ON CONTROL LYAPUNOV MODES OF LINEAR CONTROL SYSTEMS

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Abstract:
For both discrete-time and continuous-time linear multi-variable control systems, we introduce a concept of control Lyapunov modes associated with a control Lyapunov function in a quadratic form. We prove that the number of *unstable* control Lyapunov modes is at most equal to the number of input of the system.

Keywords: Control Lyapunov function, generalized eigenvalues, singular values, stability, stabilization

1. INTRODUCTION

Control Lyapunov function is a concept first introduced for the study of stabilization of nonlinear control systems (Artstein, 1983; Sontag, 1989), and quickly applied to various control design problems of nonlinear systems (Krstic, Kanellakopoulos and Kokotovic, 1995; Bacciotti and Rosier, 2001). When the concept is applied to linear stabilizable systems, it is reduced to a quadratic function of the form

$$V(x) = x^T P x, \tag{1}$$

in which P is a symmetric, positive definite matrix. Due to the speciality of linear systems, it has yielded many useful new results in stabilization of linear systems with, *e.g.*, limited information (Elia and Mitter, 2001) or Delta-modulated feedback (Xia and Chen, 2002; Gai, Xia and Chen, 2003; Xia, Gai and Chen, 2003). In this paper, we present an interesting property of a quadratic control Lyapunov function of linear control systems. We introduce a concept of control Lyapunov modes associated with a control Lyapunov function in a quadratic form, for both discrete-time and continuous-time linear multivariable control systems. We prove that the number of *unstable* control Lyapunov modes is at most equal to the number of input of the system. This property shows that a control Lyapunov function assembles the "bad behaviour" of the system in the directions that are directly controlled by the input channels, while *normalizing* the behaviour in all other directions. The "bad behaviour" is intrinsically defined in the following by the unstable control Lyapunov modes. Therefore, the existence of a control Lyapunov function restricts the "badness" of behaviour: the number of unstable control Lyapunov modes cannot be greater than the number of available independent input control channels.

To present clearly the development, we will make use of linear algebraic techniques of generalized eigenvalues, Rayleigh Quotients and materials on simultaneous diagonalization of symmetric matrices (in §2). §3 is devoted to the main results and their proofs. Examples are included in §4 to show detailed calculations. Lastly, conclusions are given in §5.

2. ALGEBRAIC PRELIMINARIES

General materials on generalized eigenvalues can be found in (Boble and Daniel, 1988).

Given two matrices $A, B \in \mathbb{R}^{n\times n}$, a vector $e \in \mathbb{C}^n$, $e \neq 0$ and a scalar $\lambda \in \mathbb{C}$ satisfying

$$Ae = \lambda Be \tag{2}$$

are called a generalized eigenvector and a generalized eigenvalue of the matrix pair (A, B), respectively.

In particular, if $B = I_n$ (the identity matrix), we obtain the standard eigenvalue problem. If B is non-singular, the generalized eigenvalue problem can be reduced to the equivalent standard eigenvalue problem by solving

$$B^{-1}Ae = \lambda e.$$

It is easily seen that, as in the case of the standard eigenvalue problem, any linear combination of two generalized eigenvectors, e_1, e_2, associated with the same generalized eigenvalue λ, yields another generalized eigenvector $\mu_1 e_1 + \mu_2 e_2$ associated with λ.

Now let us assume that both A and B are symmetric and, in addition, B is positive definite. The ratio

$$r(w) = \frac{w^T Aw}{w^T Bw},$$

which is known as the *Rayleigh quotient*, is closely related to the generalized eigenvalue problem stated above. To see this, let us determine the extremum (stationary) points of $r(w)$, *i.e.*, the points w^* such that $\nabla r(w^*) = 0$. The gradient $\nabla r(w)$ is calculated as

$$\begin{aligned}\nabla r(w) &= \frac{2w^T BwAw - 2w^T AwBw}{(w^T Bw)^2} \\ &= \frac{2Aw - 2r(w)Bw}{w^T Bw}.\end{aligned}$$

Setting $\nabla r(w)$ to 0, we obtain

$$Aw = r(w)Bw,$$

which is in the form of equation (2). Thus, the extremum points w^* and the extremum values $r(w^*)$ of the Rayleigh quotient $r(w)$ are obtained as the generalized eigenvectors e and eigenvalues $\lambda(e)$, respectively, of the corresponding generalized eigenvalue problem.

An important consequence of the symmetry of A and B is that generalized eigenvectors e_i, e_j associated with different eigenvalues λ_i and λ_j, respectively, are orthogonal with respect to the inner products induced by A and B, *i.e.*,

$$e_i A e_j = e_i B e_j = 0.$$

This property also implies that e_i and e_j are linearly independent.

Proofs of these facts can be found in (Borga, 1998).

The generalized eigenvalue problem is also related to the simultaneous diagonalization problem (see (Fukunaga, 1990) and the references therein). Given two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, the simultaneous diagonalization problem is to seek a non-singular transformation matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned}T^T AT &= \Phi, \\ T^T BT &= I_n,\end{aligned}$$

where Φ is a diagonal matrix and I_n is the identity matrix.

Simultaneous diagonalization starts by finding an intermediate transformation T' that transforms B into the identity matrix. This step is referred to as *whitening* in signal processing: if the eigenvalue decomposition of B is given by $F^T BF = \Lambda_B$, in which Λ_B is a diagonal matrix whose diagonal elements are the eigenvalues of B, then the whitening transformation matrix is obtained as $T' = F\Lambda_B^{-\frac{1}{2}}$. During the second step, the simultaneous diagonalization algorithm determines an orthonormal transformation matrix T'' that diagonalizes $(T')^T AT'$, and, due to its orthonomality, has no effect on the identity matrix. The final transformation matrix is then obtained as

$$T = T'T''.$$

As can be easily verified, this implies that

$$\begin{aligned}AT &= BT\Phi, \\ B^{-1}AT &= T\Phi,\end{aligned}$$

i.e., Φ and T represent the generalized eigenvalues and the associated generalized eigenvectors of

(A, B), respectively. Consequently, they are the extremum values and the extremum points of the corresponding Rayleigh quotient, respectively.

3. MAIN RESULTS

3.1 *Discrete-time systems*

We first consider a class of discrete-time linear control systems of the form

$$x^+ = Ax + Bu, \tag{3}$$

where $x \in I\!R^n$ is the state, $u \in I\!R^m$ is the input, x^+ denotes the system state at the next discrete-time, $A \in I\!R^{n\times n}$ is an $n \times n$ matrix of real numbers, and $B \in I\!R^{n\times m}$ is an $n \times m$ matrix of real numbers.

Assume that system (3) is stabilizable. It is therefore quadratically stabilizable in the sense that there is a control input u, which is a function of x, that makes a quadratic function of the state a valid Lyapunov function for the closed-loop system. Such Lyapunov functions are the Control Lyapunov Functions (CLFs) which we have interest about in this paper.

Given a quadratic CLF, $V(x) = x^T Px$ with $P > 0$, where P is always assumed to be symmetric in this paper, we look for a control input u such that $V(x)$ is decreasing along the trajectories of system (3), *i.e.*, for $x \neq 0$,

$$\begin{aligned}
&\Delta V(x) = V(x^+) - V(x)\\
&= x^T(A^T PA - P)x + 2u^T B^T PAx + u^T B^T PBu\\
&< 0.
\end{aligned} \tag{4}$$

Given x, it is easily verified that the following input

$$u \stackrel{\text{def}}{=} -(B^T PB)^{-1} B^T PAx \stackrel{\text{def}}{=} k_{GD}^T x \tag{5}$$

defines the gradient descent direction making $V(x)$ decrease the most along the trajectories. Under feedback (5), we have, for the closed-loop system,

$$\Delta V(x) = -x^T Qx,$$

where for convenience, denote

$$Q = P - A^T PA + A^T PB(B^T PB)^{-1} B^T PA. \tag{6}$$

By the assumption that $V(x)$ is a CLF, $Q > 0$, which is implied by the following result whose proof is obvious, thus omitted.

Lemma 1. A quadratic form $V(x) = x^T Px$ with $P > 0$ is a CLF for system (3) if and only if it solves (6) for some positive definite matrix Q.

Definition *1.* Given a CLF $V(x) = x^T Px$ of system (3), the generalized eigenvalues of the matrix pair $(A^T PA, P)$ is defined to be the Control Lyapunov Modes (CLMs) of system (3).

Denote as $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ the CLMs of system (3) corresponding to a CLF $V(x) = x^T Px$. Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then P and $A^T PA$ can be simultaneously diagonalized. Denote by M the matrix such that

$$\begin{aligned}
M^T PM &= I_n,\\
M^T A^T PAM &= \mathrm{diag}(\lambda_1, \lambda_2 \ldots, \lambda_n) \stackrel{\text{def}}{=} \Lambda.
\end{aligned}$$

Lemma 2. The CLMs of system (3) corresponding to a CLF, $V(x) = x^T Px$, are the squares of the singular values of $M^{-1}AM$. They are therefore all non-negative.

Proof: By definition of M, $P = M^{-T} M^{-1}$. Therefore,

$$\Lambda = M^T A^T PAM = (M^{-1}AM)^T (M^{-1}AM).$$

Theorem 1. Let $V(x) = x^T Px$ be a CLF for system (3), and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the CLMs corresponding to $V(x)$. Then, no more than m CLMs can be greater or equal to 1.

Proof: Since $M^T k_{GD}(B^T PB)k_{GD}^T M \geq 0$, and from (5), where $B \in I\!R^{n\times m}$, we have

$$\mathrm{rank}(M^T k_{GD}(B^T PB)k_{GD}^T M) \leq m.$$

There is an orthonormal matrix $Y \in I\!R^{n\times n}$ such that $Y^T M^T k_{GD}(B^T PB)k_{GD}^T MY = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m, 0, 0, \ldots, 0)$, for some non-negative real numbers $\sigma_1, \sigma_2, \ldots, \sigma_m$. Decompose $Y = [Y_m, Y_{n-m}]$ where $Y_{n-m} \in I\!R^{n\times(n-m)}$ and $Y_m \in I\!R^{n\times m}$. Then

$$\begin{aligned}
&Y^T M^T QMY\\
&= Y^T M^T(P - A^T PA + k_{GD}(B^T PB)k_{GD}^T)MY\\
&= Y^T(I_n - \Lambda + M^T k_{GD}(B^T PB)k_{GD}^T M)Y\\
&= \begin{bmatrix} I_m + \Sigma - Y_m^T \Lambda Y_m & -Y_m^T \Lambda Y_{n-m} \\ -Y_{n-m}^T \Lambda Y_m & Y_{n-m}^T(I_n - \Lambda)Y_{n-m} \end{bmatrix} > 0,
\end{aligned}$$

where $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_m)$. Hence, $Y_{n-m}^T(I_n - \Lambda)Y_{n-m} > 0$, or equivalently,

$$y^T(I_n - \Lambda)y > 0, \tag{7}$$

for any $0 \neq y \in \Im(Y_{n-m})$, where $\Im(Y_{n-m})$ denotes the subspace spanned by the columns of Y_{n-m}. Clearly, $\dim \Im(Y_{n-m}) = n - m$.

Further, decompose $Y_{n-m} = [Y_{1,n-m}^T, Y_{2,n-m}^T]^T$, where $Y_{1,n-m} \in I\!R^{(m+1)\times(n-m)}$ and $Y_{2,n-m} \in I\!R^{(n-m-1)\times(n-m)}$. Then, there is $0 \neq \xi \in I\!R^{n-m}$

such that $Y_{2,n-m}\xi = 0$. Since the $(n-m)$ columns of Y_{n-m} are linearly independent, we have $\eta \stackrel{\text{def}}{=} Y_{n-m}\xi \neq 0$.

If $\lambda_{m+1} \geq 1$, then by assumption, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m+1} \geq 1$, and

$$\begin{aligned}&\xi^T Y_{n-m}^T (I_n - \Lambda) Y_{n-m}\xi \\ &\quad = \xi^T Y_{1,n-m}^T \Gamma Y_{1,n-m}\xi \leq 0,\end{aligned}$$

in which

$$\Gamma = \begin{bmatrix} 1-\lambda_1 & 0 & & & 0 \\ 0 & 1-\lambda_2 & & & 0 \\ & & \ddots & & \\ 0 & & & 1-\lambda_m & 0 \\ 0 & & & 0 & 1-\lambda_{m+1} \end{bmatrix}.$$

This is a contradiction to (7). So $\lambda_{m+1} < 1$.

3.2 *Continuous-time systems*

Consider a continuous-time linear control system of the form

$$\dot{x} = Ax + Bu, \tag{8}$$

where $x \in I\!\!R^n$ is the state, $u \in I\!\!R^m$ is the input, $A \in I\!\!R^{n\times n}$ is an $n\times n$ matrix of real numbers, and $B \in I\!\!R^{n\times m}$ is an $n \times m$ matrix of real numbers.

A quadratic Control Lyapunov Function (CLF), $V(x) = x^T Px$, in which P is a symmetric, positive definite matrix, exists for the continuous-time system (8) if for any $x \in I\!\!R^n$, $x \neq 0$, there is a control input $u \in I\!\!R^m$ such that the derivative of $V(x)$ along the trajectories of (8) is strictly negative, *i.e.*,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = x^T(PA + A^T P)x + 2x^T PBu < 0. \tag{9}$$

First of all, a CLF for a continuous-time linear system has the following characterization.

Lemma 3. A quadratic form, $V(x) = x^T Px$, in which P is a symmetric, positive definite matrix, is a CLF for system (8) if and only if there exist a positive real number $\alpha > 0$ and a positive definite matrix $Q \in I\!\!R^{n\times n}$ such that

$$PA + A^T P - \alpha PBB^T P = -Q. \tag{10}$$

Proof: Sufficiency is easy. Since (10) holds, one can choose the following feedback as a control input

$$u = -\frac{\alpha}{2}B^T Px \stackrel{\text{def}}{=} kx. \tag{11}$$

With this choice of the control input, the derivative of $V(x)$ along the trajectories of (8) becomes, for any $n \neq 0$,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= x^T(PA + A^T P)x + 2x^T PBu \\ &= x^T(PA + A^T P - \alpha PBB^T P)x < 0.\end{aligned}$$

To prove necessity, we need to show that there is an $\alpha > 0$ such that the Q defined in (11) is positive definite, *i.e.*, for any $x \in I\!\!R^n$, $x \neq 0$,

$$x^T Qx > 0. \tag{12}$$

Obviously, we need only to show that (12) holds on the unit ball defined by $B \stackrel{\text{def}}{=} \{x \mid ||x|| \leq 1\}$.

Denote $S = \{x \mid B^T Px = 0\}$. When $x \in S \cap B$, and $x \neq 0$,

$$-x^T(PA + A^T P)x > 0.$$

The last step is implied by (9).

When $x \notin S$, $x^T PB \neq 0$, so for any $x \in B\backslash S$, there is an $\alpha_x > 0$ such that

$$-x^T(PA + A^T P - \alpha_x PBB^T P)x > 0.$$

By the continuity of the above expression, there is a neighborhood N_x of x such that for any $\bar{x} \in N_x\backslash S$,

$$-\bar{x}^T(PA + A^T P - \alpha_x PBB^T P)\bar{x} > 0.$$

Combining the above two cases, it is easy to see that for any $x \in B$, there is a neighborhood N_x of x such that for any $\bar{x} \in N_x$, $\bar{x} \neq 0$,

$$-\bar{x}^T(PA + A^T P - \alpha_x PBB^T P)\bar{x} > 0.$$

By the compactness of B, there are a finite number of such neighborhoods, $\{N_{x_1}, N_{x_2}, \ldots, N_{n_t}\}$, which constitute a *covering* of B (Rudin, 1987). Taking $\alpha = \max(\alpha_{x_1}, \alpha_{x_2}, \ldots, \alpha_{x_t}\}$, it is easy to see that Q defined by $-(PA + A^T P - \alpha PBB^T P)$ satisfies $x^T Qx > 0$, for any $x \in B$, $x \neq 0$. That is, Q is positive definite.

Definition *2.* Given a CLF, $V(x) = x^T Px$, of system (8), the generalized eigenvalues of the matrix pair $(PA + A^T P, P)$ is defined to be the Control Lyapunov Modes (CLMs) of system (8).

Denote by $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ the CLMs of system (8) corresponding to CLF $V(x) = x^T Px$. Then P and $PA + A^T P$ can be simultaneously diagonalized. Denote by M the matrix such that

$$\begin{aligned} M^T PM &= I_n, \\ M^T(PA + A^T P)M &= \mathrm{diag}(\lambda_1, \lambda_2 \ldots, \lambda_n) \stackrel{\text{def}}{=} \Lambda.\end{aligned}$$

Lemma 4. The CLMs of system (8) corresponding to CLF $V(x) = x^T Px$ are real numbers.

Proof: The CLMs of system (8) corresponding to $V(x)$ are the eigenvalues of $B \overset{\text{def}}{=} A + P^{-1}A^TP$. Note that

$$B^T = A^T + PAP^{-1},$$

so it is easily verified that

$$PB = B^TP. \tag{13}$$

If M is the matrix such that $M^TPM = I_n$, then substituting $P = M^{-T}M^{-1}$ into (13), one obtains

$$M^{-T}M^{-1}B = B^TM^{-T}M^{-1},$$

or

$$M^{-1}BM = (M^{-1}BM)^T.$$

That is, $M^{-1}BM$ is a symmetric matrix, therefore it has real eigenvalues. So does B.

Theorem 2. Let $V(x) = x^TPx$ be a CLF for system (8), and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the CLMs corresponding to $V(x)$. Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then, no more than m CLMs can be greater or equal to 0.

Proof: Since $\operatorname{rank}(PBB^TP) \leq m$, there exists an orthonormal matrix $Y \in \mathbb{R}^{n\times n}$ such that

$$Y^TM^TPBB^TPMY = \operatorname{diag}(\sigma_1, \ldots, \sigma_m, 0, \ldots, 0),$$

for some non-negative real numbers $\sigma_1, \sigma_2, \ldots, \sigma_m$. Decompose $Y = [Y_m, Y_{n-m}]$ where $Y_{n-m} \in \mathbb{R}^{n\times(n-m)}$ and $Y_m \in \mathbb{R}^{n\times m}$. Then

$$\begin{aligned}
&Y^TM^TQMY \\
&= Y^TM^T(-PA - A^TP + \alpha PBB^TP)MY \\
&= Y^T(-\Lambda + \alpha M^TPBB^TPM)Y \\
&= \begin{bmatrix} \alpha\Sigma - Y_m^T\Lambda Y_m & -Y_m^T\Lambda Y_{n-m} \\ -Y_{n-m}^T\Lambda Y_m & -Y_{n-m}^T\Lambda Y_{n-m} \end{bmatrix} > 0,
\end{aligned}$$

where $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_m)$. Hence, $Y_{n-m}^T\Lambda Y_{n-m} < 0$, or equivalently,

$$y^T\Lambda y < 0, \tag{14}$$

for any $0 \neq y \in \Im(Y_{n-m})$, where $\Im(Y_{n-m})$ denotes the subspace spanned by the columns of Y_{n-m}. Clearly, $\dim \Im(Y_{n-m}) = n - m$.

Further, decompose $Y_{n-m} = [Y_{1,n-m}^T, Y_{2,n-m}^T]^T$, where $Y_{1,n-m} \in \mathbb{R}^{(m+1)\times(n-m)}$ and $Y_{2,n-m} \in \mathbb{R}^{(n-m-1)\times(n-m)}$. Then, there is $0 \neq \xi \in \mathbb{R}^{n-m}$ such that $Y_{2,n-m}\xi = 0$. Since the $(n-m)$ columns of Y_{n-m} are linearly independent, we have $\eta \overset{\text{def}}{=} Y_{n-m}\xi \neq 0$.

If $\lambda_{m+1} \geq 0$, then by assumption, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m+1} \geq 0$, and

$$\begin{aligned}
&\xi^TY_{n-m}^T\Lambda Y_{n-m}\xi \\
&= \xi^TY_{1,n-m}^T \begin{bmatrix} \lambda_1 & 0 & & & 0 \\ 0 & \lambda_2 & & & 0 \\ & & \ddots & & \\ 0 & & & \lambda_m & 0 \\ 0 & & & 0 & \lambda_{m+1} \end{bmatrix} Y_{1,n-m}\xi \\
&\geq 0,
\end{aligned}$$

which is a contradiction to (14). So $\lambda_{m+1} < 0$.

4. EXAMPLES

Example 1. Consider a single input, third-order discrete-time system,

$$\begin{aligned}
x^+ &= Ax + bu \\
&\overset{\text{def}}{=} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.
\end{aligned}$$

It can be verified that $V(x) = x^TPx$, in which $P = \operatorname{diag}(1, 2, 3)$ is a CLF for the system. As a matter of fact, P solves equation (6) for $Q = I_3$.

The CLMs of this CLF are the eigenvalues of $P^{-1}A^TPA$, they are $\{5.7728, 0.6098, 0.2841\}$.

Also, $V(x) = x^TPx$, in which

$$P = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1.9142 & 0 \\ 0.5 & 0 & 2.9142 \end{bmatrix}$$

is another CLF for the system, with P satisfying (6) for

$$Q = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix}.$$

The CLMs for this CLF are $\{5.5163, 0.697, 0.2601\}$.

Note that in both cases, only the first CLM is greater than 1.

Example 2. Consider a 2-input, fifth-order continuous-time system,

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
&\overset{\text{def}}{=} \begin{bmatrix} 0 & 1 & -1 & 3 & 2 \\ 8 & -2 & 4 & 5 & 0 \\ 2 & 0 & 0 & 3 & -2 \\ 1 & -1 & 0 & 1 & -1 \\ 2 & 1 & -2 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 2 & -1 \\ 1 & -3 \end{bmatrix} u
\end{aligned}$$

It can be verified that matrix A is unstable with poles $\{2.9594, 1.6367 \pm 2.2225i, -4.1675, -3.0652\}$.

A CLF for the system is $V(x) = x^T Px$, in which P is given by

$$\begin{bmatrix} 1.388 & 0.574 & -0.478 & -0.040 & 0.964 \\ 0.574 & 0.365 & -0.210 & -0.071 & 0.454 \\ -0.476 & -0.210 & 0.762 & 0.191 & -0.608 \\ -0.040 & -0.071 & 0.191 & 0.559 & -0.321 \\ 0.964 & 0.454 & -0.608 & -0.321 & 1.057 \end{bmatrix}$$

This chosen P actually satisfies (10) for $\alpha = 3$ and $Q = I_5$.

The CLMs corresponding to this CLF are given by the eigenvalues of $A + P^{-1}A^T P$, they are $\{12.8186, 5.0277, -2.3840, -7.5361, -9.9261\}$.

Also, $V(x) = x^T Px$, in which P is

$$\begin{bmatrix} 2.466 & 1.055 & -0.658 & 0.219 & 1.919 \\ 1.055 & 0.624 & -0.352 & -0.158 & 0.953 \\ -0.658 & -0.352 & 1.093 & 0.391 & -1.137 \\ 0.219 & -0.158 & 0.391 & 1.269 & -0.607 \\ 1.919 & 0.953 & -1.137 & -0.607 & 2.269 \end{bmatrix}$$

is another CLF such that the matrix P solves (10) for $\alpha = 1$ and

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The CLMs corresponding to this CLF are $\{11.2817, 4.0982, -1.0527, -6.3095, -10.0178\}$.

We note that in both cases only two CLMs are positive.

5. CONCLUDING REMARKS

In this paper, we have presented an interesting property of a quadratic control Lyapunov function of linear control systems. For both discrete-time and continuous-time linear multi-variable control systems, we have introduced a concept of control Lyapunov modes associated with a control Lyapunov function in a quadratic form. We have proven that the number of *unstable* control Lyapunov modes is at most equal to the number of input of the system. This property shows that a control Lyapunov function assembles the "bad behaviour" of the system in the directions that are directly controlled by the input channels, while normalizing the behaviour in all other directions. The "bad behaviour" is intrinsically defined by the unstable control Lyapunov modes. Therefore, the existence of a control Lyapunov function restricts the "badness" of behaviour: the number of unstable control Lyapunov modes cannot be greater than the number of available control channels.

The generalized eigenvalue problem is useful in our discussion. One interesting observation is that the σ's appeared in the proofs of Theorem 1 and Theorem 2 are also generalized eigenvalues of $(k_{GD}^T B^T PBk_{GD}, P)$ and $(PBB^T P, P)$, respectively. It appears that they constitute a set of what may be called *secondary control Lyapunov modes*, and they can lead to further necessary conditions for a quadratic form to become a CLF.

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