ON THE STOCHASTIC CONTROLLABILITY OF HO-LEE, HULL-WHITE, BLACK-KARASINSKI AND COX-INGERSOLL-ROSS INTEREST RATE MODELS

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Abstract: In the recent contributions Bosch and Petersen (2003) and Petersen, *et al.*, (2003) the notion of stochastic controllability within the framework of linear interest rate models of Heath-Jarrow-Morton-Musiela (HJMM) type was introduced. There the aforementioned interest rate model was represented by an infinite dimensional stochastic differential equation whose drift term could be influenced by a special type of (additive) control function. As was the case in that paper, our contribution will concentrate exclusively on derived differential models that have mild solutions. Interest rate models belonging to the HJMM class that are widely used in practice are the Ho-Lee, Hull-White, Black-Karasinsky and Cox-Ingersoll-Ross models. An investigation into the stochastic controllability of these specific models is therefore of considerable interest to practitioners in the financial sector.

Keywords: Stochastic Control, Financial Systems.

1. INTRODUCTION

This contribution investigates the controllability of linear interest rate models in practice and is a natural extension of the paper Petersen, *et al.* (2003). This paper provided necessary an sufficient conditions for the stochastic controllability of linear stochastic interest rate models of the Heath-Jarrow-Morton-Musiela (HJMM) type (see Heath, *et al.*, 1992; Brace and Musiela 1994 and Musiela, 1994). In the sequel, we shall investigate four examples of the use of stochastic optimal control in solving problems related to interest rate models of HJMM type that are used in practice.

The motivation for such research is that in order to make the contribution Petersen *et al.* (2003) of greater practical significance one must be able to apply the results to models currently used in the financial industry. Although this short paper does not furnish a numerical example, it attempts to narrow the gap between the theory and practice in the field of continuous time interest rate theory.

2. THE CONTROLLABILITY OF HJMM INTEREST RATE MODELS

In this section, we provide a brief description of the HJMM interest rate model that we will consider. In addition, in Subsection 2.2, we decide on the most economic Hilbert space to be considered as a state space of forward rate curves. In Subsection 2.3 we construct the subclass of HJMM interest rate models that is amenable to stochastic controllability. Finally, in Subsection 2.4 we collect important information that will be used in the main section.

2.1 *Basic Description*

As was described in Heath, *et al.*, (1992), the HJM interest rate model for the forward curve $x \mapsto r(t,x)$ is fixed by the structure of its volatility σ and the market price of risk. In this case, $r(t,x)$ is the notation used to denote the forward rate at time t with maturation date $t+x$. In this model, we consider a default free, frictionless bond market with perfectly divisible bonds on a filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{t\geq 0},\ \mathbf{P})$. Furthermore, we denote the price at time t of a zero coupon bond maturing at $t+x$ by $p(t,x)$, where x is time to maturity and

$$p(t,x) = \exp\{-y(t,x)\},$$

where the period yield $y(t,x)$ is defined by

$$y(t,x) = \int_0^x r(t,s)ds.$$

The expression $\int_0^x \cdot(t,s)ds$ denotes integration with respect to time to maturity x. Also, $r(t,x)$ the forward rate contracted at t maturing at $t+x$ has the form

$$r(t,x) = -\frac{\partial \log p(t,x)}{\partial x}.$$

Moreover, we denote the short rate by $R(t)$, where $R(t) = r(t,0)$. As is well-known the HJMM approach addresses the question of the modelling of the dynamics for the entire forward rate curve. Here the yield curve r is the state variable rather than the short rate R.

As regards notation, in the ensuing discussion the forward rate at time t with maturation date $t+x$, is simply denoted by $r(t)$. From Filipović, (2001), we know that every classical HJM model can more or less be realized as a stochastic differential equation (SDE) of the form

$$\begin{cases} dr(t) = (Ar(t) + D(t))\,dt + \sigma(r(t))dW(t), \\ r(0) = r^*(0), \end{cases} \tag{1}$$

where W is an m-dimensional Wiener process, $\sigma(r(t))dW(t) = \sum_{j=1}^{m} \sigma_j(r(t))dW_j(t)$ and the initial curve $\{r^*(0,x) :\ x \geq 0\}$ is interpreted as the observed forward rate curve. This equation evolves on some open convex subset $\mathcal{U}$ in a separable Hilbert state space $\mathcal{H}$ (to be specified in Subsection 2.2 below) of forward rate curves. More specifically, we have for $A,\ D \in \mathcal{L}(\mathcal{U},\mathcal{H})$ and σ that

$$A = \frac{\partial}{\partial x} : \mathrm{Dom}\ A \subset \mathcal{H} \to \mathcal{H};\quad D : \mathcal{U} \subset \mathcal{H} \to \mathcal{H};$$
$$\sigma = (\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_m) : \mathcal{U} \subset \mathcal{H} \to \mathcal{H}^m,$$

respectively. Also, the model must be arbitrage-free which leads to the existence of an equivalent local martingale measure $\mathbf{Q} \sim \mathbf{P}$. In this case the drift term in the equation (1) can be written in terms of the volatility, σ, and be specified as

$$\frac{\partial}{\partial x} r(t) + D(t) = \frac{\partial}{\partial x} r(t) + \sigma(r(t)) \int_0^x \sigma(r(s))ds. \tag{2}$$

Here (1) is commonly referred to as the **HJMM equation** and (2) is called the **HJMM drift condition**. This means that the pricing formula for interest rate sensitive contingent claims only depend on σ. Furthermore, the deterministic counterpart of the stochastic linear HJMM interest rate model (1) may be represented as

$$\begin{cases} dr(t) = (Ar(t) + D(t))\,dt, \\ r(0) = r^*(0). \end{cases} \tag{3}$$

2.2 *State Space*

Our choice of state space of forward rate curves may be described as follows. Firstly, we have to assume that our state space $\mathcal{H}$ is separable. Furthermore from Filipović, (2001) we know that it is preferable that $\mathcal{H}$ is continuously embedded in $C([0,\infty);\mathbf{R})$. In other words, for any choice of $x \in [0,\infty)$ the pointwise evaluation $r \longmapsto r(x)$ is a linear functional on $\mathcal{H}$ that is continuous. Furthermore, $\mathcal{H}$ contains the constant function 1. We also insist that the family of right shifts

$$S_t r = r(t+x) \text{ for } t \in [0,\infty) \tag{4}$$

forms a strongly continuous semigroup on $\mathcal{H}$ with generator $\dfrac{\partial}{\partial x}$. Furthermore, we may assume that the domain of $\dfrac{\partial}{\partial x}$ has the form

$$\left\{ h \in \mathcal{H} \cap C^1([0,\infty);\mathbf{R}) : \ \frac{\partial}{\partial x} h \in \mathcal{H} \right\}.$$

The state space described above has all the properties needed to perform our analysis in the sequel in an economic setting. For the purposes of our discussion, however, we need to bear in mind that the stochastic differential equation (1) evolves on some open convex subset $\mathcal{U}$ of the state space $\mathcal{H}$ and not on $\mathcal{H}$ itself. Furthermore, since every subset of a separable inner product space is separable we have that $\mathcal{U}$ described above is also separable.

2.3 *An Appropriate Subclass of HJMM Interest Rate Models*

From Zabczyk, (1991) we know that if $u(\cdot) \in \mathcal{U}_{ad} \subset \mathcal{U}$ then for $\widetilde{r}(t)$ from the partially observable infinite dimensional HJMM model of the form

$$\left\{ \begin{array}{rcl} d\widetilde{r}(t) &=& (A\widetilde{r}(t) + D(t)u(t))\,dt + \sigma^{'}\,d\widetilde{W}(t), \\ dz(t) &=& C\widetilde{r}(t) + Fd\widehat{W}(t), \\ r(0) &=& r^*(0), \end{array} \right. \tag{5}$$

where $D(t)$ in (1) acts on u that belongs to some admissible control set $\mathcal{U}_{ad} \subset \mathcal{U}$, the Kalman filter

$$r(t) = \mathbf{E}\{\widetilde{r}(t)|\mathcal{Z}_t\} = \mathbf{E}\{\widetilde{r}(t)|\mathcal{Z}_t^0\}$$

is the mild solution of the linear HJMM interest rate model

$$\left\{ \begin{array}{rcl} dr(t) &=& (Ar(t) + D(t)u(t))\,dt + \sigma(r(t))dW(t), \\ r(0) &=& r^*(0), \end{array} \right. \tag{6}$$

where $u(t)$ is given as

$$u(t) = w(t) + \int_0^t K(t,s)dz(s) \tag{7}$$

(see Petersen, *et al.*, 2003). Also, we have that

$$D(t) = \sigma(r(t)) \int_0^x \sigma(r(s))ds \tag{8}$$

and

$$\sigma(r(\cdot)) = P(\cdot)C^*(FQF^*)^{-1}F, \tag{9}$$

with the linear operator $P(\cdot)$ satisfying the algebraic Riccati equation

$$\begin{gathered} \frac{d}{dt}\langle P(t)h,k\rangle - \langle P(t)h, A^*k\rangle - \langle A^*h, P(t)k\rangle \\ -\langle \sigma^{'}\widetilde{Q}\sigma^{'*}h,k\rangle + \langle P(t)C^*(FQF^*)^{-1}CP(t)h,k\rangle = 0, \\ P(0) = P_0, \quad h,k \in D(A^*). \end{gathered}$$

We note that this special subclass of HJMM interest rate models for the forward curve $r(t)$ given by (6) is fixed by the structure of its volatility σ in (9) that, in turn, may be expressed in terms of the coefficients and covariance associated with a partially observable interest rate model of the type given by (5). Furthermore, we assume that P may be chosen in such a way that the volatility (9) is (locally) Lipschitz continuous in r. From Filipović, (2001) we have that in this case, σ can be shown to be (locally) bounded. An important consequence of this is that the linear operator $D(t)$ in (6) is (locally) bounded which is a prerequisite for the analysis in the sequel.

As was demonstrated in Da Prato and Zabczyk, (1992) and Filipović, (2001) developing a notion of solution for general infinite dimensional stochastic systems can be a tricky business. For instance, it is well-known that we are able to distinguish between mild, weak and strong solutions of such systems. In particular, strong solutions are very seldom encountered in the context of interest rate models. Under the conditions specified in the previous paragraph (see Zabczyk, 1991, for more details), we are able to write the mild solution $r(t;r_0,u)$ of (6) explicitly as

$$\begin{aligned} r(t;r_0,u) = S_t r^*(0) &+ \int_0^t S_{t-s}D(s)u(s)ds \\ &+ \int_0^t S_{t-s}\sigma(r(s))dW(s). \end{aligned} \tag{10}$$

In addition, we note that a deterministic counterpart of (6) may be given as

$$\begin{array}{rcl} \dfrac{dr_d(t)}{dt} &=& Ar_d(t) + D(t)v(t), \\ r_d(0) &=& r_d^*(0), \end{array} \tag{11}$$

with a solution of the form

$$r_d(t) = S_t r_d^*(0) + \int_0^t S_{t-s} D(s) v(s) ds. \quad (12)$$

2.4 *Important Applicable Results*

The following information from Petersen *et al.* (2003) will be utilised in our subsequent discussions:

Under the conditions specified in Subsection 2.3 and Section 3 of Petersen *et al.* (2003), the mild solution $r(t; r_0, u)$ of the HJMM interest rate model (6) is given explicitly as in (10), where $D(t)$ in the drift term from (6) can be written in terms of the volatility σ as

$$D(t) = \sigma(r(t)) \int_0^x \sigma(r(s)) ds. \quad (13)$$

and the optimal control, $u^0(t)$, is found to be

$$u^0(t) = -D(t)^* S_{T-t}^* \left\{ (\Gamma_0^T)^{-1} (S_T r_0 - \mathbf{E}(h)) + \int_0^t (\Gamma_0^T)^{-1} [S_{T-s} \sigma(r(s)) - h(s)] dW(s) \right\}$$

where Γ represents the deterministic controllability operator given by

$$\Gamma_s^T = \int_s^T S_{T-t} D(t) D(t)^* S_{T-t}^* dt. \quad (14)$$

3. APPLICATIONS TO HO-LEE, HULL-WHITE, BLACK-KARASINKI AND COX-INGERSOLL-ROSS MODELS

The main question that we hope to answer in this our main section, may be formulated as follows:

What results do we obtain when we apply the general arguments in Petersen et al. *(2003) to specific interest rate models like Ho-Lee, Hull-White, Black-Karasinski and Cox-Ingersoll-Ross models ?*

3.1 *The Ho-Lee Model*

Historically, the Ho-Lee model (see Ho and Lee, 1986) was the first of the deterministic no-arbitrage models for interest rates. Ho and Lee pioneered the approach that proposed an exogeneous evolution for the term structure of rates in contrast with the endogeneous models that reproduced current term structures. Their model is based on the assumption that rates evolve as a binomial tree. In our discussion, we use the continuous time version as derived by Dybvig (Dybvig, 1988) and Jamshidian (see Jamshidian, 1988). It cannot, however, be regarded as a proper extension of the Vasicek model (see Vasicek, 1977), since in the continuous case it does not retain the mean reversion property in the short rate dynamics.

The Ho-Lee model for the instantaneous short rate is given by

$$dr(t) = \theta(t) dt + \sigma dW(t). \quad (15)$$

Furthermore, in this case, it follows that the diffusion coefficient may be represented as

$$\sigma(r(t)) = \sigma.$$

This information leads us to the following conclusions about the linear operators $D(t)$, Γ_s^T and the optimal control $u^0(t)$ for the Ho-Lee model with mild solutions.

$$D(t) = \sigma^2 x; \quad (16)$$

$$\Gamma_s^T = \sigma^4 x^2 (T - s); \quad (17)$$

$$u^0(t) = -\frac{1}{\sigma^2 x T} \left\{ (r(T) - \mathbf{E}(h)) + \int_0^t [\sigma - h(s)] dW(s) \right\}.$$

3.2 *The Hull-White Model*

Due to the poor fitting of the initial term structure as implied by the Vasicek model (see Vasicek, 1977) extensions were addressed in subsequent papers. Although Ho and Lee (1986) did, to some extent, address the deficiencies in the Vasicek model, it cannot be regarded as a proper extension as it does not retain the mean reversion property in the dynamics of the Vasicek model. Hull and White (1990) proposed to include an additional deterministic parameter to the drift coefficient. This determinsitic model is still being used for risk management purposes today, and is of historical importance as procedures developed for it are easily applied to some of the other models in this class. A drawback, as in the case of the Vasicek model, is that this model allows for negative interest rates and is thus hardly applicable to concrete pricing.

The Hull-White model (see Hull and White, 1990a, 1990b, 1993a, 1993b, 1994a, 1994b and 1994c) for the instantaneous short rate is given by

$$dr(t) = (\theta(t) - ar(t)) dt + \sigma dW(t) \quad (18)$$

The diffusion coefficient for the Hull-White model is

$$\sigma(r(t)) = \sigma e^{-cx}.$$

We make the following conclusions about the linear operators $D(t)$, Γ_s^T and the optimal control $u^0(t)$ in terms of the mild solution for the Hull-White model

$$D(t) = \frac{\sigma^2}{c} e^{-cx}[1 - e^{-cx}]; \tag{19}$$

$$\Gamma_s^T = \frac{\sigma^4}{c^2} e^{-2cx}[1 - e^{-cx}]^2 (T - s); \tag{20}$$

$$u^0(t) = -\frac{1}{\frac{\sigma^2}{c} e^{-cx}[1 - e^{-cx}]T} \left\{ (r(T) - \mathbf{E}(h)) \right. \\ \left. + \int_0^t [\sigma e^{-cx} - h(s)] dW(s) \right\}.$$

3.3 *The Black-Karasinski Model*

The Black-Karasinski model (see Black and Karasinski, 1991) is a generalisation of the Black-Derman-Toy model (see Black, *et al.*, 1990). It addresses the problem of a negative short rate by utilising a lognormal short rate model. This seems a reasonable choice as the lognormal distribution is often chosen in market formulas, e.g. Black and Scholes, (1973). On the negative side this model in not analytically tractable. This makes for more difficult calibration than the Gaussian models like Hull-White, as explicit models for bonds do not exist. Another and more fundamental drawback is that the expected value of a money market account is infinite no matter what maturity is considered. This problem can partly be overcome by approximating the model using a tree in a similar fashion to when we are working with a finite number of states and by implication finite expectation.

The Black-Karasinski model for the instantaneous short rate may be represented by

$$dr(t) = r(t) \left(\theta(t) + \frac{\sigma^2}{2} - a \ln r(t) \right) dt \\ + \sigma r(t) dW(t). \tag{21}$$

The diffusion coefficient for this model is given by

$$\sigma(r(t)) = \sigma r(t).$$

For the Black-Karasinksi model we make the following conclusions about the linear operators $D(t)$, Γ_s^T and the optimal control $u^0(t)$ in terms of the mild solution.

$$D(t) = \sigma^2 r(t) \int_0^x r(s) ds; \tag{22}$$

$$\Gamma_s^T = \sigma^4 \left(\int_0^x r(s) ds \right)^2 r(T)^2 (T - s); \tag{23}$$

$$u^0(t) = -\sigma^2 r(T) \int_0^x r(s) ds \\ \times \left\{ \left(\sigma^4 \left(\int_0^x r(s) ds \right)^2 r(T)^2 T \right)^{-1} \right. \\ \times (r(T) - \mathbf{E}(h)) \\ + \int_0^t \left(\sigma^4 \left(\int_0^x r(s) ds \right)^2 r(T)^2 T \right)^{-1} \\ \left. \times [\sigma r(t + T - s) - h(s)] dW(s) \right\}.$$

3.4 *The Cox-Ingersoll-Ross Model*

The Cox-Ingersoll-Ross model (see Cox, *et al.* 1985) aims to address the deficiencies in the Vasicek model (see Vasicek, 1977), namely allowing negative interest rates with positive probability. Negative interest rates would correspond to being paid for borrowing. While this has happened in the past, e.g., in Japan, it is irregular and unlikely. This led to the introduction of the "square-root" term in the diffusion coefficient of the instantaneous short rate. The CIR model has been regarded as a benchmark for many years, because the instantaneous short rate is always positive and it is analytically tractable. The CIR model is however less tractable than the Vasicek model, especially where the extension to the multi factor case is concerned.

The CIR model for the instantaneous short rate is given by

$$dr(t) = a(b - r(t))dt + \sigma \sqrt{r(t)} dW(t) \tag{24}$$

The diffusion coefficient may be represented by

$$\sigma(r(t)) = \sigma \sqrt{r(t)}.$$

For the CIR model we deduce the following about the linear operators $D(t)$, Γ_s^T and the optimal control $u^0(t)$ in terms of the mild solution.

$$D(t) = \sigma^2 \sqrt{r(t)} \int_0^x \sqrt{r(s)} ds; \tag{25}$$

$$\Gamma_s^T = \sigma^4 \left(\int_0^x \sqrt{r(s)} ds \right)^2 r(T)(T - s); \tag{26}$$

$$
\begin{aligned}
u^0(t) = &-\sigma^2 \sqrt{r(T)} \int_0^x \sqrt{r(s)} ds \\
&\times \left\{ \left(\sigma^4 \left(\int_0^x \sqrt{r(s)} ds \right)^2 r(T)T \right)^{-1} \right. \\
&\times (r(T) - \mathbf{E}(h)) \\
&+ \int_0^t \left(\sigma^4 \left(\int_0^x \sqrt{r(s)} ds \right)^2 r(T)T \right)^{-1} \\
&\left. \times \, [\sigma \sqrt{r(t+T-s)} - h(s)] dW(s) \right\}.
\end{aligned}
$$

4. CONCLUSION AND ONGOING INVESTIGATIONS

In this paper we have investigated the notion of stochastic controllability for the Ho-Lee, Hull-White, Black-Karasinsky and Cox-Ingersoll-Ross linear interest rate models that belong to the general class of HJMM models.

The results in this paper are by no means definitive and may be extended to include more areas of application for linear interest rate models of HJMM type. Also, we would like to derive numerical examples to illustrate the theory developed in this paper more clearly.

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