilistic design approach that will be explored in this paper is a contradiction in terms but design with hard boundaries can emerge from the soft-edged template. The method developed in the paper is based on application of simple approximations for the mean and variance behaviour of functions of a random variable, for example, from Papoulis (1965).

In system identification based on measured data, a probabilistic description of the plant may be given, in the form of either a non-parametric frequency response with variances, or a parametric transfer function with mean and variances given for the parameters. More complicated linear system models may emerge from first principles modelling and possibly linearisation, where the underlying parameterisation of the system is uncertain and is described by means and variance. In many of these problems, there is no practical prospect of knowing the density of the underlying random variables and even if they were known, attempting to calculate the resulting density of the closed loop transfer function (as a function of the unknown controller and the mapping from the parameters to the plant and from the plant to the closed loop transfer function of interest) would be unreasonable. What is proposed here is to calculate the approximate propagation of the known plant mean and variance to the closed loop transfer function of interest. The designer may then have a more accurate view of the relationship between the mean open loop and closed loop system and an estimate of the variance. This would allow the designer to assign a number of standard deviations of tolerance to the design, depending on the robustness required.

The use of multiple plant cases (for example at different operating conditions) is not excluded by the approach

introduced in the paper. The method is per plant element and different (individual) elements of the plant set may have different levels of probabilistic uncertainty.

We will discuss only sensitivity design for single-input, single-output (SISO) plants in detail but the ideas presented here can be expanded to situations that are more general.

2 Preliminary results

2.1 Problem statement

We will consider a linear SISO plant, $P(s, \alpha)$, where s is the usual frequency variable and α is a vector of underlying, uncertain plant parameters. In usual QFT, the design task is to achieve client-specified tracking and regulating behaviour for all $\alpha \in \{\alpha\}$, a set of known elements. For practical calculations, $\{\alpha\}$ is finite or can be adequately approximated by a finite set. In this work, we will investigate the situation where either α is uncertain or where $P(s, \alpha)$ is uncertain directly. We then require that approximate, client-specified mean, or mean plus standard deviation tolerance, closed loop behaviour be obtained.

2.2 Background approximations from probability theory

Developing the scalar results of Papoulis (1965, Section 5.4), the following approximations are available for the mean and variance of, $y = h(x)$, a function of a random vector, $x \in \mathfrak{R}^{n \times 1}$, with density, $f(x)$. The great benefit of the approximations that follow is that they are independent of the density.

Mean

Approximate y using the first three terms of the Taylor series expansion around \bar{x} the mean of x . This is reasonable if $h(x)$ is smooth as the density $f(x)$ takes significant values near the mean.

$$y \approx h(\bar{x}) + \left. \frac{\partial h}{\partial x} \right|_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2} (x - \bar{x})^T \left. \frac{\partial^2 h}{\partial x^2} \right|_{x=\bar{x}} (x - \bar{x}) \quad (1)$$

$\left. \frac{\partial h}{\partial x} \right|_{x=\bar{x}}$ is the Jacobian and $\left. \frac{\partial^2 h}{\partial x^2} \right|_{x=\bar{x}}$ is the Hessian of h with respect to x .

Using eq(1), the mean of y can be approximated by,

$$\begin{aligned} \mathcal{E}\{y\} &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(\mathbf{x}) f(\mathbf{x}) dx_1 \cdots dx_n \\ &\approx h(\bar{\mathbf{x}}) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \left[\frac{\partial^2 h}{\partial \mathbf{x}^2} \Big|_{\bar{\mathbf{x}}} \circ \mathbf{C}_x \right]_{ij} \end{aligned} \quad (2)$$

where \circ denotes the point-wise (Schur or Hadamard) product of the matrices, $[\cdot]_{ij}$ refers to the individual matrix elements, and $\mathbf{C}_x = \text{cov}(\mathbf{x})$ is the covariance of \mathbf{x} . Notice that when calculating the expectation from eq(1) to eq(2), the linear term with $(\mathbf{x} - \bar{\mathbf{x}})$ vanishes and the quadratic term is calculated by observing that the result is a scalar, allowing the order of calculations to be swapped to isolate terms which give $\text{cov}(\mathbf{x})$.

Variance

The variance of y is approximated using a similar approach,

$$\begin{aligned} \sigma_y^2 &= \mathcal{E}\{(y - \bar{y})^2\} \\ &\approx \frac{\partial h}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}} \mathbf{C}_x \left(\frac{\partial h}{\partial \mathbf{x}} \Big|_{\bar{\mathbf{x}}} \right)^T \end{aligned} \quad (3)$$

2.3 Sensitivity design for SISO systems using mean and variance information from the plant elements

Typical SISO design using QFT examines performance specifications that can be written as linear fractional mappings of the design parameter, G , typically the feedback controller, at design frequency, ω_i

$$\left| \frac{a + bG}{c + dG} \right| \leq \gamma \quad (4)$$

Each of a , b , c , and d is a known complex number, e.g., $a = r_a e^{j\phi_a}$, and the specification, γ , is a given positive number. At a fixed controller angle, ϕ_g , the magnitude r_g is calculated as,

$$\begin{aligned} (r_a^2 + 2r_a r_b r_g (\cos \phi_a \cos(\phi_b + \phi_g) + \sin \phi_a \sin(\phi_b + \phi_g)) + r_b^2 r_g^2) \leq \\ (r_c^2 + 2r_c r_d r_g (\cos \phi_c \cos(\phi_d + \phi_g) + \sin \phi_c \sin(\phi_d + \phi_g)) + r_d^2 r_g^2) \gamma^2 \end{aligned} \quad (5)$$

Clearly, this is a quadratic inequality in r_g for which a closed form solution can be found at equality and some logic applied to discover if the acceptable solution lies above or below the corresponding gains required for equality

A special case is sensitivity design where $a = c = 1$, $b = 0$, and $d = P = P(j\omega_i)$, the plant transfer function at the design frequency, ω_i , giving,

$$\left| \frac{1}{1 + PG} \right| \leq \gamma \quad (6)$$

and,

$$(1 + 2r_p r_g \cos(\phi_p + \phi_g) + r_p^2 r_g^2) \geq 1/\gamma^2 \quad (7)$$

From eq(2) it is clear that if a sensitivity design is attempted based on a “mean” plant alone, the “mean” sensitivity will not be correct (i.e. will be biased) because of the neglected second and higher order terms in eq(2).

Using eq(2) and eq(3), the expected value and variance of $|1+GP|^2$ required for specifications from eq(6) can be estimated. The reason for choosing $|1+GP|^2$ rather than the sensitivity directly is the complications that arise in the calculation of the square-root and inverse. (Although only useful *a posteriori*, for $y = 1/\sqrt{x}$,

$$\mathcal{E}\{y\} \approx \frac{1}{\sqrt{\bar{x}}} \left(1 + \frac{3}{8\bar{x}^2} \sigma_x^2 \right) \text{ and } \sigma_y^2 \approx \frac{1}{4\bar{x}^3} \sigma_x^2.$$

The calculation of $|1+GP|^2$ gives a quadratic inequality in the controller magnitude, r_g , that is easy to solve for bounds and will be familiar to QFT practitioners. For this, define $\mathbf{x} = (r_p \ \phi_p)^T$ and assume that the co-variance of the uncertainty in magnitude and phase of the plant is given as $\mathbf{C}_x = \text{cov}(\mathbf{x})$. Finally,

$$\begin{aligned} h &= |1 + L|^2 \\ &= (1 + r_p r_g \cos(\phi_p + \phi_g))^2 + (r_p r_g \sin(\phi_p + \phi_g))^2 \end{aligned} \quad (8)$$

The partial derivatives required are given by,

$$\begin{aligned} \frac{\partial h}{\partial r_p} &= 2(r_g \cos(\phi_p + \phi_g) + r_p r_g^2) \\ \frac{\partial h}{\partial \phi_p} &= -2r_p r_g \sin(\phi_p + \phi_g) \\ \frac{\partial^2 h}{\partial r_p^2} &= 2r_g^2 \\ \frac{\partial^2 h}{\partial \phi_p^2} &= -2r_p r_g \cos(\phi_p + \phi_g) \\ \frac{\partial^2 h}{\partial \phi_p \partial r_p} &= -2r_g \sin(\phi_p + \phi_g) \end{aligned} \quad (9)$$

The expected value of $|1+GP|^2$ is then approximately,

$$\begin{aligned} \mathcal{E}\{|1 + L|^2\} &\approx \\ &(1 + \bar{r}_p r_g \cos(\bar{\phi}_p + \phi_g))^2 + (\bar{r}_p r_g \sin(\bar{\phi}_p + \phi_g))^2 \\ &- 2r_g \sin(\bar{\phi}_p + \phi_g) \sigma_{r_p \phi_p}^2 - \bar{r}_p r_g \cos(\bar{\phi}_p + \phi_g) \sigma_{\phi_p}^2 \\ &+ r_g^2 \sigma_{r_p}^2 \end{aligned} \quad (10)$$

with,

$$\mathbf{C}_x = \begin{pmatrix} \sigma_{r_p}^2 & \sigma_{r_p\phi_p}^2 \\ \sigma_{r_p\phi_p}^2 & \sigma_{\phi_p}^2 \end{pmatrix} \quad (11)$$

The variance of $|1+GP|^2$ is approximated by,

$$\begin{aligned} \sigma_y^2 &\approx 4r_g^2 (\cos(\phi_p + \phi_g) + r_p r_g)^2 \sigma_{r_p}^2 \\ &+ 4r_g^2 (r_p \sin(\phi_p + \phi_g))^2 \sigma_{\phi_p}^2 \\ &- 8r_g^2 (\cos(\phi_p + \phi_g) + r_p r_g) (r_p \sin(\phi_p + \phi_g)) \sigma_{r_p\phi_p}^2 \end{aligned} \quad (12)$$

As mentioned above, eq(10) is quadratic in r_g so the usual approach in QFT computer aided design using quadratic inequalities can be followed. For more general specification equations this is unlikely to be the case as one would typically have to apply the approximations eq(2) and eq(3) more than once (firstly to calculate the magnitude squared and then inverse, ratios and square-root), resulting in an explosion in complexity and tedium.

3. Example

Suppose that system identification has resulted in a plant description,

$$P = \frac{k e^{-sT}}{s+a} \quad (13)$$

with

$$\begin{pmatrix} \bar{k} \\ \bar{a} \\ \bar{T} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0.1 \end{pmatrix} \text{ and } \text{cov} \begin{pmatrix} k \\ a \\ T \end{pmatrix} = \begin{pmatrix} 0.2^2 & 0 & 0 \\ 0 & 0.5^2 & 0 \\ 0 & 0 & 0.01^2 \end{pmatrix} \quad (14)$$

Further, suppose it is required to achieve $|1/(1+L)| \leq -6\text{dB}$ at $\omega = 1 \text{ rad/s}$. A first application of the results of eqs (2) and (3) to the plant alone gives the following mean and covariance for the magnitude, r_p , and phase angle, ϕ_p :

$$\begin{aligned} r_p(j1) &= \frac{k}{\sqrt{1+a^2}} \\ \bar{r}_p(j1) &= \frac{\bar{k}}{\sqrt{1+\bar{a}^2}} + \\ &+ \frac{1}{2} \left(\left(\frac{\bar{k}}{\sqrt{1+\bar{a}^2}} \frac{2\bar{a}^2-1}{(1+\bar{a}^2)^2} \right) \sigma_a^2 + 0\sigma_k^2 + 0\sigma_T^2 \right) \\ &= 1.4140 + 0.0443 = 1.4583 \end{aligned} \quad (15)$$

$$\begin{aligned} \phi_p(j1) &= -T - \arctan \frac{1}{a} \\ \bar{\phi}_p(j1) &\approx -T - \arctan \frac{1}{a} + \\ &+ \frac{1}{2} \left(\frac{-2\bar{a}}{(1+\bar{a}^2)^2} \sigma_a^2 + 0\sigma_k^2 + 0\sigma_T^2 \right) \\ &= -0.8854 - 0.0625 \\ &- 0.9479 \text{ radians} \end{aligned} \quad (16)$$

$$\begin{aligned} \text{cov} \begin{pmatrix} r_p \\ \phi_p \end{pmatrix} &= \\ &\begin{pmatrix} \frac{\partial r_p}{\partial k} & \frac{\partial r_p}{\partial a} & \frac{\partial r_p}{\partial T} \\ \frac{\partial \phi_p}{\partial k} & \frac{\partial \phi_p}{\partial a} & \frac{\partial \phi_p}{\partial T} \end{pmatrix} \text{cov} \begin{pmatrix} k \\ a \\ T \end{pmatrix} \begin{pmatrix} \frac{\partial r_p}{\partial k} & \frac{\partial \phi_p}{\partial k} \\ \frac{\partial r_p}{\partial a} & \frac{\partial \phi_p}{\partial a} \\ \frac{\partial r_p}{\partial T} & \frac{\partial \phi_p}{\partial T} \end{pmatrix} \\ &= \begin{pmatrix} 0.1450 & -0.0884 \\ -0.0884 & 0.0626 \end{pmatrix} \end{aligned} \quad (17)$$

Using values from eq(15) to eq(17) in eq(10), with nominal plant, $P_o = \frac{2e^{-s0.1}}{s+1}$, results in the nominal bound shown in Figure 1. Instead of including the variance result, eq(12) and obtaining a non-linear equation in r_g , an estimate of $\sigma_{(1+L)^2}^2$ is obtained using the r_g , already obtained and the specification (on $(1+L)^2$) is modified by 1σ and a new r_g is calculated using eq(10) to illustrate the effect. Because the variance is small, this procedure (calculate bounds on L for $(1+L)^2 + 1\sigma$, use these to estimate σ and re-calculate ...) can be iterated successfully to a fixed point.

4 Conclusions

This paper has introduced an approach for treating individual plant elements that are themselves uncertain in the QFT framework. The approach uses approximations for the mean and covariance of a function of a random variable of with arbitrary density.

The approach has been illustrated on a simple sensitivity specification for a SISO plant with a single plant element.

5 References

Horowitz I, "Survey of Quantitative Feedback Theory (QFT)", International Journal of Control, Vol. 53, No. 2, pp. 255-291, 1991.

Paupolis A, (1965) *Probability, Random Variables and Stochastic Processes*, McGraw Hill, New York.

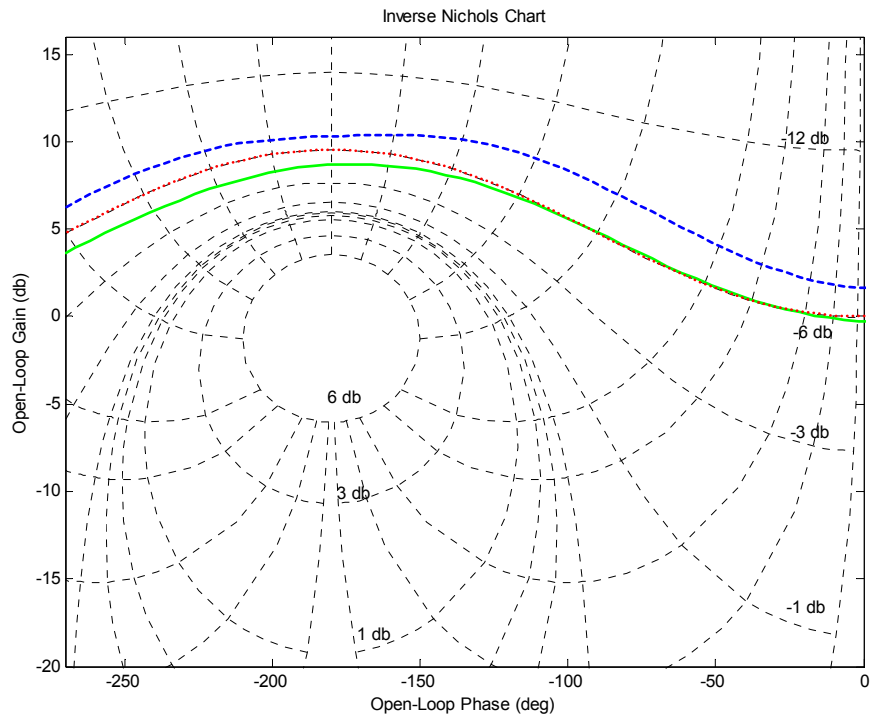


Figure 1 – Inverse Nichols chart showing the effect of including second order knowledge on a -6dB sensitivity specification. Solid line – nominal bounds using second order knowledge; dashed line – nominal bound using second order knowledge and one standard deviation; dotted line – nominal bound using only mean parameters (i.e. nominal = plant)