# EXTERNAL DISTURBANCE REJECTION IN UNCERTAIN MIMO SYSTEMS WITH QFT NON-DIAGONAL CONTROLLERS

**M. García-Sanz[1], M. Barreras[1], I. Egaña[1], C.H. Houpis[2]**

[1] *Automatic Control and Computer Science Department, Public University of Navarre 31006 Pamplona, SPAIN. Email: mgsanz@unavarra.es*

[2] *Air Force Institute of Technology, Air Force Research Laboratory, Wright-Patterson AFB, Ohio, U.S.A.*

Abstract: This paper focuses on the analysis of the external disturbance rejection problem in MIMO systems under the presence of plant model uncertainty. A methodology to design robust non-diagonal QFT controllers for such a problem is developed in this work. The definition of specific coupling matrices allows to quantify the amount of interaction and to design the non-diagonal controllers. The new technique is validated with a highly coupled multivariable system example.

Keywords: Multivariable systems, disturbance rejection, model uncertainty, non-diagonal controller.

## 1. INTRODUCTION

Disturbance attenuation in multivariable systems is often an important and difficult problem to deal with. This work circumvents rejection of external disturbances in MIMO systems with plant model uncertainties. The Quantitative Feedback Theory (QFT) is applied to design fully populated matrix controllers to attenuate the effect of external disturbances at plant input and plant output.

The study starts with some previous ideas suggested by Garcia-Sanz and Egaña (2002) about the design of non-diagonal QFT controllers to reduce the loop coupling in tracking problems, and considers the approach introduced by Houpis and Rassmussen (1999) about MIMO systems with external disturbances.

Based on these ideas, the present work introduces a sequential design methodology for non-diagonal QFT controllers to achieve disturbance rejection specifications, taking into account the reduction of interactions among loops. The method presents the definition of a specific coupling matrix, which comes in useful to quantify the amount of interaction.

Consider a general square multivariable *n* x *n* system -Fig. 1-, made up of a plant $\boldsymbol{P}$, where $\boldsymbol{P} \in \mathcal{P}$ and $\mathcal{P}$ is a set of possible plants due to uncertainty, $\boldsymbol{G}$ is the full populated matrix controller, and $\boldsymbol{P}_{di}$ and $\boldsymbol{P}_{do}$ are the plant input and output disturbance transfer functions respectively. In the same way, $\boldsymbol{d}_i^{'}$ and $\boldsymbol{d}_o^{'}$ are the external disturbances.

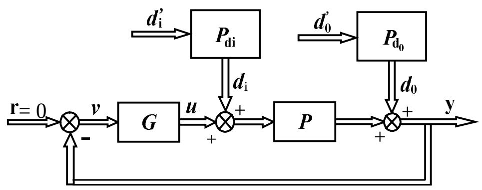



**Fig. 1** General structure of a MIMO system with plant output and input external disturbances.

The paper begins with two sections that formulate the procedure to design a non-diagonal controller for external disturbance rejection. These sections also develop the basic theoretical principles and include a detailed description of the system transmission matrix $\boldsymbol{T}$, which relates the outputs ($\boldsymbol{y}$) to input /output external disturbances ($\boldsymbol{d}_i$ and $\boldsymbol{d}_o$ respectively). Later, the stages of the developed methodology are thoughtfully explained. Section 5 leaps right to illustrate the advantages of the proposed method and analyses the practical use of the technique by means of one example. Finally, the most relevant ideas of the paper are summarized.

## 2. REJECTION OF EXTERNAL DISTURBANCES AT PLANT INPUT

Consider the $n$ x $n$ linear multivariable system shown in Fig.1. By denoting the external disturbance at plant input by $\boldsymbol{d_i} = \boldsymbol{P_{di}}\ \boldsymbol{d_i'}$, the closed loop transfer matrix $\boldsymbol{T}_{Y/di}$ that relates the disturbance at plant input $\boldsymbol{d_i'}$ to the output $\boldsymbol{y}$ becomes,

$$\boldsymbol{y} = (\boldsymbol{I} + \boldsymbol{P}\ \boldsymbol{G})^{-1}\ \boldsymbol{P}\ \boldsymbol{d_i} = \boldsymbol{T_{Y/di}}\ \boldsymbol{d_i} = \boldsymbol{T_{Y/di}}\ \boldsymbol{P_{di}}\ \boldsymbol{d_i'} \qquad (1)$$

Hence,

$$\boldsymbol{T_{Y/di}} = (\boldsymbol{I} + \boldsymbol{P}\ \boldsymbol{G})^{-1}\ \boldsymbol{P} \qquad (2)$$

The expression of $\boldsymbol{T_{Y/di}}$, - Eq. (2) - is the starting point of the mathematical developments that lead to solve this disturbance rejection problem.

Multiplying Eq.(2) by ($\boldsymbol{I} + \boldsymbol{P}\ \boldsymbol{G}$), it follows,

$$(\boldsymbol{I} + \boldsymbol{P}\ \boldsymbol{G})\ \boldsymbol{T_{Y/di}} = \boldsymbol{P} \qquad (3)$$

Then, Eq. (3) is premultiplied by $\boldsymbol{P}^{-1}$ to obtain,

$$\left(\boldsymbol{P}^{-1} + \boldsymbol{G}\right)\boldsymbol{T_{Y/di}} = \boldsymbol{I} \qquad (4)$$

From now on, the plant inverse $\boldsymbol{P}^{-1}$ will be denoted as $\boldsymbol{P}^* = [\,p_{ij}^*\,]$ and it is partitioned to the form $\boldsymbol{P}^* = \Lambda + \boldsymbol{B}$, where $\Lambda$ and $\boldsymbol{B}$ are the diagonal part and the balance of $\boldsymbol{P}^{-1}$, respectively. In the same way, the fully populated controller $\boldsymbol{G} = [g_{ij}]$ is divided into two terms; $\boldsymbol{G_d}$ and $\boldsymbol{G_b}$, which represent the diagonal part -subscript d- and balance -subscript b- of $\boldsymbol{G}$.

Substituting these matrixes in Eq. (4) and rearranging it, yields the next expressions,

$$(\Lambda + \boldsymbol{B} + \boldsymbol{G_d} + \boldsymbol{G_b})\ \boldsymbol{T_{Y/di}} = \boldsymbol{I} \qquad (5)$$

$$(\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})\ \boldsymbol{T_{Y/di}} = \Lambda^{-1} - \Lambda^{-1}(\boldsymbol{B} + \boldsymbol{G_b})\ \boldsymbol{T_{Y/di}} \qquad (6)$$

As a result, the following equation, which describes a $n\ x\ n$ matrix, holds,

$$\boldsymbol{T_{Y/di}} = (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1}\Lambda^{-1} - (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1}\Lambda^{-1}[(\boldsymbol{B} + \boldsymbol{G_b})\ \boldsymbol{T_{Y/di}}] \qquad (7)$$

By inspecting Equation (7) two different terms can be found:

i. A diagonal term $\boldsymbol{T_{Y/di\text{-}d}}$, which neither depends on the non-diagonal part of the plant nor on the non-diagonal part of the controller.

$$\boldsymbol{T_{Y/di\text{-}d}} = (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1}\Lambda^{-1} \qquad (8)$$

Where,

$$\boldsymbol{T_{Y/di\text{-}d}} = [t_{di\text{-}ii}] = \left[\frac{1}{g_{ii} + p_{ii}^*}\right] \quad i = 1,2,\ldots,n \qquad (9)$$

As illustrated in Fig. 2, this diagonal term is equivalent to a set of $n$ MISO systems so that,

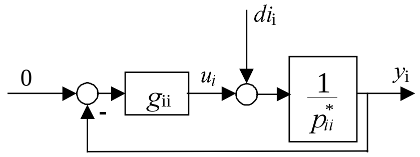



**Fig. 2** i-th equivalent MISO system. $1 \le i \le n$

ii. A non-diagonal term $\boldsymbol{T_{Y/di\text{-}b}}$

$$\boldsymbol{T_{Y/di\text{-}b}} = (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1}\ \Lambda^{-1}[(\boldsymbol{B} + \boldsymbol{G_b})\ \boldsymbol{T_{Y/di}}] \qquad (10)$$

The last term in square brackets in Eq. (10) is the only part which has a non-diagonal structure. Since it depends on the balance of the controller and plant, it comprises the coupling and represents the interaction between loops. Consequently this term will be call the coupling matrix for the rejection of external disturbances at plant input and will be denoted as $\boldsymbol{C_{di}}$.

$$\boldsymbol{C_{di}} = (\boldsymbol{B} + \boldsymbol{G_b})\ \boldsymbol{T_{Y/di}} \qquad (11)$$

i.e.

$$\boldsymbol{T_{Y/di\text{-}b}} = (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1}\ \Lambda^{-1}\boldsymbol{C_{di}} \qquad (12)$$

where

$$\boldsymbol{C_{di}} = [\,c_{di-ij}\,] \qquad i,j = 1,2,\ldots,n \qquad (13)$$

and

$$\boldsymbol{T_{Y/di\text{-}b}} = \left[\frac{c_{di-ij}}{g_{ii} + p_{ii}^*}\right] \quad i,j = 1,2,\ldots,n \qquad (14)$$

Eq.(14) represents the i[th] MISO loop with external disturbances at plant input (See Figure 3).

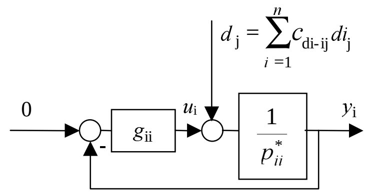



**Fig. 3** Equivalent MISO regulator with disturbances at the plant input. $1 \le i \le n$

The coupling matrix is essential to analyse the reduction of the cross-coupling effects. Therefore, it will be thoroughly studied to quantify the loop interaction effects properly. From Eq.(11) it can be seen that each element of $\boldsymbol{C}_{di}$ obeys,

$$c_{di-ij} = \sum_{k=1}^{m}(p_{ik}^* + g_{ik})\ t_{kj}\,(1 - \delta_{ik}) \qquad (15)$$

where $\delta_{ki}$ is the delta of Kronecker, defined as,

$$\delta_{ki} = \begin{cases}\delta_{ki} = 1 \Leftrightarrow k = i \\ \delta_{ki} = 0 \Leftrightarrow k \ne i\end{cases} \qquad (16)$$

The influence of the non-diagonal elements $g_{ik}$ ($i \ne j$) on the cross-coupling elements described in Eq.(15) is difficult to analyse directly because of the complexity of the expression. However the study of the effect of each $g_{ik}$ is essential to design a fully populated MIMO controller. For this reason, one hypothesis and two simplifications are stated in order to make the quantification of coupling effects easier.

Hypothesis H1: The diagonal elements $t_{jj}$ in Eq. (15) are assumed to be much larger than the non-diagonal ones $t_{kj}$,

$$\left| t_{jj}\left(p_{ij}^{*}+g_{ij}\right)\right| >> \left| t_{kj}\left(p_{ik}^{*}+g_{ik}\right)\right| \quad \text{for } k \neq j \qquad (17)$$

Simplification S1: Applying Hypothesis H1, Eq. (15) can be rewritten as,

$$c_{di-ij} = t_{jj}(p_{ij}^{*}+g_{ij}) \;\; ; \; i \neq j \qquad (18)$$

Simplification S2: The elements $t_{jj}$ can be replaced using the expression obtained from the equivalent system, $[t_{d-ii}]$ in Eq. (9).

Due to the above two considerations the coupling effect $c_{di-ij}$ can be defined as,

$$c_{di-ij} = \frac{(p_{ij}^{*}+g_{ij})}{(p_{jj}^{*}+g_{jj})} \;\; ; \; i \neq j \qquad (19)$$

Note, that every uncertain plant $p_{ij}^{*}$ is represented by the following family,

$$\left\{p_{ij}^{*}\right\} = p_{ij}^{*N}\left(1+\Delta_{ij}\right) \qquad 0 \le \Delta_{ij} \le \Delta p_{ij}^{*}$$

for i,j=1,…, n (20)

Where $p_{ij}^{*N}$ is the nominal plant and $\Delta_{ij}$ the non parametric uncertainty radii.

In order to find out the optimum non-diagonal controller, Eq. (19) is made equal to zero, and a nominal plant that minimises the maximum non-parametric uncertainty radii $\Delta p_{ij}^{*}$ in Eq. (20) is chosen,

$$g_{ij}^{opt} = -p_{ij}^{*N} \qquad (21)$$

The minimum achievable coupling effects can be computed substituting the optimum controller, Eq. (21), in the coupling expression of Eq. (19) and taking into account the uncertainty radii of Eq. (20).

$$\left| c_{ij} \right|_{g_{ij}=g_{ij}^{OPT}} = \left| \frac{p_{ij}^{*N}}{(1+\Delta_{jj})\, p_{jj}^{*N}+g_{jj}} \Delta_{ij} \right| \qquad (22)$$

In the same manner, the maximum coupling effect without any non-diagonal controller -pure diagonal controller cases- can be computed substituting $g_{ij}$=0 in Eq. (19),

$$\left| c_{ij} \right|_{g_{ij}=0} = \left| \frac{p_{ij}^{*N}}{(1+\Delta_{jj})\, p_{jj}^{*N}+g_{jj}} \left(1+\Delta_{ij}\right) \right| \qquad (23)$$

where the uncertainty radii is,

$$0 \le \Delta_{ij} \le \Delta p_{ij}^{*} \;,\; 0 \le \Delta_{jj} \le \Delta p_{jj}^{*} \;,\; \text{for } i,j=1,...,n$$

## 3. REJECTION OF EXTERNAL DISTURBANCES AT PLANT OUTPUT

Consider the *n x n* linear multivariable system shown in Fig.1. The external disturbances at plant output is represented by $\boldsymbol{d_o} = \boldsymbol{P_{do}}\, \boldsymbol{d_o'}$. The closed loop transfer function matrix from external disturbances at plant output $\boldsymbol{d_o'}$ to the output $\boldsymbol{y}$ is called $\boldsymbol{T}_{Y/do}$ and it is obtained from,

$$\boldsymbol{y} = (\boldsymbol{I}+\boldsymbol{P}\,\boldsymbol{G})^{-1}\,\boldsymbol{d_o} = \boldsymbol{T_{Y/do}}\,\boldsymbol{d_o} = \boldsymbol{T_{Y/do}}\,\boldsymbol{P_{do}}\,\boldsymbol{d_o'} \qquad (24)$$

Hence,

$$\boldsymbol{T_{Y/do}} = (\boldsymbol{I}+\boldsymbol{P}\,\boldsymbol{G})^{-1} \qquad (25)$$

Repeating a procedure similar to the previous section, now with external disturbances at plant output, the following results are obtained,

$$\begin{aligned}\boldsymbol{T_{Y/do}} &= (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G})^{-1} + \\ &+ (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1}\Lambda^{-1}\,[\,\boldsymbol{B} - (\boldsymbol{B} + \boldsymbol{G_b}\;\boldsymbol{T_{Y/do}})\,] \qquad (26)\end{aligned}$$

Eq. (26) is divided into a diagonal term and a non-diagonal term.

i. Diagonal term $\boldsymbol{T}_{Y/do-d}$

$$\boldsymbol{T_{Y/do-d}} = (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1} \qquad (27)$$

where

$$\boldsymbol{T}_{Y/do-d} = [t_{do-ii}] = \left[\frac{p_{ii}^{*}}{g_{ii}+p_{ii}^{*}}\right] \qquad (28)$$

Again the MIMO problem is decomposed into *n* MISO systems. This corresponds to the scheme of Fig. 4.

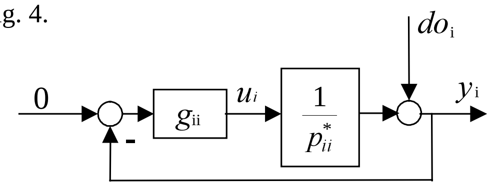



**Fig. 4** Equivalent MISO regulator. $1 \le i \le n$

ii. Non-diagonal term $\boldsymbol{T}_{Y/do-b}$

$$\boldsymbol{T_{Y/do-b}} = (\boldsymbol{I} + \Lambda^{-1}\boldsymbol{G_d})^{-1}\Lambda^{-1}\;[\boldsymbol{B} - (\boldsymbol{B} + \boldsymbol{G_b})]\,\boldsymbol{T}_{Y/do} \qquad (29)$$

where,

$$\boldsymbol{T_{Y/do-b}} = [t_{do-ij}] = \frac{c_{do-ij}}{g_{ii}+p_{ii}^{*}} \qquad (30)$$

Now the coupling matrix for the rejection of external disturbances at plant output is denoted as $\boldsymbol{C}_{do}$.

$$\boldsymbol{C_{do}} = [\,c_{do-ij}\,] = \boldsymbol{B} - (\boldsymbol{B} + \boldsymbol{G_b}\;\boldsymbol{T}_{Y/do}) \qquad (31)$$

Fig. 5 presents the block diagram of the i^th control loop.

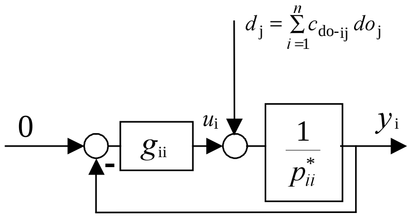



**Fig. 5** Equivalent MISO regulator with disturbances at the plant input and output. $1 \le i \le n$

Each element of the coupling matrix obeys,

$$c_{do-ij} = p^*_{ij}(1-\delta_{ij}) - \sum_{k=1}^{m}(p^*_{ik} + g_{ik})\, t_{kj}\,(1-\delta_{ik}) \qquad (32)$$

where $\delta_{ki}$ is the delta of Kronecker defined in Eq.(16).

Applying again Hypothesis H1 and Simplifications S1 and S2 with Eq.(28), the final expression of the coupling effect can be written as,

$$c_{do-ij} = p^*_{ij} - \frac{p^*_{jj}\,(p^*_{ij} + g_{ij})}{(p^*_{jj} + g_{jj})} \quad ;\ i \neq j \qquad (33)$$

Taking into account Eq.(20) and making Eq.(33) equal to zero, the optimum controller in this case is,

$$g_{ij}^{opt} = g_{jj}\,\frac{p_{ij}^{*N}}{p_{jj}^{*N}} \qquad (34)$$

Finally, the minimum and the maximum achievable coupling effects are computed using an analogous procedure to that presented in the previous section.

$$\left| c_{ij} \right|_{g_{ij}=g_{ij}^{OPT}} = \left| \frac{p_{ij}^{*N}}{(1+\Delta_{jj})\, p_{jj}^{*N} + g_{jj}} \left(\Delta_{ij} - \Delta_{jj}\right) \right| \qquad (35)$$

In the same manner, the maximum coupling effect without any non-diagonal elements in the controller expression is,

$$\left| c_{ij} \right|_{g_{ij}=0} = \left| \frac{p_{ij}^{*N}}{(1+\Delta_{jj})\, p_{jj}^{*N} + g_{jj}} \left(1 + \Delta_{ij}\right) \right| \qquad (36)$$

## 4. DESIGN METHODOLOGY

The proposed design method is a sequential procedure closing loops (Francheck,, *et al.,* 1997) that uses fully populated matrix controllers.

In order to use the design equations developed in the preceding sections, firstly it is necessary to fulfil the Hypothesis H1. And secondly, another Hypothesis H2 is stated.

Hypothesis H2: The plant ***P*** and its inverse ***P***$^*$should be stable and do not have any hidden unstable mode. This is only a sufficient condition to guarantee the stability of the system. Consequently, the designer must pay close attention to systems with non minimum phase or unstable elements (Francheck,, *et al.,* 1997), (De Bedout and Francheck, 2002).

In the last few years, several works have studied the stability problem of MIMO systems with uncertainty, using inverse plants in the QFT methodology. A deep analysis of the subject can be found in three excellent references: Chait and Yaniv, 1991, Houpis and Rasmussen, 1999, Yaniv, 1999. The first paper proofs that it is necessary and sufficient that the plant of each successive loop is stabilised. The second and third references expand the analysis.

In addition, before starting the sequential procedure, it is advisable to analyse the effect of interactions in the system and identify input-output pairings using the Relative Gain Array (RGA), (Bristol, 1966). Afterwards, matrix ***P***$^*$ is rearranged so that $(p^*_{11})^{-1}$ has the smallest phase margin frequency, $(p^*_{22})^{-1}$ the next smallest phase margin frequency, and so on, (Houpis and Rasmussen, 1999).

Then, the design technique, composed of n stages, as many as loops, performs the following steps for every column of the matrix controller ***G***.

Step A*:* Design the diagonal element of the controller $g_{kk}$ for the inverse of equivalent plant described in Eq.(37), using a standard QFT loop-shaping method, (Horowitz, 1982), (Houpis and Rasmussen,1999).

$$\left[p^{*e}_{ii}\right]_k = \left[p^*_{ii}\right]_{k-1} - \frac{\left(\left[p^*_{i(i-1)}\right]_{k-1} + \left[g_{i(i-1)}\right]_{k-1}\right)\left(\left[p^*_{(i-1)i}\right]_{k-1} + \left[g_{(i-1)i}\right]_{k-1}\right)}{\left[p^*_{(i-1)(i-1)}\right]_{k-1} + \left[g_{(i-1)(i-1)}\right]_{k-1}}$$

$$i \ge k;\quad \left[\boldsymbol{P}^*\right]_{k=1} = \boldsymbol{P}^* \qquad (37)$$

Eq.(37) represents the equivalent open-loop transfer function of the channel $i^{th}$ assuming the previous ones have been closed. Note that the expression depends on both diagonal and non-diagonal elements of the controller.

Step B: Design the (n-1) non-diagonal elements $g_{ik}$ ($i \neq k$, i = 1,2,...n) of the $k^{th}$ controller column, minimising the coupling $c_{di-ik}$ or $c_{do-ik}$ described in Eqs. (19), (33), and applying the optimum non-diagonal controller equations Eq. (21) and Eq. (34) respectively.

## 5. EXAMPLE

The present section introduces an example to improve the understanding of the former methodology. Likewise it illustrates the suitability of the technique for the disturbance rejection problem.

**Problem:** Let ***P*** be the *2 x 2* matrix whose elements are transfer functions from inputs **u** to outputs **y** – Eq. (38) –. Each element is described by a set of plants due to the parametric uncertainty showed in Table 1.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{k_{11}}{\tau_{11}+1} & \dfrac{k_{12}}{\tau_{12}+1} \\ \dfrac{k_{21}}{\tau_{21}+1} & \dfrac{k_{22}}{\tau_{22}+1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{38}$$

Table1 – Coefficients of parametric uncertainties

| $k$ | $k_{min}$ | $k_{max}$ | $\tau$ | $\tau_{min}$ | $\tau_{max}$ |
|---|---|---|---|---|---|
| $k_{11}$ | 0.5 | 3 | $\tau_{11}$ | 0.5 | 3 |
| $k_{12}$ | -2.2 | -1.8 | $\tau_{12}$ | 8 | 12 |
| $k_{21}$ | 1 | 5 | $\tau_{21}$ | 3 | 8 |
| $k_{22}$ | 2 | 7 | $\tau_{22}$ | 5 | 10 |

This example is meant to design the controller ***G*** using the former methodology, such that the desired specifications, mainly related to disturbance rejection and stability, must be met.

The desired closed-loop performance specifications are:

- Robust stability in each channel:

$$\left| \frac{L_i}{1+L_i} \right| \le 1.2 \quad \text{i = 1, 2} \tag{39}$$

This means at least 50º lower phase margin and at least 1.833 (5.26 dB) lower gain margin.

- Reduction of coupling effect as much as possible.
- Robust output disturbance rejection so that,

$$\frac{y_i(s)}{d_i(s)} \le \frac{s}{s+10} \qquad \omega < 50\ rad/s, \quad i = 1, 2 \tag{40}$$

The RGA is calculated before starting the iterative controller design method. Computing it for more than 3000 plants generated due to uncertainty, the results show that the best possible pairing are [$y_1$-$u_1$] and [$y_2$-$u_2$].

**Design Procedure:**

*Step A-1:* Standard QFT loop-shaping for $\dfrac{1}{p^*_{11}}$

$$g_{11} = \frac{s+0.36}{s(s+302)} \tag{41}$$

*Step B-1*: By substituting in Eq. (34) the optimum non-diagonal controller results,

$$g_{21} = -\frac{(s+0.3055)(s+0.2)}{s(s+326.7)(s+0.33)} \tag{42}$$

*Step A-2:* Once the first column ($g_{11}$ and $g_{21}$) has been designed, the equivalent plant of the second channel Eq. (43) is calculated.

$$[p^{*e}_{22}]_2 = [p^*_{22}]_1 - \frac{([p^*_{21}]_1 + [g_{21}]_1)([p^*_{12}]_1 + [g_{12}]_1)}{[p^*_{11}]_1 + [g_{11}]_1} \tag{43}$$

Now, the diagonal controller $g_{22}$ for $\dfrac{1}{p^{*e}_{22}}$ using a standard QFT loop- shaping method is,

$$g_{22} = \frac{(s+45)(s+0.14)}{s(s+185)(s+92)} \tag{44}$$

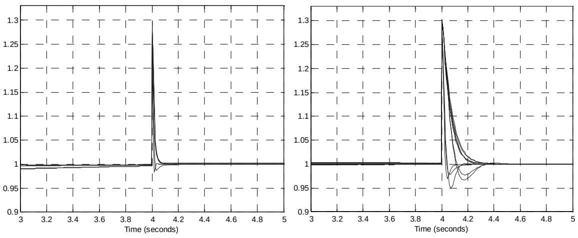
*Step B-2:* Design $g_{12}$ from Eq. (34)

$$g_{12} = \frac{(s+0.14)(s+0.666)(s+45)}{s(s+92)(s+185)(s+0.1)} \tag{45}$$

**Results:**

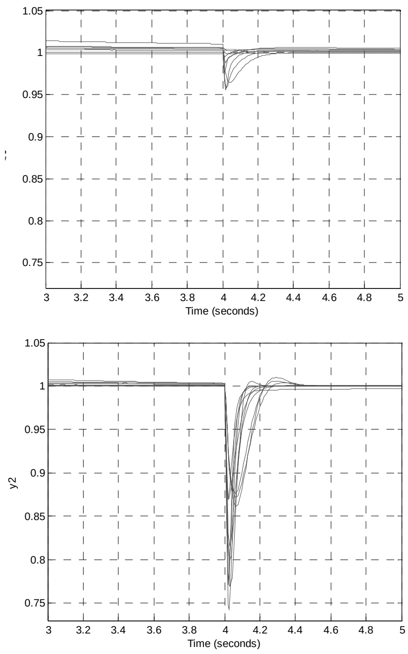
The transient responses of the closed-loop system to a unit step reference and an external disturbance at plant input in the first loop are shown in Fig. 6 and 7. In case (a), a fully populated matrix controller is implemented with the above new methodology, whereas in case (b) an only diagonal controller is applied.

At t = 1 sec. a unit step reference input $r_1$ is given. At t = 4 sec. a 0.3 step disturbance $d_1$ is added at plant output $y_1$. As can be seen in Fig. 7, the closed-loop response to the disturbance is much more satisfactory in the case (a) that is to say, the non-diagonal controller.




(a) (b)

**Fig. 6** Response $y_1$. Disturbance at plant input in the same channel, (a) fully populated controller, (b) diagonal controller.



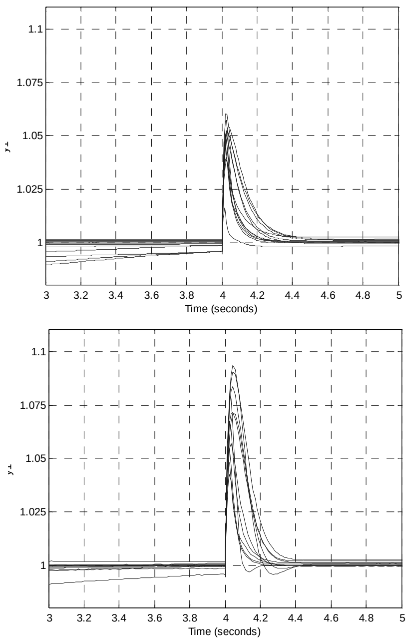

(a)

(b)

**Fig. 7** Response $y_2$ of the 2x2 MIMO system with a disturbance at plant output in first channel, (a) fully populated controller, (b) diagonal controller.

Figures 8 and 9 show the transient responses of the closed-loop system to a unit step reference and a disturbance at plant output in the second loop with a fully populated matrix controller (a) and with an only diagonal controller (b) respectively.

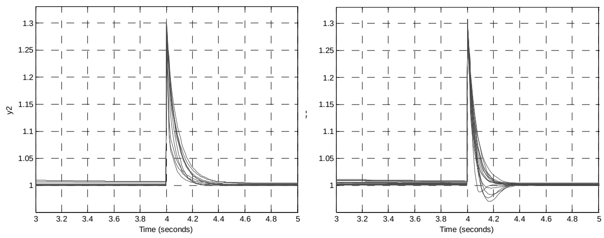
At t = 1 sec. a unit step reference input $r_2$ is applied. At t = 4 sec. a 0.3 step disturbance $d_2$ is added at plant output $y_2$. The results yield that the closed-loop response to the disturbance input is better, once again, with a fully populated controller.




(a)

(b)

**Fig. 8** Response of the first channel ($y_1$) of the 2x2 MIMO system with a disturbance at plant output in the second channel, (a) fully populated controller, (b) diagonal controller.




(a) (b)

**Fig. 9** Response of the second channel ($y_2$) of the 2x2 MIMO system with a disturbance input in the same channel, (a) fully populated controller, (b) diagonal controller.

## 5. CONCLUSIONS

A QFT methodology to design fully populated matrix controllers to solve the MIMO external disturbance rejection problem at both, plant input and output, and in the presence of model plant uncertainty, was presented. The definition of a specific coupling matrix allows the statement of the controller design methodology addressed in this paper.

**Acknowledgement**

Authors wish to gratefully appreciate the support given by the Spanish “Ministerio de Ciencia y Tecnología” (MCyT) under grants CICYT DPI’2000-0785 and DPI’2003-08580-C02-01.

**Referentes**

Boje E, Nwokah ODI. Quantitative multivariable feedback design for a turbofan engine with forward path decoupling. International Journal of Robust and Nonlinear Control 1999; 9 (12), pp. 857-882.

Bristol EH. On a new measure of interactions for multivariable process control. Transactions on Automatic Control 1966; 11: 133-134.

Chait, Y. and Yaniv, O. (1991). Disturbance rejection in flexible structures via the quantitative feedback theory. Dynamics and control of large structures: Proceedings of the 8th VPI & SU Symposium,pp.445-446.

De Bedout, J.M. and Francheck, M.A.,(2002). Stability conditions for the sequential design of non-diagonal multivariable feedback controllers. International Journal of Control. Vol **75**, No 12, 910-922.

Francheck, M.A., P.Herman, and O.D.I.Nwokah (1997). Robust nondiagonal controller design for uncertain multivariable regulating systems. ASME Journal of Dynamic Systems, Measurement and Control; 119:80-85.

García-Sanz, M. and I. Egaña (2002). Quantitative Non-diagonal Controller Design for Multivariable Systems with Uncertainty. International Journal of Robust and Non-Linear Control. Isaac Horowitz Special Issue, Part 2. Nº 12, pp. 321-333. John Wiley & Sons. USA. ISSN: 1049-8923.

Houpis, C.H. and S. J. Rasmussen (1999). Quantitative Feedback Theory Fundamentals and Applications. Marcel Dekker, New York, ISBN: 0-8247-7872-3.

Horowitz I. Quantitative synthesis of uncertain multiple input-output feedback systems, International Journal of Control 1979; 30: 81-106.

Horowitz I. Improved design technique for uncertain multiple input-output feedback systems. *International Journal of Control* 1982; **36**: 977-988.

Horowitz I. Survey of quantitative feedback Theory (QFT), *International Journal of Control* 1991; **53:** 255-291.

Horowitz I. Quantitative feedback Theory (QFT), Vol. 1. QFT Publications: Boulder Colorado, 80303, 1992.

Yaniv, O. (1995). MIMO QFT using non-diagonal controllers. International Journal of Control 1995; 61 (1), pp. 245-253.

Yaniv, O.(1999).Quantitative Feedback Design of Linear and Non-linear Control Systems. Kluuler Academic Publishers, Norwell,M.A