ON JORDAN CONTROLLABLE AND OBSERVABLE CANONICAL FORMS

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Abstract: This paper deals with the problem of transforming an observable pair and a controllable pair of matrices into a so-called Jordan observable pair and a Jordan controllable pair respectively. We provide the necessary algorithms for deriving the similarity matrices that permit such transformation. The Jordan canonical form is, in fact, an extension of the classical Brunowsky canonical form. We also show how such canonical forms can be used as an alternative tool for designing observers and controllers for linear time varying systems.
Keywords: Brunovsky and Jordan observable and controllable canonical forms.

1. Introduction

Consider the single input single output (SISO) linear system

$$\begin{cases} \dot{x} = Fx + Gu \\ y = Hx \end{cases} \tag{1}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}$, $y \in \mathcal{R}$, and, $F \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 1}$ and $H \in \mathcal{R}^{1\times n}$ are constant matrices. We assume that the pairs (F, H) and (F, G) are observable and controllable respectively. It is well-known that if the pair of matrices (F, H) is observable, then (F, H) is equivalent to the following pair of matrices

$$\begin{cases} A_o = \begin{bmatrix} \psi_1 & 1 & 0 & \cdots & 0 \\ \psi_2 & 0 & 1 & \ddots & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ \psi_n & 0 & \cdots & 0 & 0 \end{bmatrix}, \\ C = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} \end{cases} \tag{2}$$

More precisely, there exists a nonsingular matrix P_o such that $A_o = P_o F P_o^{-1}$ and $C = HP_o^{-1}$. The pair (A_o, C) is known as the Brunowsky observable canonical pair (see eg. Chen, 1970). Note that the matrix A_o can be further decomposed as $A_o = A + \Psi C$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & 0 & 0 & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \text{ and } \Psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix}. \tag{3}$$

Similarly, if (F, G) is controllable, then (F, G) is equivalent to the following pair of matrices

$$\begin{cases} A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & 0 & \ddots & 0 \\ 0 & \cdots & & 0 & 1 \\ \varphi_1 & \varphi_2 & \cdots & \cdots & \varphi_n \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \end{cases} \tag{4}$$

More precisely, there exists a nonsingular matrix P_c such that $A_c = P_c F P_c^{-1}$ and $B = P_c G$. The pair (A_c, B) is known as the Brunowsky controllable canonical pair (see eg. Chen, 1970). Again, it can be observed that the matrix A_c can be decomposed as $A_c = A + B\Phi$ where $\Phi = \begin{pmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \end{pmatrix}$. Such canonical pairs are extremely useful for the analysis and synthesis of control systems. In effect, the Brunowsky canonical forms provide a basis for controllers and observers design for linear time-invariant systems (Brogan, 1982). On the other hand, it is well-known that any square matrix is similar to a Jordan matrix (see eg. Gantmacher, 1959, Horn, 1985). Notice that the matrix A is nothing more than a Jordan block with its diagonal entries being 0. Consequently, one might ask if it is possible to extend the Brunowsky canonical form into an extended Jordan canonical form where all the main diagonal entries are not necessarily 0 and all the entries on the diagonal just above the main diagonal are not necessarily equal to 1.

In effect, in this paper, we consider the problem of transforming the observable pair (F, H) and the controllable pair (F, G) into the following pairs

$(J_o(\alpha,\beta),C)$ and $(J_c(\alpha,\beta),B)$ respectively where

$$J_o(\alpha,\beta)=\begin{pmatrix}\gamma_1+\alpha & \beta & 0 & \cdots & 0\\ \gamma_2 & \alpha & \beta & \ddots & \vdots\\ \vdots & 0 & \alpha & \ddots & 0\\ \vdots & & \ddots & \ddots & \beta\\ \gamma_n & 0 & \cdots & 0 & \alpha\end{pmatrix}, \quad (5)$$

$$J_c(\alpha,\beta)=\begin{pmatrix}\alpha & \beta & 0 & \cdots & 0\\ 0 & \alpha & \beta & \ddots & \vdots\\ \vdots & \ddots & \alpha & \ddots & 0\\ 0 & \cdots & 0 & \ddots & \beta\\ \delta_1 & \delta_2 & \cdots & \delta_{n-1} & \delta_n+\alpha\end{pmatrix} \quad (6)$$

and the matrices C and B are as above; α and β are coefficients that are chosen arbitrarily with the restriction that $\beta \neq 0$. Note that the matrix $J_o(\alpha,\beta)$ and $J_c(\alpha,\beta)$ can be respectively decomposed as $J_o(\alpha,\beta) = J_{\alpha\beta} + \Gamma_{\alpha\beta}C$ and $J_c(\alpha,\beta) = J_{\alpha\beta} + B\Delta_{\alpha\beta}$ where

$$J_{\alpha\beta} = \begin{pmatrix}\alpha & \beta & 0 & \cdots & 0\\ 0 & \alpha & \beta & \ddots & \vdots\\ \vdots & \ddots & \alpha & \ddots & 0\\ \vdots & & \ddots & \ddots & \beta\\ 0 & \cdots & \cdots & 0 & \alpha\end{pmatrix}, \Gamma_{\alpha\beta} = \begin{pmatrix}\gamma_1\\ \gamma_2\\ \vdots\\ \gamma_n\end{pmatrix}$$

$$\Delta_{\alpha\beta} = \begin{pmatrix}\delta_1 & \delta_2 & \cdots & \delta_n\end{pmatrix}. \quad (7)$$

The matrix $J_{\alpha\beta}$ is a Jordan block where all its diagonal entries are equal to α all the entries on the diagonal just above the main diagonal are equal to β. We shall call the pair $(J_o(\alpha,\beta),C)$ a *Jordan observable canonical pair*, and, the pair $(J_c(\alpha,\beta),B)$ a *Jordan controllable canonical pair*. It is important to note that the pairs $(J_o(\alpha,\beta),C)$ and $(J_c(\alpha,\beta),B)$ are indeed observable and controllable pairs respectively since their corresponding observability and controllability matrices are both of rank n for $\beta \neq 0$.

We shall show that any controllable and observable pairs can be transformed into a Jordan controllable and a Jordan observable pair, for arbitrary values of α and $\beta \neq 0$, respectively. Indeed, we give the construction of a matrix $M_o(\alpha,\beta)$ and a matrix $M_c(\alpha,\beta)$ such that

$$J_o(\alpha,\beta) = M_o(\alpha,\beta)FM_o^{-1}(\alpha,\beta) \text{ and } C = HM_o^{-1}(\alpha,\beta)$$

and,

$$J_c(\alpha,\beta) = M_c(\alpha,\beta)FM_c^{-1}(\alpha,\beta) \text{ and } B = M_c(\alpha,\beta)G.$$

The fact, that the coefficients α and β can be chosen arbitrarily - with the only restriction that $\beta \neq 0$ - will turn out to be particularly useful in the design of controllers and observers. Indeed, we shall show that the coefficients α and β can be used as controllers and observers gain tuning parameters.

An outline of the paper is as follows: In the next section, some preliminary results on the equivalence of controllable and observable systems are given. In Section 3, the equivalence of an observable pair to the Jordan observable canonical pair is discussed. In Section 4, the equivalence of an controllable pair to the Jordan controllable canonical pair is demonstrated. In Section 5, the application of the Jordan controllable canonical form to design a controller for linear time varying rank controllable systems is considered. Section 6 deals with the application of the Jordan canonical observable form to design an observer for linear time varying rank observable systems. Finally, some conclusions are drawn.

2. Some preliminaries

In this section, we are going to recall some preliminary results, on the equivalence of controllable and observable pairs, which are used in the subsequent sections.

Definition 1. *Let (F,G) be a controllable pair. Then, the pair $(\bar{F},\bar{G})$ is said to be an equivalent controllable pair of (F,G) if there exists a nonsingular matrix P_c such that $\bar{F} = P_cFP_c^{-1}$ and $\bar{G} = P_cG$.*

Definition 2. *Let (F,H) be an observable pair. Then, the pair $(\bar{F},\bar{H})$ is said to be an equivalent observable pair of (F,H) if there exists a nonsingular matrix P_o such that $\bar{F} = P_oFP_o^{-1}$ and $\bar{H} = HP_o^{-1}$.*

An important result on observable and controllable equivalent pairs is given in the following lemmas:

Lemma 1 (Chen, 1970) *Let (F,H) and $(\bar{F},\bar{H})$, with $F,\bar{F} \in \mathcal{R}^{n\times n}$ and $H,\bar{H} \in \mathcal{R}^{1\times n}$, be two equivalent observable pairs. Then, the matrix P_o such that $\bar{F} = P_oFP_o^{-1}$ and $\bar{H} = HP_o^{-1}$ is given by $P_o = W_o^{-1}\Upsilon_o$ where $\Upsilon_o = \left[H^T, F^TH^T, \ldots, (F^{n-1})^TH^T\right]^T$ and $W_o = \left[\bar{H}^T, \bar{F}^T\bar{H}^T, \ldots, (\bar{F}^{n-1})^T\bar{H}^T\right]^T$.*

Lemma 2 (Chen, 1970) *Let (F,G) and $(\bar{F},\bar{G})$, with $F,\bar{F} \in \mathcal{R}^{n\times n}$ and $G,\bar{G} \in \mathcal{R}^{n\times 1}$, be two equivalent controllable pairs. Then, the matrix P_c such that $\bar{F} = P_cFP_c^{-1}$ and $\bar{G} = P_cG$ is given by $P_c = Y_cU_c^{-1}$ where $U_c = \left[G, FG, \ldots, F^{n-1}G\right]$ and $Y_c = \left[\bar{G}, \bar{F}\bar{G}, \ldots, \bar{F}^{n-1}\bar{G}\right]$.*

Remark 1. Note that, since F and $\bar{F}$ are similar, the characteristic polynomials of F and $\bar{F}$ are equal.

3. The Jordan observable canonical form

The main objective of this section is to give the algorithm for the construction of a similarity matrix $M_o(\alpha,\beta)$ such that the observable pair (F,H) of system (1) is transformed into a Jordan observable pair $(J_o(\alpha,\beta),C) = (J_{\alpha\beta} + \Gamma_{\alpha\beta}C, C)$ as described in (5) for any given value of α and $\beta \neq 0$. It is clear that the column matrix $\Gamma_{\alpha\beta}$ will depend on the particular choice of α and β. As a result, to construct the matrix $M_o(\alpha, \beta)$, we need to know what is the relationship between the components of the column matrix $\Gamma_{\alpha\beta}$ and the parameters α and β. For this, we define the observability matrices corresponding to the pair $(J_o(\alpha, \beta), C)$ and (F, H) that is

$$W(\alpha, \beta) = \begin{bmatrix} C \\ CJ_o(\alpha, \beta) \\ \vdots \\ CJ_o^{n-1}(\alpha, \beta) \end{bmatrix} \quad (8)$$

and

$$\Upsilon = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}. \quad (9)$$

According to Lemma 1, it is obvious that the matrix $M_o(\alpha, \beta)$ is defined by

$$M_o(\alpha, \beta) = W^{-1}(\alpha, \beta)\Upsilon. \quad (10)$$

Lemma 3. *Let (F,H) ; $F \in \mathcal{R}^{n\times n}$, $H \in \mathcal{R}^{1\times n}$ be an observable pair and let $(J_o(\alpha, \beta), C) = (J_{\alpha\beta}+\Gamma_{\alpha\beta}C, C)$ be a Jordan observable canonical pair as described by (5) for any given value of α and $\beta \neq 0$ with $\Gamma_{\alpha\beta}^T = (\ \gamma_1,\ \gamma_2,\ \ldots,\ \gamma_n\ )$. Then, the matrix $M_o(\alpha, \beta)$ define by (10) satisfies*

$$J_o(\alpha, \beta) = M_o(\alpha, \beta)FM_o^{-1}(\alpha, \beta) \text{ and } C = HM_o^{-1}(\alpha, \beta) \quad (11)$$

if

$$\gamma_k = \frac{1}{\beta^{k-1}}\left[\sum_{i=1}^{k} \psi_i C_{n-i}^{k-i}\alpha^{k-i} - C_n^k\alpha^k\right] \quad (12)$$

where $C_n^k = \dfrac{n!}{(n-k)!k!}$ and the ψ_i's are the coefficients the characteristic equation of F

$$\det[\lambda I_n - F] = \lambda^n - \psi_1\lambda^{n-1} - \cdots - \psi_{n-1}\lambda - \psi_n.$$

In other words, the pair (F, H) is equivalent to the pair $(J_{\alpha\beta} + \Gamma_{\alpha\beta}C, C)$ if the entries of $\Gamma_{\alpha\beta}$ are as above.

Proof

First of all, it is easy to check that the pair $(J_{\alpha\beta} + \Gamma_{\alpha\beta}C, C)$ is observable whatever the vector $\Gamma_{\alpha\beta}$. In fact, $\det W(\alpha, \beta) = \beta^{\frac{n(n-1)}{2}}$ which is nonzero since $\beta \neq 0$ and is independent of $\Gamma_{\alpha\beta}$. Consequently, $W(\alpha, \beta)$ is invertible which also justify the definition of $M_o(\alpha, \beta) = W^{-1}(\alpha, \beta)\Upsilon$. As mentioned previously, if (F, H) is equivalent to $(J_o(\alpha, \beta), C)$, then

$$\begin{aligned} \det[\lambda I_n - J_o(\alpha, \beta)] &= \det[\lambda I_n - (J_{\alpha\beta} + \Gamma_{\alpha\beta}C)] \\ &= \det[\lambda I_n - F]. \end{aligned}$$

Note that $J_{\alpha\beta} = \alpha I_n + \beta A$ where A is as in (3). It can be checked that for any $\Gamma_{\alpha\beta} \in \mathcal{R}^n$

$$\det[\lambda I_n - (\beta A + \Gamma_{\alpha\beta}C)] = \lambda^n - \gamma_1\lambda^{n-1}\beta^0 - \cdots - \beta^{n-1}\gamma_n.$$

Consequently, since $\lambda I_n - J_o(\alpha, \beta) = (\lambda - \alpha)I_n - (\beta A + \Gamma_{\alpha\beta}C)$, we have

$$\det[\lambda I_n - J_o(\alpha, \beta)] = \bar{\lambda}^n - \gamma_1\bar{\lambda}^{n-1}\beta^0 - \cdots - \beta^{n-1}\gamma_n \quad (13)$$

where $\bar{\lambda} = \lambda - \alpha$. Now,

$$\begin{aligned} \det[\lambda I_n - F] &= \lambda^n - \psi_1\lambda^{n-1} - \cdots - \psi_{n-1}\lambda - \psi_n \\ &= (\bar{\lambda} + \alpha)^n - \psi_1(\bar{\lambda} + \alpha)^{n-1} - \cdots - \psi_n \end{aligned}$$

Using the binomial expansion of $(\bar{\lambda} + \alpha)^{n-i}$ one can show that

$$\det[\lambda I_n - F] = \bar{\lambda}^n + r_1\bar{\lambda}^{n-1} + \cdots + r_{n-1}\bar{\lambda} + r_n \quad (14)$$

where

$$r_k = C_n^k\alpha^k - \sum_{i=1}^{k} C_{n-i}^{k-i}\alpha^{k-i}\psi_i.$$

By comparing (13) to (14), it is easy to see that $\det[\lambda I_n - F] = \det[\lambda I_n - (J_{\alpha\beta} + \Gamma_{\alpha\beta}C)]$ if

$$\gamma_k = -\frac{r_k}{\beta^{k-1}} = \frac{1}{\beta^{k-1}}\left[\sum_{i=1}^{k} \psi_i C_{n-i}^{k-i}\alpha^{k-i} - C_n^k\alpha^k\right] \quad (15)$$

□

Remark 2

i) In the above lemma, the Brunowski canonical observable form consists of the particular case where $\alpha = 0$ and $\beta = 1$. In fact, in such a case $\gamma_k = \psi_k$.

ii) The above result can also be applied to parametrised observable pairs as long as the rank observability condition is obeyed. In particular, if we have a time dependent observable pair $(F(t), H(t))$, then provided that its observability matrix $\Upsilon(t) = \begin{bmatrix} H(t) \\ H(t)F(t) \\ \vdots \\ H(t)F^{n-1}(t) \end{bmatrix}$ is of full rank for all t, we can use Lemma 3 to construct a matrix $M_o(\alpha, \beta, t)$ such that $J_o(\alpha, \beta, t) = M_o(\alpha, \beta, t)F(t)M_o^{-1}(\alpha, \beta, t)$ and $C = H(t)M_o^{-1}(\alpha, \beta, t)$ for any given value of α and $\beta \neq 0$.

4. The Jordan controllable canonical form

In this section, we are going to give the construction of a similarity matrix $M_c(\alpha, \beta)$ such that the controllable pair (F, G) of system (1) is transformed into a Jordan controllable pair $(J_c(\alpha, \beta), B) = (J_{\alpha\beta} + B\Delta_{\alpha\beta}, B)$ as described in (6) for any given value of α and $\beta \neq 0$. As before, this amounts to deriving the relationship between the components of $\Delta_{\alpha\beta}$ and the parameters α and β. According to Lemma 2 the similarity matrix $M_c(\alpha,\beta)$ is given by

$$M_c(\alpha,\beta) = Y(\alpha,\beta)U^{-1} \tag{16}$$

where

$$Y(\alpha,\beta) = \left[B,\ J_c(\alpha,\beta)B,\ \ldots,\ J_c^{n-1}(\alpha,\beta)B\right] \tag{17}$$

and

$$U = \left[G,\ FG,\ \ldots,\ F^{n-1}G\right]. \tag{18}$$

Lemma 4. *Let (F,G) ; $F \in \mathcal{R}^{n\times n}$, $G \in \mathcal{R}^{n\times 1}$ be a controllable pair and let $(J_c(\alpha,\beta),\ B) = (J_{\alpha\beta} + B\Delta_{\alpha\beta}, B)$ be a Jordan controllable canonical pair as described by (6) for any given value of α and $\beta \neq 0$ with $\Delta_{\alpha\beta} = \begin{pmatrix} \delta_1 & \delta_2 & \cdots & \delta_n \end{pmatrix}$. Then, the matrix $M_c(\alpha,\beta)$ defined by (16) satisfies*

$$J_c(\alpha,\beta) = M_c(\alpha,\beta)FM_c^{-1}(\alpha,\beta) \text{ and } B = M_c(\alpha,\beta)G \tag{19}$$

if

$$\begin{aligned}\delta_m &= \frac{1}{\beta^{n-m}} \sum_{i=1}^{n-m+1} \varphi_{n-i+1} C_{n-i}^{m-1} \alpha^{n+1-m-i} \\ &\quad - \frac{1}{\beta^{n-m}} C_n^{m-1} \alpha^{n-m+1}\end{aligned} \tag{20}$$

where $C_n^k = \dfrac{n!}{(n-k)!k!}$ and the φ_i's are the coefficients the characteristic equation of F

$$\det[\lambda I_n - F] = \lambda^n - \lambda^{n-1}\varphi_n - \cdots - \lambda\varphi_2 - \varphi_1.$$

In other words, the pair (F,G) is equivalent to the pair $(J_{\alpha\beta} + B\Delta_{\alpha\beta}, B)$ if the component of $\Delta_{\alpha\beta}$ are as above.

Proof

First of all, it is easy to check that the pair $(J_{\alpha\beta} + B\Delta_{\alpha\beta}, B)$ is controllable for every vector $\Delta_{\alpha\beta} = \begin{pmatrix} \delta_1 & \delta_2 & \cdots & \delta_n \end{pmatrix}$. Indeed, the controllability matrix $Y(\alpha,\beta)$ of the pair $(J_c(\alpha,\beta),\ B)$, given by (17), is lower triangular and $\det(Y(\alpha,\beta)) = \beta^{\frac{n(n-1)}{2}}$. Therefore, the determinant of $Y(\alpha,\beta)$ is independent of $\Delta_{\alpha\beta}$ and is nonzero since $\beta \neq 0$; that is, $Y(\alpha,\beta)$ is nonsingular. This also ensures the invertibility of $M_c(\alpha,\beta)$; that is $M_c^{-1}(\alpha,\beta) = UY^{-1}(\alpha,\beta)$. Again, since $J_{\alpha\beta} = \alpha I_n + \beta A$, it can be checked that for any $\Delta_{\alpha\beta} \in \mathcal{R}^n$

$$\det[\lambda I_n - (\beta A + B\Delta_{\alpha\beta})] = \lambda^n - \lambda^{n-1}\delta_n\beta^0 - \cdots - \delta_1\beta^{n-1}.$$

Consequently,

$$\det[\lambda I_n - J_c(\alpha,\beta)] = \bar{\lambda}^n - \bar{\lambda}^{n-1}\delta_n\beta^0 - \cdots - \delta_1\beta^{n-1} \tag{21}$$

where $\bar{\lambda} = \lambda - \alpha$. Now,

$$\begin{aligned}\det[\lambda I_n - F] &= \lambda^n - \lambda^{n-1}\varphi_n - \cdots - \lambda\varphi_2 - \varphi_1 \\ &= \left(\bar{\lambda} + \alpha\right)^n - \cdots - \left(\bar{\lambda} + \alpha\right)\varphi_2 - \varphi_1.\end{aligned}$$

Using the binomial expansion of $(\bar{\lambda} + \alpha)^{n-i}$ one can show that

$$\det[\lambda I_n - F] = \bar{\lambda}^n + r_1\bar{\lambda}^{n-1} + \cdots + r_{n-1}\bar{\lambda} + r_n \tag{22}$$

where

$$r_k = C_n^k\alpha^k - \sum_{i=1}^{k} \varphi_{n-i+1} C_{n-i}^{k-i} \alpha^{k-i}$$

As noted in Remark 1, if (F,G) is equivalent to $(J_c(\alpha,\beta),\ B)$ then, the characteristics equations of F and $J_c(\alpha,\beta)$ are the same. By identifying the coefficients of (22) with those of (21), we can see that $\det[\lambda I_n - F] = \det[\lambda I_n - J_c(\alpha,\beta)]$ if

$$\delta_{n-k+1} = -\frac{r_k}{\beta^{k-1}} = \frac{1}{\beta^{k-1}}\left(\sum_{i=1}^{k} \varphi_{n-i+1} C_{n-i}^{k-i}\alpha^{k-i} - C_n^k\alpha^k\right).$$

By a change of variable, we get

$$\begin{aligned}\delta_m &= \frac{1}{\beta^{n-m}} \sum_{i=1}^{n-m+1} \varphi_{n-i+1} C_{n-i}^{m-1} \alpha^{n+1-m-i} \\ &\quad - \frac{1}{\beta^{n-m}} C_n^{m-1} \alpha^{n-m+1}\end{aligned}$$

□

Remark 3. The above results can also be applied to parametrised controllable pairs as long as the rank controllability condition is obeyed. In particular, if we have a time dependent controllable pair $(F(t), G(t))$, then, provided that its controllability matrix $U(t) = \begin{pmatrix} G(t) & F(t)G(t) & \cdots & F^{n-1}(t)G(t) \end{pmatrix}$ is of full rank for all t, we can use Lemma 4 to construct a matrix $M_c(\alpha,\beta,t)$ such that

$$\begin{aligned}J_c(\alpha,\beta,t) &= M_c(\alpha,\beta,t)F(t)M_c^{-1}(\alpha,\beta,t) \\ \text{and } B &= M_c(\alpha,\beta,t)G(t)\end{aligned}$$

for any given value of α and $\beta \neq 0$ and where $J_c(\alpha,\beta,t) = J_{\alpha\beta} + B\Delta_{\alpha\beta}(t)$ with $\Delta_{\alpha\beta}(t) = \begin{bmatrix} \delta_1(t) & \delta_2(t) & \cdots & \delta_n(t) \end{bmatrix}$.

5. Application to observer and controller design

The previous results can be used to design observers and controllers for particular classes of dynamical systems. The Jordan controllable form was used by Busawon (2000) to design a controller for a class of nonlinear systems. In this section, we are going to use the previous results to design observers and controllers for linear time-varying systems.

5.1. *Control design for linear time varying systems*

Consider the single-input linear time varying system

$$\dot{x}(t) = F(t)x(t) + G(t)u(t) \tag{23}$$

where $x \in \mathcal{R}^n$, $u \in \mathcal{R}$. $F(t) = (f_{ij}(t))$ and $G(t) = (g_{ij}(t))$ are time varying matrices of appropriate dimensions. We make the following assumptions:

A1) The entries $f_{ij}(t)$ and $g_{ij}(t)$ of $F(t)$ and $G(t)$ are continuously differentiable and bounded for all $t \geq 0$.

A2) The controllability matrix $U(t) = \begin{pmatrix} G(t) & F(t)G(t) & \cdots & F^{n-1}(t)G(t) \end{pmatrix}$ is of full rank for all $t \geq 0$.

Due to Assumption A2) and Remark 3, one can use the procedure of Lemma 4 to construct a matrix $M_c(\alpha, \beta, t)$ such that

$$\begin{aligned} J_c(\alpha,\beta,t) &= J_{\alpha\beta} + B\Delta_{\alpha\beta}(t) \\ &= M_c(\alpha,\beta,t)F(t)M_c^{-1}(\alpha,\beta,t) \end{aligned}$$

and

$$B = M_c(\alpha,\beta,t)G(t).$$

Consider the feedback law defined by

$$u(x(t)) = \left(-\Delta_{\alpha\beta}(t) + \beta K_c\right) M_c(\alpha,\beta,t)x(t) \tag{24}$$

where K_c is a vector which is chosen such that the matrix $(A + BK_c)$ is Hurwitz.

Theorem 1. *Assume that system (23) satisfies assumptions A1) and A2). Then, for all $\beta > 0$ there exists $\tilde{\alpha}_0 > 0$ such that for all $\alpha \in [-\tilde{\alpha}_0, 0]$ the origin of the closed-loop system*

$$\dot{x}(t) = F(t)x(t) + G(t)u(x(t)) \tag{25}$$

where $u(x(t))$ is as in (24), is globally asymptotically stable.

Proof:

Consider closed-loop system (25) and let $\bar{x} = M_c(\alpha,\beta,t)x$. Then,

$$\begin{aligned} \dot{\bar{x}} &= M_cFM_c^{-1}\bar{x} + M_cGu(x) + \dot{M}_cx \\ &= (J_{\alpha\beta} + B\Delta_{\alpha\beta})\bar{x} + Bu(x) + \dot{M}_cM_c^{-1}\bar{x} \end{aligned}$$

The arguments of the various time varying matrices are dropped for the sake of convenience. Since $u(x) = -\Delta_{\alpha\beta}\bar{x} + \beta K_c\bar{x}$, we have

$$\dot{\bar{x}} = \alpha\bar{x} + \beta\left(A + BK_c\right)\bar{x} + \dot{M}_cM_c^{-1}\bar{x}$$

Now, since $(A + BK_c)$ is Hurwitz, there exists a symmetric positive definite matrix P such that:

$$(A + BK_c)^TP + P(A + BK_c) = -I_n.$$

Consider the following candidate Lyapunov function $V(\bar{x}) = \bar{x}^TP\bar{x}$. Then,

$$\begin{aligned} \dot{V} &= 2\bar{x}^TP\,\dot{\bar{x}} \\ &= 2\alpha\bar{x}^TP\bar{x} + 2\beta\bar{x}^TP(A + BK_c)\bar{x} + 2\bar{x}^TP\dot{M}_cM_c^{-1}\bar{x} \\ &\leq 2\alpha\bar{x}^TP\bar{x} - \beta\bar{x}^T\bar{x} + 2\bar{x}^TP\dot{M}_cM_c^{-1}\bar{x} \\ &\leq 2\alpha\bar{x}^TP\bar{x} - \beta\left\|\bar{x}\right\|^2 + 2c_1\left\|\dot{M}_cM_c^{-1}\right\|\left\|\bar{x}\right\|^2 \end{aligned}$$

where $c_1 = \|P\|$. Now, by Assumption A1) there exist a positive constant c_2 such $\left\|\dot{M}_cM_c^{-1}\right\| \leq c_2\left|\alpha\right|^{m_1}\left|\beta\right|^{m_2}$ for some constant m_1 and m_2. Let $\alpha = -\tilde{\alpha}$ with $\tilde{\alpha} > 0$, then

$$\dot{V} \leq -2\tilde{\alpha}\lambda_{\min}(P)\left\|\bar{x}\right\|^2 - (\beta - 2\sigma\tilde{\alpha}^{m_1}\beta^{m_2})\left\|\bar{x}\right\|^2$$

where $\lambda_{\min}(P)$ is the smallest eigenvalue of P. Now, for a fixed value of $\beta > 0$, it suffices to choose $\tilde{\alpha}$ such that $\beta - 2\sigma\tilde{\alpha}^{m_1}\beta^{m_2} > 0$ that is $0 < \tilde{\alpha} < \left(\frac{\beta}{\sigma\beta^{m_2}}\right)^{\frac{1}{m_1}} = \tilde{\alpha}_0$. This completes the proof of Theorem 1.

5.2 *Observer for linear time varying systems*

Consider the single-output linear time varying system

$$\begin{cases} \dot{x}(t) = F(t)x(t) + G(t)u(t) \\ y(t) = H(t)x(t) \end{cases} \tag{26}$$

where $x = (x_1, \ldots, x_n)^T \in \mathcal{R}^n$, $u \in \mathcal{R}^m$ and $y \in \mathcal{R}$. The matrices $F(t) = (f_{ij}(t))$, $G(t) = (g_{ij}(t))$ and $H(t) = (h_{ij}(t))$ are time varying matrices of appropriate dimensions. We make the following assumptions:

A3) The entries $f_{ij}(t)$, $g_{ij}(t)$ and $h_{ij}(t)$ of $F(t)$, $G(t)$ and $H(t)$ are continuously differentiable and bounded for all $t \geq 0$.

A4) The observability matrix

$$\Upsilon(t) = \begin{bmatrix} H(t) \\ H(t)F(t) \\ \vdots \\ H(t)F^{n-1}(t) \end{bmatrix}$$

of the pair $(F(t), H(t))$ is of full rank for all $t \geq 0$.

Due to Assumption A4) and Remark 2, one can use the procedure of Lemma 3 to construct a matrix $M_o(\alpha,\beta,t)$ such that

$$J_o(\alpha,\beta,t) = J_{\alpha\beta} + \Gamma_{\alpha\beta}(t)C = M_o(\alpha,\beta,t)F(t)M_o^{-1}(\alpha,\beta,t)$$

and

$$C = H(t)M_o^{-1}(\alpha,\beta,t).$$

Consider the system defined by

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + G(t)u(t) + L(\alpha,\beta,t)(y(t) - H(t)\hat{x}(t)) \tag{27}$$

where

$$L(\alpha,\beta,t)=M_o^{-1}(\alpha,\beta,t)\left(\Gamma_{\alpha\beta}(t)+\beta K_o\right)$$

with K_o is a vector which is chosen such that the matrix $(A+K_oC)$ is Hurwitz.

Theorem 2. *Assume that system (26) satisfies assumptions A3) and A4). Then, for all $\beta>0$ there exist $\tilde{\alpha}_1>0$ such that for all $\alpha\in[-\tilde{\alpha}_1,0]$ the system (27) is an exponential observer for system (26).*

Proof

Set $e=x-\hat{x}$, then the error dynamics is given by

$$\dot{e}(t)=F(t)e(t)-L(\alpha,\beta,t)H(t)e(t)$$

Let $\bar{e}(t)=M_o(\alpha,\beta,t)e(t)$. Then,

$$\dot{\bar{e}}=M_oFM_o^{-1}\bar{e}-M_oLHM_o^{-1}\bar{e}+\dot{M}_oM_o^{-1}\bar{e}$$

The arguments of the various time varying matrices are dropped for the sake of convenience. Therefore,

$$\dot{\bar{e}}=J_o(\alpha,\beta,t)\bar{e}-M_oLC\bar{e}+\dot{M}_oM_o^{-1}\bar{e}$$

Since $L(\alpha,\beta,t)=M_o^{-1}\left(\Gamma_{\alpha\beta}(t)+\beta K_o\right)$, we have

$$\dot{\bar{e}}=\alpha\bar{e}+\beta\left(A-K_oC\right)\bar{e}+\dot{M}_oM_o^{-1}\bar{e}$$

Now, as $(A-KC)$ is Hurwitz, there exists a symmetric positive definite matrix S such that

$$\left(A-K_oC\right)^TS+S\left(A-K_oC\right)=-I_n$$

Consider the candidate Lyapunov function $V\left(\bar{e}\right)=\bar{e}^TS\bar{e}$. Then,

$$\begin{aligned}\dot{V}\left(\bar{e}\right) &= 2\bar{e}^TS\dot{\bar{e}}\\ &= 2\alpha\bar{e}^TS\bar{e}+2\beta\bar{e}^TS\left(A-K_oC\right)\bar{e}+2\bar{e}^TS\dot{M}_oM_o^{-1}\bar{e}\\ &= 2\alpha\bar{e}^TS\bar{e}-\beta\bar{e}^T\bar{e}+2\bar{e}^TS\dot{M}_oM_o^{-1}\bar{e}\\ &\leq 2\alpha\bar{e}^TS\bar{e}-\beta\left\|\bar{e}\right\|^2+2\left\|S\right\|\left\|\dot{M}_oM_o^{-1}\right\|\left\|\bar{e}\right\|^2\end{aligned}$$

where $l_1=\left\|S\right\|$. Now, by Assumption A3) there exist a positive constant l_2 such $\left\|\dot{M}_oM_o^{-1}\right\|\leq c_2\left|\alpha\right|^{n_1}\left|\beta\right|^{n_2}$ for some constant n_1 and n_2. Let $\alpha=-\tilde{\alpha}$ with $\tilde{\alpha}>0$, then

$$\dot{V}\leq-2\tilde{\alpha}\lambda_{\min}(S)\left\|\bar{e}\right\|^2-\left(\beta-2\sigma\tilde{\alpha}^{n_1}\beta^{n_2}\right)\left\|\bar{e}\right\|^2$$

where $\lambda_{\min}(S)$ is the smallest eigenvalue of S. Now, for a fixed $\beta>0$, it suffices to choose $\tilde{\alpha}$ such that $\beta-2\sigma\tilde{\alpha}^{n_1}\beta^{n_2}>0$ that is $0<\tilde{\alpha}<\left(\frac{\beta}{\sigma\beta^{n_2}}\right)^{\frac{1}{n_1}}=\tilde{\alpha}_1$. This completes the proof of Theorem 2.

6. Conclusions

In this paper, we have shown that any observable pair and controllable pair of matrices can be transformed into a so-called Jordan observable and a Jordan controllable pair respectively. We have provided algorithms for deriving the similarity matrices that permit such transformation. The Jordan canonical form is, in fact, an extension of the Brunowsky canonical form. We have also shown how such canonical form can be used to design observers and controllers for linear time varying systems. We have treated only single input and single-output systems. The extension of the Jordan canonical form to multi-input and multi-output case can be done, *a priori*, in similar fashion as in the SISO case and is currently under investigation.

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