TIME-OPTIMAL CONTROL OF ROBOTIC MANIPULATORS MODELLED WITH ACTUATOR DYNAMICS

J. O. Pedro [*,1] **M. Mthethwa** [*] **O. T. Nyandoro** [**]

[*] *School of Mechanical, Industrial and Aeronautical Engineering*
[**] *School of Electrical and Information Engineering, University of the Witwatersrand, Johannesburg.*

Abstract: The paper presents algorithms to compute time-optimal solutions for a two-link robotic manipulator operating in the horizontal plane subject to control (motor input voltage) constraints. The system dynamic equations with the inclusion of actuator dynamics are derived using Lagrange's formalism. Application of Pontryagin Maximum Principle (PMP) results in a nonlinear two-point boundary value problem (TPBVP). The singular control problem is solved by using an ϵ-transformation method. The solution of the voltage-constrained model relied on good initial guesses of the adjoint variables' boundary conditions and the final time.

Keywords: Robotic Manipulators, Actuator Dynamics, Time-Optimal Control, Energy-perturbation Method, Point-to-point Motion, Continuation and Multiple-shooting Method.

1. INTRODUCTION

There is a great demand for applications of robots in manufacturing processes and medical applications especially for high precision laser eye surgery and cancer treatment. It is expected that these applications should be accompanied by an increase in efficiency, precision, safety and cost savings. Two categories of motion can be identified during the operation of a robotic manipulator: a) point-to-point motion, and b) prescribed path tracking (Chettibi and Lehtihet, 2002). The present work, which is an extension of the work in (Pedro, 2003), concentrates on time optimal control of robot point to point motion as the later has been extensively researched. Attempts have been made by several authors to solve the time optimal point-to-point problems for robots as in (Fotouhi-C and Szyszkowski, 1998; von Stryk, 1994; Chen and Huang, 1993) to name a few.

The approaches for solving the optimal control problem are twofold which are: a) direct, which is based on nonlinear programming, and indirect which is based on the application of PMP. Most of the work done on the indirect approach is restricted to bang-bang solutions, therefore avoiding the more challenging singular control cases (Fotouhi-C and Szyszkowski, 1998; Chen and Jr, 1990), while very few address the time-optimal control problem with both state and control contraints (von Stryk, 1994; Shiller and Lu, 1992). The case of singular control arising in the solution of the problem has been dealt with by means of an energy perturbation method as suggested in (Chen and Huang, 1993; Chen and Desrochers, 1993).

[1] All correspondence should be addressed to this author, Email: jpedro@mech.wits.ac.za, Tel: 011 717 7317

An indirect collocation method to solve the time-optimal cotrol problem for robot point-to-point motion is developed in this work. The method is based on the application of PMP to a modified optimal control problem. The modification is based on converting the singular time-optimal control problem to a sequence of non-singular ones by introducing energy pertubation parameters (denoted by ϵ) in the performance index. The proposed method can handle both singular and non-singular controls, and a wide spectrum of performance indices and constraints on the control variables. This paper however does not consider the displacement, velocity and acceleration constraints on the manipulator's arms, joints and end effector.

2. PROBLEM FORMULATION

Owing to the complexity of the mechanical structure, mathematical models of the dynamic behaviour of the robot are too complicated to be completely solved with all the coefficients to be considered. The following simplifying assumptions are used in this paper: a) frictional forces are neglected, b) all components and links are rigid, c) thermal expansion of components is not considered, d) viscous friction factor in the motor assembly is small and can therefore be neglected, and e) gear backlash is neglected.

The problem considered here is to develop a simple algorithm to analyze the point-to-point motion of the two-link robotic manipulator taking into account the dynamics of the actuators as well as their limitations. The system model can be realized by the generalized state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{t})$. The initial and final positions of the manipulator links are given as: $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$. The problem is to find the control vector, $\mathbf{u}$, subject to constraints $\mathbf{u} \in \mathbf{U}$ such that the performance index, $J = \int_0^{t_f} L dt$ is minimised. The perfomance index in this case is the minimum time i.e., $J = t_f$, where $L = 1$.

3. THE MATHEMATICAL MODEL

In deriving the dynamic equations of the robotic manipulator, we include the dynamics of the dc motors that supply the torques at the joints. A thourough discussion on the inclusion of actuator dynamics is given in (Sage *et al.*, 1999; Mahmoud, 1993; Tarn *et al.*, 1989). Figure 1 shows the robotic manipulator degrees of freedom. Application of Lagrange formalism to the two-link robotic manipulator gives the dynamic equations of motion as follows:

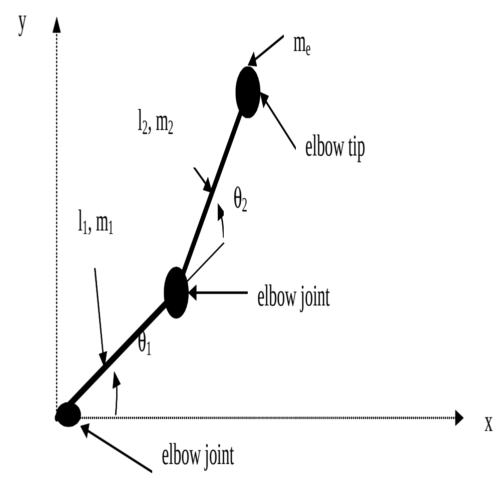

Fig. 1. Degrees of freedom of a rigid two-link robotic manipulator.

$$\mathbf{M}(\theta)\ddot{\theta} + \mathbf{C}(\theta, \dot{\theta}) + \mathbf{K}(\theta) = \tau(\theta, \dot{\theta}, \ddot{\theta}) \qquad (1)$$

where $\theta = [\theta_1, \theta_2]^T$ is the generalized angular joints position vector, $\tau = [\tau_1, \tau_2]^T$ is the vector of the generalized torques applied at the joints, $\mathbf{M}(\theta, \dot{\theta})$ is the links' inertia matrix, $\mathbf{C}(\theta, \dot{\theta})$ is the vector of the Coriolis and centripetal torques, and $\mathbf{K}(\theta)$ is the vector of the gravitational torques. For the particular case being considered in this paper, $\mathbf{K}(\theta) = 0$, as the motion is assumed to be taking place in the horizontal plane. Therefore, equation (1) becomes:

$$\mathbf{M}(\theta)\ddot{\theta} + \mathbf{C}(\theta, \dot{\theta}) = \tau(\theta, \dot{\theta}, \ddot{\theta}) \qquad (2)$$

The dynamic equations for the armature-controlled dc motors at the joints can be described as follows (Engelmann and Middendorf, 1995):

$$\mathbf{Ri} + \mathbf{L}\frac{d\mathbf{i}}{dt} + \mathbf{K_e}\dot{\theta}_\mathbf{m} = \mathbf{u} \qquad (3)$$

where: $\mathbf{R}$ is the diagonal matrix containing the resistances of the armature circuits, $\mathbf{i}$ is the vector of armature currents, $\mathbf{L}$ is the diagonal matrix of the armature circuits' inductances, $\mathbf{K_e}$ is the diagonal matrix containing the back EMF constants, θ_m is the positions of the actuators' shafts, $\mathbf{u}$ represents the armature input voltages. Vector of the supplied torques by the dc motors is $\tau_m = \mathbf{K_{Tm}}\mathbf{i}$, where $\mathbf{K_{Tm}}$ is the diagonal matrix containing the motor torque constants. The relations between the motors' angular positions and torques to the joints' angular positions and torques are as follows: $\theta_\mathbf{m} = \mathbf{N}\theta, \quad \tau = \mathbf{N}\tau_m$.

Combining equations (2) and (3) gives the dynamic equations of motion of the robotic manipulator with inclusion of the actuators dynamics as follows:

$$\mathbf{J_N}\frac{d^3\theta}{dt^3} + \mathbf{D_N}\frac{d^2\theta}{dt^2} + \mathbf{C_N}\frac{d\theta}{dt} + \mathbf{R_N} + \mathbf{L_N} = \mathbf{u} \quad (4)$$

where:

$$\begin{aligned}
\mathbf{J_N} &= [\mathbf{L}(\mathbf{NK_{Tm}})^{-\mathbf{1}}\mathbf{M}(\theta)] \\
\mathbf{D_N} &= [\mathbf{R}(\mathbf{NK_{Tm}})^{-\mathbf{1}}\mathbf{M}(\theta) + \mathbf{L}(\mathbf{NK_{Tm}})^{-\mathbf{1}}\dot{\mathbf{M}}(\theta)] \\
\mathbf{C_N} &= [\mathbf{NK_e}] \\
\mathbf{R_N} &= [\mathbf{R}(\mathbf{NK_{Tm}})^{-\mathbf{1}}\mathbf{C}(\theta,\dot{\theta})] \\
\mathbf{L_N} &= [\mathbf{L}(\mathbf{NK_{Tm}})^{-\mathbf{1}}\dot{\mathbf{C}}(\theta,\dot{\theta})]
\end{aligned}$$

The control constraints for the robotic manipulator are as follows:

$$U_{1min} \leq u_1(t) \leq U_{1max} \quad (5)$$

$$U_{2min} \leq u_2(t) \leq U_{2max} \quad (6)$$

The dynamic equations must then be rearranged so that they represent the rate of change of angular accelerations $\frac{d^3\theta_1}{dt^3}$ and $\frac{d^3\theta_2}{dt^3}$. Let the following state vector be defined:

$$\begin{aligned}
\mathbf{x} &= (X_1, X_2, X_3) = (x_1, x_2, x_3, x_4, x_5, x_6)^T, \\
X_1 &= (\theta_1, \theta_2) \\
X_2 &= (\dot{\theta}_1, \dot{\theta}_2) \\
X_3 &= (\ddot{\theta}_1, \ddot{\theta}_2)
\end{aligned}$$

The system equation (4) can be written in state-space form by defining the generalised position, velocity, and acceleration vectors as the state vector (von Stryk, 1994):

$$\begin{cases} x_1 = \theta_1 \\ x_2 = \theta_2 \\ x_3 = \dot{\theta}_1 \\ x_4 = \dot{\theta}_2 \\ x_5 = \ddot{\theta}_1 \\ x_6 = \ddot{\theta}_2 \end{cases} \Rightarrow \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = X_2, \quad \text{and} \quad (7)$$

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_5 \\ x_6 \end{pmatrix} = X_3 \quad (8)$$

Making $\dot{X}_3$ the subject of equation (4):

$$\begin{aligned}
\begin{pmatrix} \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} &= -\mathbf{J_N^{-1}}(X_1)[\mathbf{D_N}(X_1, X_2)X_3 + \mathbf{C_N}X_2 \\
&\quad +\mathbf{R_N}(X_1, X_2) + \mathbf{L_N}(X_1, X_2, X_3)] \\
&\quad +J_N^{-1}(X_1)\mathbf{u} \qquad (9)
\end{aligned}$$

Hence, the Hamiltonian of the problem stated above is:

$$\begin{aligned}
H &= 1 + \lambda^T\dot{\mathbf{x}} = 1 + (\Lambda_1)^T X_2 + (\Lambda_2)^T X_3 \\
&\quad -(\Lambda_3)^T[\mathbf{J_N^{-1}}(X_1)[\mathbf{D_N}(X_1, X_2)X_3 \\
&\quad +\mathbf{C_N}X_2 + \mathbf{R_N}(X_1, X_2)+ \\
&\quad \mathbf{L_N}(X_1, X_2, X_3)] + \mathbf{J_N^{-1}}(X_1)\mathbf{u}]
\end{aligned} \quad (10)$$

Necessary optimality conditions are obtained by applying PMP (Bryson Jr. and Y-C., 1975), (Pontryagin *et al.*, 1962). Defining the adjoint variables as:

$\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)^T$, where $\Lambda_1 = (\lambda_1, \lambda_2)^T, \Lambda_2 = (\lambda_3, \lambda_4)^T$, and $\Lambda_3 = (\lambda_5, \lambda_6)^T$

The necessary optimality conditions from PMP are as follows:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i}, \quad \dot{\lambda}_i = -\frac{\partial H}{\partial x_i}, \quad \frac{\partial H}{\partial u_j} = 0, \quad i = 1, ...6, j = 1, 2 \quad (11)$$

$$H(\mathbf{x}^*(t), \lambda^*(t), \mathbf{u}^*(t)) \leq H(\mathbf{x}^*(t), \lambda^*(t), \mathbf{u}(t)) \quad \text{for} \quad t \in (0, t_f) \quad (12)$$

In addition to the conditions in (11) and (12), the final time, t_f, can be determined from:

$$H(\mathbf{x}(t), \lambda^*(t), \mathbf{u}^*(t)) \equiv 0 \quad \text{for all} \quad t \in (0, t_f) \quad (13)$$

Since the Hamiltonian does not depend explicitly on time.
The third condition in (11) gives the jth-component of the switching functions $\kappa(t)$ as:

$$\kappa_j(t) = (\Lambda_3)^T(\mathbf{J_N}(X_1))^{-1} \quad (14)$$

and the control sequence for the jth-input voltage is:

$$u_j = \begin{cases} U_{jmax} & \text{if } \kappa_j(t) < 0 \\ U_{jmin} & \text{if } \kappa_j(t) > 0 \\ \text{singular} & \text{if } \kappa_j(t) = 0 \end{cases} \quad (15)$$

The existence of singular control is as a result of the control variables u_1 and u_2 appearing linearly in the Hamiltonian (10). The singular control in (15) can be handled by introducing a perturbed energy term to the cost function and later use this to modify the system Hamiltonian:

$$\begin{aligned}
H &= 1 + \frac{1}{2}\epsilon\mathbf{u}^T\mathbf{W}_u\mathbf{u} + \lambda^T\dot{\mathbf{x}} = 1 + \frac{1}{2}\epsilon\mathbf{u}^T\mathbf{W}_u\mathbf{u} \\
&\quad +(\Lambda_1)^T X_2 + (\Lambda_2)^T X_3 - (\Lambda_3)^T[\mathbf{J_N^{-1}}(X_1) \\
&\quad [\mathbf{D_N}(X_1, X_2)X_3 + \mathbf{C_N}X_2 + \mathbf{R_N}(X_1, X_2) \\
&\quad +\mathbf{L_N}(X_1, X_2, X_3)] + \mathbf{J_N^{-1}}(X_1)\mathbf{u}]
\end{aligned} \quad (16)$$

where $\epsilon = (\epsilon_1, \epsilon_2)^T$ is the vector of the energy perturbation terms. $\mathbf{W}_u$ is a real diagonal, positivedefinite weighting matrix, or $\mathbf{W}_u = diag r_j$. Applying PMP to the modified Hamiltonian gives a similar set of necessary conditions to the one in (15). The control sequence of the modified time-optimal control is given as follows:

$$u_j = \begin{cases} U_{jmax} & \text{if } \kappa_j(t) < 0 \\ U_{jmin} & \text{if } \kappa_j(t) > 0 \\ \kappa_{jm}(t) & \text{if } \kappa_{jm} \in (U_{jmin}, U_{jmax}) \end{cases} \tag{17}$$

where:

$$\kappa_{jm}(t) = -\frac{(\Lambda_3)^T (\mathbf{J_N})^{-1}}{\epsilon_j} \tag{18}$$

The solution to the modified time-optimal control problem converges to the solution of the original time-optimal control problem as the perturbed parameters ϵ_1, ϵ_2 approaches zero.

4. NUMERICAL SIMULATIONS AND DISCUSSION OF RESULTS

Numerical analysis of time-optimal control is performed for a two-link robotic manipulator with the effects of actuator dynamics. The resulting TPBVP problem is coded in MATLAB Problem Solving Environment (MPSE) using a library routine called *bvp4c* (Kierzenka and Shampine, 2001). (Kierzenka and Shampine, 2001) give the full theoretical and software developments for this TPBVP solver. The free-final time problem is converted to a fixed-final time one by changing the independent variable t to $\tau = t/t_f$, then $\tau \in [0, 1]$. The differential equations then become:

$$\frac{d\mathbf{x}}{d\tau} = \tau\frac{d\mathbf{x}}{dt} \quad \text{and} \quad \frac{d\lambda}{d\tau} = \tau\frac{d\lambda}{dt} \tag{19}$$

with the boundary condition for computing the final time t_f given as:

$$H(\mathbf{x}(1), \lambda(1), \mathbf{u}(1)) = 0 \tag{20}$$

The physical parameters for the robotic manipulator and the dc motors (the motor used in this case is the Faulhaber GNM70130 Engel DC Motor) are given as follows: $m_1 = m_2 = 5$ kg, $m_{j2} = 6.5$ kg, $m_e = 6$ kg, $m_{m1} = m_{m2} = 13.2$ kg, $I_{m1} = I_{m2}$ $=0.00485$ kgm^2, $l_1 = 0.4$ m, $l_2 = 0.3$ m, $N_1 = N_2 = 10$, $L_1 = L_2 = 0.006$ Henry, $R_1 = R_2 = 1.45$ Ohm, $K_{e1} = K_{e2} = 0.535$ Volt/(radian/second), and $K_{Tm_1} = K_{Tm_2} = 0.5298$ Nm/ampere. The initial and final conditions for the robotic manipulator are $\mathbf{x}(0) = [0, 0, 0, 0]^T$ and $\mathbf{x}(t_f) = [45^o, 60^o, 0, 0]^T$ respectively.

The energy-perturbation parameters (ϵ_1 and ϵ_2) are used as continuation (homotopy) parameters because of the sensitive nature of the solution to the initial guesses for the adjoint variables, final time, and the location of the switching points. The quality of the initial guesses for the adjoint variables is very critical to the performance of the TPBVP solver. An initial value of 10 is chosen for both ϵ_1 and ϵ_2. This enables faster convergence of the numerical process for the TPBVP solver in the MPSE. Finally, the homotopy parameters are then gradually reduced to a value of 0.002. The corresponding optimal final time is $t_f^* = 1.1058$ sec. Figure 2 shows the time histories of the optimal angular positions of the links. Angular velocities of the links are shown in Figure 3. Figure 4 shows the links' angular accelerations. The time-optimal adjoint variables for the robotic manipulator are shown in Figure 5.

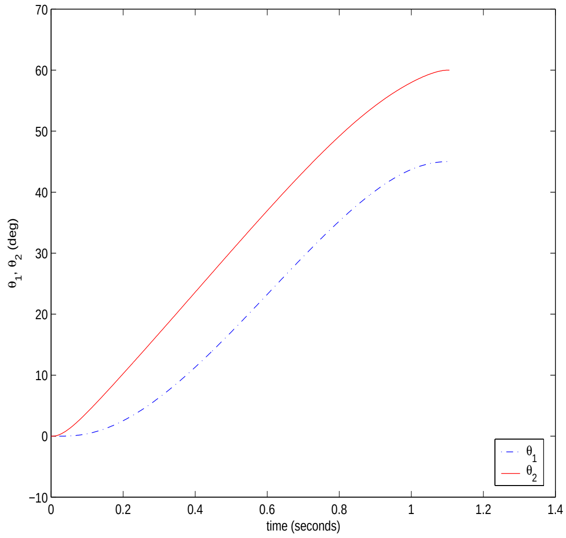

Fig. 2. Time-optimal trajectories for the robotic manipulator links.

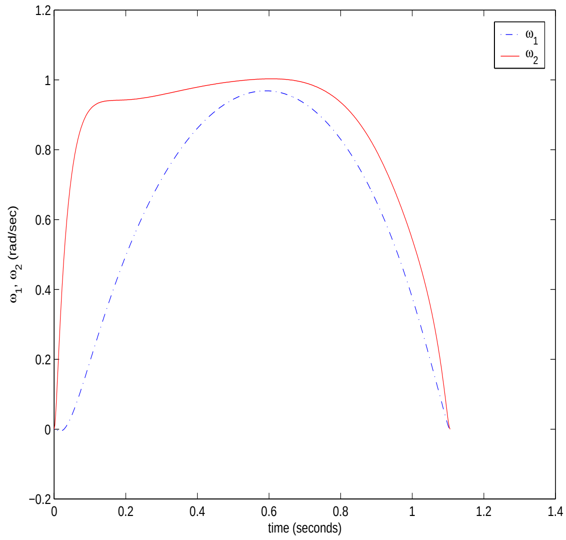

Fig. 3. Time-optimal angular velocities for the robotic manipulator links.

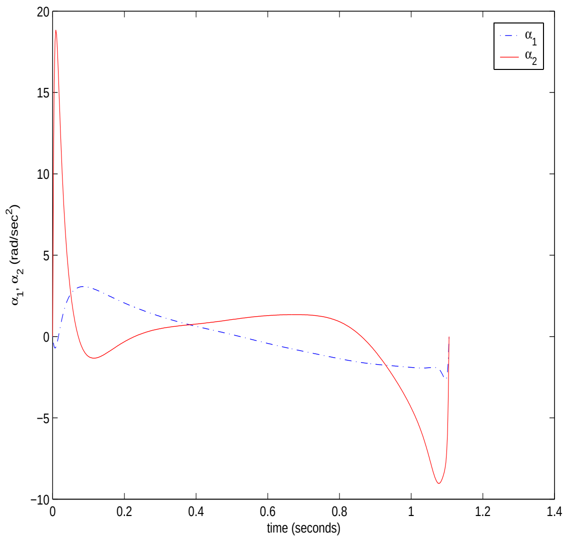

Fig. 4. Time-optimal angular accelerations for the robotic manipulator links.

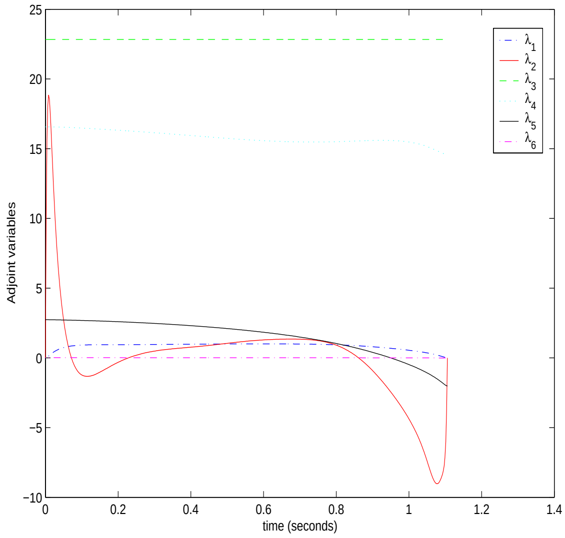

Fig. 5. Time-optimal adjoint variables for the robotic manipulator.

5. CONCLUSIONS

The effects of actuator dynamics on the time-optimal point-to-point motion of a two-link robotic manipulator subject to inequality control constraints have been investigated in this paper. Presence of the actuator dynamics increases the degree of the complexities of the mathematical model. The difficult singular optimal control problem was converted to a set of nonsingular optimal control problems by introducing energy-perturbation terms in the performance index. The computer simulations, using a combination of indirect collocation methods with continuation algorithms, have validated the efficiency and feasibility of the proposed approach for solving this highly complex, nonlinear time-optimal control problem. The time-optimal control problem solutions converge very quickly for the modified problem and the initial guesses for the adjoint variables need not be close to the solution. However, as the values of the perturbation parameters approach zero some convergence problems were encountered.

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