ON THE ROBUST STABILITY AND ROBUST CONTROL VIA REFLECTION COEFFICIENTS

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Abstract: The problems of robust stability of linear discrete-time systems and controller design via reflection coefficients of the system characteristic polynomial is studied. It is shown, first, that reflection vectors are placed on the stability boundary with specific roots placement depending on the reflection vector number and the argument sign and, second, that the line segments between an arbitrary Schur polynomial and its reflection vectors are Schur stable. Then a Schur stable polytope is obtained around a given stable point and a robust controller is found by quadratic programming approach.

Keywords: robust stability , robust control, discrete-time systems

1. INTRODUCTION

The stability of linear dynamic systems is a well studied topic of linear differential equations. However, some serious problems of so-called robust stability arise when the parameters of systems are not exactly known (Ackermann,1993; Bhattacharyya *et al.*,1995) .That is why several stability margins are defined in different domains: gain and phase margin in frequency domain, minimal distance from imaginary axis in pole domain, stability radius in system parameter domain.

For robust pole placement the domain of characteristic polynomial coefficients is of interest (Ackermann,1993). Here some kind of stability margin can be obtained by the Kharitonov theorem (Kharitonov,1978) or edge theorem (Bartlett *et al.*,1988). Unfortunately, the first one does not hold for discrete-time systems.

In this paper the reflection coefficient stability criteria (Oppenheim *et al.*,1989) is used to define a Schur stability margin in polynomial coefficient space. The reflection vectors of an n-th order system will be introduced as $2n$ specific points on the stability boundary. The line segments between an arbitrary Schur polynomial (a point in coefficient space) and its reflection vectors will be Schur stable. So the minimal distance between a polynomial and its reflection vectors can be used as some kind of stability margin for linear discrete-time systems.

The more serious task is: how to find a convex subset of the stability region in system parameters domain and how to design a robust controller by reflection coefficient placement.

The paper is organized as follows. In section 2 we recall the stability condition via reflection coefficients and introduce reflection vectors of a monic Schur polynomial. Section 3 is devoted to the roots of reflection vectors. In section 4 the problem of stable simplex building around a given stable point is considered. At last, in section 5, the robust controller design problem will be solved by quadratic programming approach.

2. REFLECTION COEFFICIENTS OF SCHUR POLYNOMIALS

A polynomial $a(z)$ of degree n with real coefficients $a_i \in \mathcal{R}$, $i = 0, ..., n$

$$a(z) = a_n z^n + ... + a_1 z + a_0$$

is said to be Schur if all its roots are placed inside the unit circle. A linear discrete-time dynamical system is stable if its characteristic polynomial is Schur, i.e. if all its poles lie inside the unit circle.

Besides the unit circle criterion some other criterias are known for checking the stability of a linear system. It is interesting to mention that the well-known Jury's stability test leads precisely to the stability hypercube of reflection coefficients . In the following we use the stability criterion via reflection coefficients.

Let us recall the recursive definition of reflection coefficients $k_i \in \mathcal{R}$ of a polynomial $a(z)$ (Oppenheim *et al.*,1989):

$$k_i = -a_i^{(i)}, \tag{1}$$

$$a_i^{(n)} = \frac{a_{n-i}}{a_n}, \qquad i = 1, ..., n; \tag{2}$$

$$a_j^{(i-1)} = \frac{a_j^{(i)} + k_i a_{i-j}^{(i)}}{1 - k_i^2}, \ j = 1, ..., i-1. \tag{3}$$

Reflection coefficients are well-known in signal processing and digital filters. They are called also PARCOR coefficients and k-coefficients (Makhoul,1975). The stability criterion via reflection coefficient is as follows (Oppenheim *et al.*,1989).

Lemma 1. A polynomial $a(z)$ will be Schur stable if and only if its reflection coefficients $k_i, i = 1, ..., n$ lie within the interval $-1 < k_i < 1$.

A polynomial $a(z)$ lies on the stability boundary if some $k_i = \pm 1, i = 1, ..., n$. For monic Schur polynomials, $a_n = 1$, there is a one-to-one correspondence between the vectors $\tilde{a} = (a_0, ..., a_{n-1})^T$ and $k = (k_1, ..., k_n)^T$.

The transformation from reflection coefficients k_i to polynomial coefficients $a_{i-1}, i = 1, ..., n$ is multilinear. For monic polynomials we obtain from (1)-(3)

$$\begin{aligned} & a_i = a_{n-i}^{(n)}, \\ & a_i^{(i)} = -k_i, \\ & a_j^{(i)} = a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, \\ & \qquad\qquad i = 1, ..., n; j = 1, ..., i-1. \end{aligned} \tag{4}$$

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Lemma 2.(Nurges and Rüstern,1999) Through an arbitrary stable point $a = [a_0, a_1, ..., a_{n-1}]$ with reflection coefficients $k_i^a \in (-1, 1)$, $i = 1, ..., n$ you can put n stable line segments

$$A^i(\pm 1) = conv\{a | k_i^a = \pm 1\}$$

where $conv\{a | k_i^a = \pm 1\}$ denotes the convex hull obtained by varying the reflection coefficient k_i^a between -1 and 1.

Now let us introduce the reflection vectors of a monic polynomial $a(z)$. They will be useful for convex stable subsets building in polynomial coefficient space.

Definition. Let us call the vectors

$$a^i(1) = (a | k_i = 1), i = 1, ..., n$$

positive reflection vectors and

$$a^i(-1) = (a | k_i = -1), i = 1, ..., n$$

negative reflection vectors of a monic polynomial $a(z)$.

It means, reflection vectors are the extreme points of the Schur stable line segment $A^i(\pm 1)$ through the point a defined by Lemma 2. Due to the definition and the Lemmas 1 and 2 the following assertions hold:

1) every Schur polynomial has $2n$ reflection vectors $a^i(1)$ and $a^i(-1)$, $i = 1, ..., n$;

2) all the reflection vectors lie on the stability boundary ($k_i = \pm 1$);

3) all the innerpoints of the line segments between reflection vectors $a^i(1)$ and $a^i(-1)$ are Schur stable.

3. ROOTS OF REFLECTION VECTORS

In this section we study the reflection vectors placement on the stability boundary.

Theorem 1. Reflection vectors $a^i(\pm 1)$, $i = 1, ..., n$ of a monic Schur polynomial $a(z)$ have the following i roots r_j , $j = 1, .., i$ on the stability boundary:

1) the positive reflection vector $a^i(1)$ has
   - for i even $r_1 = 1$,
     $r_2 = -1$
     and $(i-2)/2$ pairs
     of complex roots on the unit circle,
   - for i odd $r_1 = 1$,
     and $(i-1)/2$ pairs
     of complex roots on the unit circle,
2) the negative reflection vector $a^i(-1)$ has
   - for i even $i/2$ pairs
     of complex roots on the unit circle,
   - for i odd $r_1 = -1$,
     and $(i-1)/2$ pairs
     of complex roots on the unit circle.

The proof is given in (Nurges,2003).

Now we can introduce some kind of a stability margin via reflection vectors of a Schur polynomial.

Definition : Let us call the distance between a Schur stable polynomial $a(z)$ and its reflection vector $a^i(\pm 1)$, $i = 1, ..., n$ the stability margin in direction of i-th reflection vector or simply *i-th reflection vector margin* and denote it by $d_i(\pm 1)$

$$d^i(\pm 1) = |a - a^i(\pm 1)|.$$

Our aim is to find for an arbitrary stable point a with $|k_i^a| < 1$, $i = 1, ..., n$ a point b on the stability boundary with $|k_i^b| = \pm 1$, $i \in \{1, ..., n\}$ such that the distance between a and b is minimal, i.e.

$$|a - b| = \rho$$

where ρ is the stability radius of a .

It can be easily done by a simple search procedure in directions of reflection vectors taking into account the background of reflection vectors (according to Theorem 1). Indeed:

- the first positive reflection vector margin $d_1(1)$ gives us the distance to the real positive root boundary,
- the first negative reflection vector margin $d_1(-1)$ gives us the distance to the real negative root boundary,
- the second negative reflection vector margin $d_2(-1)$ gives us the distance to the complex root boundary,
- the second positive reflection vector margin $d_2(1)$ gives us the distance to the two different real root boundary ($r_1 = 1$, $r_2 = -1$),
- the third positive reflection vector margin $d_3(1)$ gives us the distance to the real positive and complex root boundary ($r_1 = 1$, $r_{2,3} = \alpha \pm \beta\mathbf{i}, \alpha^2 + \beta^2 = 1$),
- the third negative reflection vector margin $d_3(-1)$ gives us the distance to the real negative and complex root boundary ($r_1 = -1$, $r_{2,3} = \alpha \pm \beta\mathbf{i}, \alpha^2 + \beta^2 = 1$),
- the higher reflection vector margins give the distance to the several complex root boundaries.

4. STABLE SIMPLEX BUILDING BY REFLECTION VECTORS OF POLYNOMIALS

Two different approaches can be used for stable simplex (or polytope) building via reflection vectors:

1) choose such a stable point that the linear cover of its reflection vectors is stable;
2) choose an arbitrary stable point and build the stable simplex by n edges in directions of reflection vectors of the starting point.

The possibility of the first approach is confirmed by the following lemma.

Lemma 3. The innerpoints of the polytope S^0 generated by reflection vectors of the origin $a = 0$

$$S^0 = conv\{0^i(\pm 1), \quad i = 1, ..., n\} \tag{5}$$

are Schur stable.

The proof follows from the Cohn stability criterion (Cohn,1922)

$$\sum_{i=0}^{n-1} |a_i| < 1.$$

Lemma 3 (or Cohn stability condition) is quite conservative. The question is: is it possible to relax the initial condition of Lemma 3 in some neighborhood of the origin? The answer is given by the following proposition.

Theorem 2. Let $k_1^a \in (-1, 1)$ and $k_2^a = ... = k_n^a = 0$. Then the innerpoints of the polytope

S^a generated by the reflection vectors of the point a

$$S^a = conv\{a^i(\pm 1), \quad i = 1, ..., n\} \tag{6}$$

are Schur stable.

The proof is given in (Nurges,2001).

Example 1. Let $a(z) = z^3 - 0.75z^2$. The reflection coefficients and reflection vectors of the polynomial $a(z)$ are following:

$$\begin{array}{ll} k_1^a = 0.75, & a^1(1) = [1 \ \ -1 \ \ \ 0 \ \ \ 0 \ ]^T, \\ k_2^a = 0, & a^2(1) = [1 \ \ \ 0 \ \ -1 \ \ 0 \ ]^T, \\ k_3^a = 0, & a^3(1) = [1 \ -0.75 \ 0.75 \ -1 \ ]^T, \end{array}$$

$$\begin{array}{l} a^1(-1) = [1 \ 1 \ \ \ 0 \ \ \ 0 \ ]^T, \\ a^2(-1) = [1 \ -1.5 \ \ \ 1 \ \ \ 0 \ ]^T, \\ a^3(-1) = [1 \ -0.75 \ -0.75 \ 1 \ ]^T. \end{array}$$

By Theorem 1 the polytope

$$S^a = conv\{a^1(1), a^1(-1), a^2(1), a^2(-1), a^3(1), a^3(-1)\}$$

is stable.

Remark. Theorem 2 is less conservative than Lemma 3 because for S^a we have

$$\sum_{i=0}^{n-1} |a_i| < 3.$$

Theorem 3. Let $k_1^a \in (-1,1)$, $k_2^a \in (-1,1)$ and $k_3^a = ... = k_n^a = 0$. Then the innerpoints of the simplex $\tilde{S}^a$ generated by the reflection vectors of the point a

$$\tilde{S}^a = conv\{a, a^i((-1)^{i-1}), \quad i = 1, ..., n\} \tag{7}$$

is Schur stable.

The proof is given in (Nurges,2001).

Example 2. Let $a(z) = z^3 + 0.25z^2 - 0.5z$. The reflection coefficients and reflection vectors of the polynomial $a(z)$ are following:

$$\begin{array}{ll} k_1^a = -0.5, & a^1(1) = [\ 1 \ -0.5 \ \ -0.5 \ \ \ 0 \ \ ]^T, \\ k_2^a = 0.5, & a^2(-1) = [\ 1 \ \ \ 1 \ \ \ \ 1 \ \ \ \ 0 \ \ ]^T, \\ k_3^a = 0, & a^3(1) = [\ 1 \ \ 0.75 \ \ -0.75 \ -1 \ ]^T. \end{array}$$

By Theorem 2 the simplex
$\tilde{S}^a = conv\{a, a^1(1), a^2(-1), a^3(1)\}$ is stable.

5. ROBUST CONTROLLER DESIGN

In the previous sections we have find some convex approximations of the stability region in shape of a simplex or a polytope. Now we are looking for a robust controller such that the closed-loop characteristic polynomial will be placed in the preselected convex stability region.

Consider a discrete-time linear SISO system. Let the plant transfer function $G(z)$ of dynamic order m and the controller transfer function $C(z)$ of dynamic order r be given respectively by

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_{m-1}z^{m-1} + \cdots + b_1 z + b_0}{a_m z^m + \cdots + a_1 z + a_0}$$

and

$$C(z) = \frac{q(z)}{r(z)} = \frac{q_r z^r + \cdots + q_1 z + q_0}{r_r z^r + \cdots + r_1 z + r_0}.$$

It means that the closed loop characteristic polynomial

$$f(z) = a(z)r(z) + b(z)q(z)$$

is of degree $m + r$.

Let us require that the polynomial $f(z)$ will be placed in a simplex S of coefficient space . Without any restrictions we can assume that $a_m = r_r = 1$ and deal with monic polynomials.

Let us now introduce a stability measure p in accordance with the simplex S

$$p = c^T c$$

where

$$c = S^{-1} f$$

and S is the $(m + r + 1)$x$(m + r + 1)$ matrix of vertices of the target simplex. Obviously, for monic polynomials

$$\sum_{i=1}^{n+1} c_i = 1$$

where $n = m + r$. If all coefficients $c_i > 0$, $i = 1, ..., n+1$ then the point f is placed inside the simplex S.

It is easy to see that the minimum of p is obtained by

$$c_1 = c_2 = ... = c_{n+1} = \frac{1}{n+1}.$$

Then the point f is placed in the center of the simplex S.

Now we can formulate the following problem of controller design : find a controller $C(z)$ such that the stability measure p is minimal. In other words, we are looking for a controller which places the closed loop characteristic polynomial $f(z)$ as close as possible to the center of the target simplex S.

In matrix form we have

$$f = Gx \tag{8}$$

where G is the plant Sylvester matrix

$$G = \begin{bmatrix} a_0 & 0 & ... & 0 & b_0 & 0 & ... & 0 \\ a_1 & a_0 & ... & 0 & b_1 & b_0 & ... & 0 \\ . & . & . & . & . & . & . & . \\ a_{n-1} & a_{n-2} & ... & a_0 & b_{n-1} & b_{n-2} & ... & b_0 \\ 0 & a_{n-1} & ... & a_1 & 0 & b_{n-1} & ... & b_1 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & ... & a_{n-1} & 0 & 0 & ... & b_{n-1} \end{bmatrix}$$

of dimensions $(m+r+1)$x$(2r+1)$ and x is the $(2r+1)$-vector of controller parameters $x = [q_0, ..., q_{r-1}, r_0, ..., r_r]^T$.

The above controller design problem is equivalent to the quadratic programming problem: find x such that the minimum

$$\min_x x^T G^T (SS^T)^{-1} Gx$$

is obtained by the linear restrictions

$$S^{-1}Gx > 0$$

$$1^T S^{-1} Gx = 1$$

where $1^T = [1...1]$ is an n vector.

Let us now consider the case where the plant is subject to parameter uncertainty. We represent this by supposing that the given plant transfer function coefficients $a_0, ..., a_{m-1}$ and $b_0, ..., b_{m-1}$ are placed in a polytope P with vertices $p^1, ..., p^M$

$$P = conv\{p^j, j = 1, ..., M\}.$$

Because the relations (8) are linear in plant parameters we can claim that for an arbitrary fixed controller x the vector f of closed loop characteristic polynomial coefficients is placed in a polytope F with vertices $f^1, ..., f^M$

$$F = conv\{f^j, j = 1, ..., M\}$$

where

$$f^j = P^j x$$

and P^j is a 2m x 2m matrix composed by the vertex plant $p^j = [a_0^j, ..., a_{m-1}^j, b_0 j, ..., b_{m-1}^j]$.

The problem of robust controller design can be formulated as follows : find a controller x such that all vertices f^j , $j = 1, ..., M$ are placed inside the simplex S.

This problem can be solved by quadratic programming task : find x which minimizes

$$J = \min_x x^T \tilde{P}^T (I \otimes ((S^T)^{-1})(I \otimes S^{-1})\tilde{P}x$$

by linear restrictions

$$S^{-1}P^j x > 0,$$

$$1^T S^{-1} P^j x = 1, \qquad j = 1, ..., M.$$

Here I is the unit matrix, $\otimes$ denotes the Kronecker product and $\tilde{P}^T = [P_1^T, ..., P_M^T]$.

Example 3. Let us consider an uncertain second order interval plant

$$G(z) = \frac{b_0}{z^2 + a_1 z + a_0}$$

with parameters in the intervals $1.85 \le b_0 \le 1.95$, $-1.525 \le a_1 \le -1.475$, $a_0 = 0.55$ and we are looking for a first order robust controller.

Let the nominal closed loop characteristic polynomial be

$$f^0 = z^3 - 0.25z^2 + 0.03z - 0.001.$$

Then by pole placement algorithm we can easily find the controller

$$C_0(z) = \frac{0.7132z - 0.3624}{z + 1.25}$$

for the nominal plant

$$G_0(z) = \frac{1.9}{z^2 - 1.5z + 0.55}.$$

The simplex S will be chosen according to considerations of section 3.

For the above example we obtain

$$S = \begin{bmatrix} 0 & 0 & 0 & -1 \\ -0.2 & -0.2 & 1 & -0.6 \\ 0.4 & -0.8 & 1 & 0.6 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By the use of MATLAB Optimization Toolbox and above quadratic programming formulation we have find the robust controller

$$C(z) = \frac{1.0993z - 0.6403}{z + 1.7685}.$$

The minimum of the criterion $J_{min} = 0.5467$ indicates that the closed loop characteristic polynomial is placed in the given simplex S with considerable stability margin.

6. CONCLUSIONS

A new kind of stability margin for discrete-time systems is proposed in the system characteristic polynomial coefficient space making use of, so-called, reflection vectors of monic Schur polynomials. It is shown, first, that reflection vectors are placed on the stability boundary with specific roots placement depending on the reflection vector number and the argument sign and, second, that the line segments between an arbitrary Schur polynomial and its reflection vectors are Schur stable.

To find a robust controller by quadratic programming a stable simplex must be preselected in the closed loop characteristic polynomials coefficients space. A constructive procedure for generating simplexis in polynomial coefficients space is given. This procedure of stable simplex (or polytope) building is quite straightforward because you need to choose only one stable point with some restrictions for reflection coefficients of it. Then all the vertices of the simlex will be generated by reflection vectors of this point.

The procedure of controller design by quadratic programming is based on a stability measure p which indicates the placement of a (vertex) point against the preselected stable simplex.

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