Performance Assessment of Decentralized Controllers

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Abstract: Minimum variance (MV) benchmark is useful for identifying variance reduction opportunities in industrial control systems. During the past two decades, MV benchmarks for single-input single-output (SISO) and multi-input multi-output (MIMO) systems have been proposed. These MV benchmarks do not account for the structure of the decentralized or multi-loop controllers, which are used almost exclusively for regulation purposes in process industries. Due to this drawback, the available MV benchmarks can lead to incorrect conclusions regarding the performance of decentralized controllers. This paper aims to fill this gap. For performance assessment of decentralized controllers on a loop-by-loop basis, we present a simple modification of the available MV benchmark for SISO systems. For simultaneous performance assessment of all loops, we present a method for computing a tight lower bound on the achievable output variance. In the latter approach, the non-convexity of the resulting optimization problem is handled using sums of squares programming. The usefulness of the proposed benchmarks is evaluated using examples drawn from the literature.

Keywords: Decentralized control, Minimum variance control, Performance limits, Performance monitoring, Sums of squares programming.

1. INTRODUCTION

The performance of a well-designed control system can degrade over time due to changes in operating conditions and disturbance dynamics. Controller performance assessment is useful for identifying the opportunities for performance improvement of industrial controllers. Among the various available methods (Qin, 1998; Jelali, 2006), minimum variance (MV) benchmarking is one of the most promising methods for controller performance assessment. In this approach, the controller is deemed to provide satisfactory performance, if MV benchmark (ratio of least achievable and observed output variances) is close to 1. On the other hand, reduction in output variance is considered to be feasible through controller retuning, when the MV benchmark is significantly lower than 1.

The origin of MV benchmark can be traced back to Åström (1970), who demonstrated that the achievable output variance for a single-input single-output (SISO) process under feedback control depends on the first few impulse response coefficients of the disturbance model. Harris (1989) showed that with *a priori* knowledge of time delay, MV benchmark can be estimated using closed loop operating data and established it as a tool for performance assessment of SISO systems. Using the concept of interactor matrices, Harris et al. (1996) and Huang et al. (1997) proposed MV benchmark for multi-input multi-output (MIMO) systems.

This paper focusses on performance assessment of decentralized or multi-loop controllers, which are used almost exclusively for regulation purposes in process industries. Though useful, the available MV benchmarks for SISO and MIMO processes show limitations, when applied to processes under decentralized control. The conventional approaches for performance assessment of decentralized controllers include (see *e.g.* (Harris et al., 1996) and (Huang et al., 1997) for examples):

(1) *Loop by loop analysis*: The performance of individual loops is assessed independent of each other using MV benchmark for SISO processes.
(2) *Simultaneous analysis*: The performance of all loops is assessed simultaneously using MV benchmark for MIMO processes.

The MV benchmark for SISO processes assumes that the loop under consideration is being operated in isolation from the rest of the process and thus inherently views the process as being diagonal; see Figure 1. Due to this assumption, it may be possible to improve the performance of the existing controller further than indicated by the MV benchmark; see Section 3 for details. On the other hand, MV benchmark for MIMO processes ignores the diagonal structure of the decentralized controller and thus has more degrees of freedom for variance minimization than are available in the actual controller. Using a simple 2×2 process, we demonstrate in Section 4 that the least output variance that can be achieved using a diagonal controller can be four times higher than that can be achieved using a full multivariable controller. In summary, ignoring the controller structure can lead to incorrect conclusions regarding performance assessment of decentralized controllers.

The derivation of an approach for MV benchmarking of decentralized controllers requires characterization of the

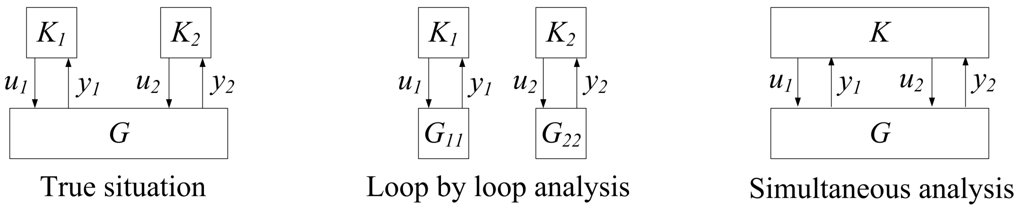

Fig. 1. Insufficiency of available MV benchmarks for performance assessment of decentralized controllers

least achievable output variance and its subsequent estimation from closed-loop data. This paper mainly focusses on the first issue. We first propose an MV benchmark for loop-by-loop analysis, where the presence of other loops is accounted for. The proposed result requires a small modification of the existing MV benchmark for SISO processes. It is further shown that the modified MV benchmark for loop-by-loop analysis can be directly estimated from closed-loop data with *a priori* knowledge of the delays of the different elements of the process model. An interesting insight is that for processes under decentralized control, the pre-whitening of output data using algorithms such as filtering and correlation (FCOR) algorithm (Huang and Shah, 1999) does not necessarily provide the first few impulse response coefficients of the disturbance model, as is traditionally believed.

The derivation of MV benchmark for simultaneous analysis of decentralized controller is more challenging. This happens as the optimization problem involving minimization of output variance becomes non-convex, once the diagonal structure is imposed on the controller (Sourlas and Manousiouthakis, 1995; Rotkowitz and Lall, 2006). With this difficulty, one may alternately look for tight upper and lower bounds on the least achievable output variance using decentralized controllers. Clearly, any suboptimal tuning strategy for the decentralized controller provides an upper bound on the least achievable output variance. Some approaches for finding upper bounds on least achievable output variance have been reported using non-convex optimization (Ko and Edgar, 1998; Jain and Lakshminarayanan, 2007) or by utilizing the structure of the optimization problem (Yuz and Goodwin, 2003; Kariwala et al., 2005). Recently, Kariwala (2007) addressed the more difficult problem involving derivation of a tight lower bound on the least achievable output variance, where an explicit bound is proposed by considering those impulse response coefficients of the closed-loop transfer function between disturbances and outputs, which depend linearly on the controller parameters. In general, however, the lower bound proposed in (Kariwala, 2007) can be conservative due to the neglected impulse response coefficients.

In this paper, we show that though nonlinear, the impulse response coefficients of the closed-loop transfer function between disturbances and outputs can be represented as polynomials in unknown controller parameters. Subsequently, the non-convex optimization problem related to the minimization of output variance is solved using sums of squares (SOS) programming (Parillo, 2000). This result is further extended to find a lower bound on the least achievable output variance, when the individual sub-controllers of the decentralized controller are restricted to be of proportional-integral-derivative (PID) type. The estimation of these lower bounds is difficult from closed-loop data only and the knowledge of process model is required. Nevertheless, the derivation of lower bound on the least achievable output variance can itself be seen as a major step towards systematic performance assessment of decentralized controllers.

2. PROBLEM FORMULATION

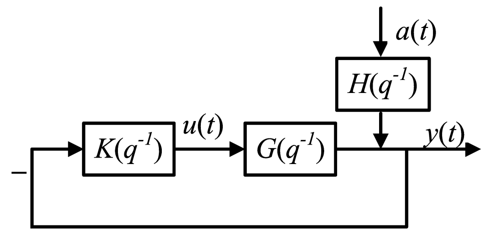

Fig. 2. Block diagram of closed-loop system

We consider the closed-loop system shown in Figure 2. For this system, we denote $G(q^{-1})$ and $H(q^{-1})$ as the process and disturbance models, respectively, such that

$$y(t) = G(q^{-1})\,u(t) + H(q^{-1})\,a(t) \tag{1}$$

Here, $y(t), u(t)$ and $a(t)$ are controlled outputs, manipulated variables and disturbances, respectively. We make the following simplifying assumptions:

(1) $G(q^{-1})$ and $H(q^{-1})$ are stable, causal transfer matrices, contain no zeros on or outside the unit circle except at infinity (due to time delays), and are square having dimensions $n_y \times n_y$.
(2) $a(t)$ is a random noise sequence with unit variance.

When $H(q^{-1})$ contains zeros outside the unit circle, these zeros can be factored through an all pass factor without affecting the noise spectrum (Huang and Shah, 1999). Further, there is no loss of generality in assuming that the system is affected by noise having unit variance, as the disturbance model can always be scaled to satisfy this assumption. For notational simplicity, we drop the arguments q^{-1} and t in the subsequent discussion.

Our objective is to find the least achievable value of $\mathrm{Var}(y)$ with respect to the controller K, *i.e.*

$$J_{\mathrm{decen}} = \min_K \mathrm{Var}(y) = \min_K E[\mathrm{tr}(y\,y^T)] \tag{2}$$

where K is assumed to have a diagonal structure, *i.e.* $K = \mathrm{diag}(K_1, K_2, \cdots, K_{n_y})$. In (2), $E(\cdot)$ and $\mathrm{tr}(\cdot)$ denote the expectation and trace operators, respectively. The pairings are considered to be selected on the diagonal elements of G.

Based on (2), the MV benchmark for decentralized controller can be defined as

$$\eta_{\text{decen}} = \frac{J_{\text{decen}}}{E[\text{tr}(y\,y^T)]} \tag{3}$$

where $E[\text{tr}(y\,y^T)]$ is the observed output variance.

A related problem involves finding the least achievable variance of the ith output, *i.e.*

$$J_{i,\text{decen}} = \min_{K_i} E[\text{tr}(y_i\,y_i^T)] \tag{4}$$

Similar to (3), the MV benchmark for the ith output can be defined as

$$\eta_{i,\text{decen}} = \frac{J_{i,\text{decen}}}{E[\text{tr}(y_i\,y_i^T)]} \tag{5}$$

where $E[\text{tr}(y_i\,y_i^T)]$ is the observed variance of the ith output.

3. LOOP-BY-LOOP ANALYSIS

We first consider performance assessment of the decentralized controller on a loop-by-loop basis. For clarity of presentation, we limit the discussion to 2×2 systems. The result can be generalized to $n_y \times n_y$ systems using block-partitioning of G and H. We have

$$y_1 = G_{11}\,u_1 + G_{12}\,u_2 + H_1\,a \tag{6}$$

$$y_2 = G_{21}\,u_1 + G_{22}\,u_2 + H_2\,a \tag{7}$$

The time delay associated with G_{ij} is denoted as d_{ij}, *i.e.*

$$G_{ij} = q^{-d_{ij}}\bar{G}_{ij} \tag{8}$$

where $\bar{G}_{ij}$ denotes the invertible part of G_{ij}. Without loss of generality, we consider that the objective is to characterize the least achievable variance of y_1.

3.1 Conventional approach

The traditional approach for loop-by-loop analysis involves using the MV benchmark for SISO systems. Here, H_1 is decomposed using Diophantine identity as

$$H_1 = F_1 + q^{-d_{11}}R_1 \tag{9}$$

Then, the least achievable variance of y_1 is taken as (Åström, 1970; Harris, 1989)

$$J_1 = \min_{K_1} E[\text{tr}(y_1\,y_1^T)] = \|F_1\|_2^2 \tag{10}$$

where $\|\cdot\|_2$ denotes the $\mathcal{H}_2$-norm. The MV benchmark for individual outputs is defined similar to (5). An inherent assumption in the derivation of (10) is that $u_2 = 0$ at all times or in other words, the first loop is being operated in isolation from the rest of the process. We next demonstrate that when the presence of other loops is accounted for, the first d_{11} elements of H_1 are not necessarily feedback invariant and thus the least achievable variance of y_1 can be lower than J_1 in (10).

3.2 Modified MV benchmark

Consider that the second loop is closed with $u_2 = -K_2 y_2$. Under partially closed loop conditions, we have (Skogestad and Postlethwaite, 2005)

$$y_1 = P_{11}\,u_1 + P_{d1}\,a \tag{11}$$

where

$$P_{11} = G_{11} - \frac{G_{12}K_2G_{21}}{1+G_{22}K_2},\ P_{d1} = H_1 - \frac{G_{12}K_2H_2}{1+G_{22}K_2} \tag{12}$$

Since $1/(1+G_{22}K_2)$ and K_2 are rational and invertible, it follows that the delay associated with P_{11} or the effective delay of the first loop is

$$d_1^{'} = \min(d_{11}, d_{12}+d_{21}) \tag{13}$$

Now, let P_{d1} be decomposed using Diophantine identity as

$$P_{d1} = F_1^{'} + q^{-d_1^{'}}R_1^{'} \tag{14}$$

Using (11) and (14), it follows that

$$y_1 = F_1^{'}\,a + q^{-d_1^{'}}(\bar{P}_{11}\,u_1 + R_1^{'}\,a) \tag{15}$$

where $P_{11} = q^{-d_1^{'}}\bar{P}_{11}$ and $\bar{P}_{11}$ denotes the invertible part of P_{11}. Since the first term in (15) cannot be affected by u_1 (invariant of K_1), it follows that

$$J_{1,\text{decen}} = \min_{K_1} E[\text{tr}(y_1\,y_1^T)] = \|F_1^{'}\|_2^2 \tag{16}$$

Note that $\|F_1^{'}\|_2^2$ represents the least achievable variance of y_1, when the presence of second loop is accounted for and can be used readily for performance assessment of decentralized controllers on a loop-by-loop basis.

Remark 1. In general, $J_{1,\text{decen}}$ may depend on K_2. This dependance, however, has no bearing on the loop-by-loop performance assessment, where the objective is to find the least achievable variance of y_i through tuning of K_i. If both controllers are allowed to be tuned simultaneously, using similar analysis as used in this section earlier, it can be shown that the first $d_1^{''} = \min(d_{11}, d_{12})$ impulse response coefficients of H_1 are feedback invariant and thus

$$\min_{K_1,K_2} E[\text{tr}(y_1\,y_1^T)] = \|F_1^{''}\|_2^2 \tag{17}$$

where $H_1 = F_1^{''} + q^{-d_1^{''}}R_1^{''}$. Note that the bound in (17) is independent of controller type (full multivariable or decentralized).

Though the result in (16) may seem entirely mathematical, a physical reasoning with this result can be associated by considering the block diagram of a 2×2 system shown in Figure 3. Here, u_1 can affect y_1 directly through G_{11}, but also through a parallel path involving G_{12} and G_{21} (shown with thick line in Figure 3). Thus, $d_1^{'} = \min(d_{11}, d_{12}+d_{21})$ represents the delay of the fastest path through which u_1 can affect y_1. In this sense, loop interaction can sometimes be beneficial for reducing output variance.

Example 2. To illustrate the findings of this section, we use the case study of a binary distillation column (Wood and Berry, 1973). The continuous-time model is discretized with a sampling time of 1 minute to get

$$G = \begin{bmatrix} \dfrac{0.744q^{-2}}{1-0.942q^{-1}} & \dfrac{-0.879q^{-4}}{1-0.954q^{-1}} \\ \dfrac{0.579q^{-8}}{1-0.912q^{-1}} & \dfrac{-1.302q^{-4}}{1-0.933q^{-1}} \end{bmatrix} \tag{18}$$

$$H = \begin{bmatrix} \dfrac{0.247q^{-9}}{1-0.935q^{-1}} \\ \dfrac{0.358q^{-4}}{1-0.927q^{-1}} \end{bmatrix} \tag{19}$$

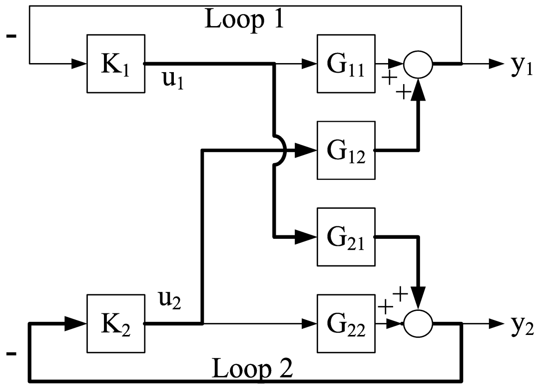

Fig. 3. Presence of parallel path (shown with thick line) from u_1 to y_1 for 2×2 systems

For diagonal pairings, the following decentralized PI controller is tuned using internal model control (IMC) method (Skogestad, 2003)

$$K = \text{diag}\left(\frac{0.652 - 0.571q^{-1}}{1-q^{-1}}, \frac{-0.124 + 0.115q^{-1}}{1-q^{-1}}\right) \quad (20)$$

which provides $\text{Var}(y_1) = 0.122$ and $\text{Var}(y_2) = 0.759$. After factoring the time delay of H, the least achievable variances of y_1 and y_2 according to the conventional approach discussed in Section 3.1 are $J_1 = 0.114$ and $J_2 = 0.413$, respectively. Thus, the conventional MV benchmarks for loops 1 and 2 are $\eta_1 = 0.932$ and $\eta_2 = 0.544$, respectively, which indicate that the variance of y_2 can be reduced significantly by re-tuning K_2, but tuning K_1 will not reduce variance of y_1 significantly. Using the modified MV benchmark, we next show that this conclusion is not entirely correct.

We have $d_{11} = 2$, $d_{12} = 4$, $d_{21} = 8$ and $d_{22} = 4$. Thus, $d_1' = \min(d_{11}, d_{12} + d_{21}) = 2$ and $d_2' = \min(d_{22}, d_{12} + d_{21}) = 4$. Since the first $d_2' = d_{22}$ impulse response coefficients of P_{d2} and H_2 are the same, we find that $J_{2,\text{decen}} = J_2 = 0.413$ and $\eta_{2,\text{decen}} = \eta_2 = 0.544$. The first $d_1' = d_{11}$ impulse response coefficients of P_{d1}, however, are different from the corresponding impulse response coefficients of H_1 and we find that $J_{1,\text{decen}} = 0.031$ and $\eta_{1,\text{decen}} = 0.251$. Thus, the modified MV benchmark identifies that the variance of y_1 can be reduced by approximately 4 times through tuning of K_1.

We point out that for this process, the first d_1' impulse response coefficients of P_{d1} and thus $J_{1,\text{decen}}$ depend on K_2. For example, when the gain of K_2 is decreased by a factor of 0.75, $J_{1,\text{decen}}$ increases to 0.037. With this controller tuning the variance of y_1 is 0.126. Thus, we have $\eta_{1,\text{decen}} = 0.293$, which indicates that the variance of y_1 can still be reduced by approximately 3 times.

Remark 3. Although for Example 2, the effective delays for both loops are the same as open-loop delays, *i.e.* $d_1' = d_{11}$ and $d_2' = d_{22}$, this is not true in general. For example, when pairings are chosen on the off-diagonal elements of G in (18), the effective delay for loop 1 is $d_1' = \min(d_{12}, d_{11} + d_{22}) = 6$, which is different from open-loop delay, $d_{12} = 8$.

Remark 4. Based on (11), under closed loop conditions

$$y_1 = \frac{P_{d1}}{1 + P_{11}K_1} a \quad (21)$$

Let $K_1 = M_1/(1 - P_{11}M_1)$, where M_1 is a stable transfer function (Youla parameterization). Then,

$$y_1 = (1 - P_{11}M_1)P_{d1}\, a \quad (22)$$
$$= F_1' a - q^{-d_1'}(\bar{P}_{11}M_1P_{d1} - R_1')a \quad (23)$$

Thus, F_1' and thus $\eta_{i,\text{decen}}$ can be estimated using closed loop data, *e.g.* using the FCOR algorithm (Huang and Shah, 1999), with *a priori* knowledge of the delays of G_{ij}. It is also interesting to note that the use of FCOR algorithm for loop-by-loop analysis of decentralized controllers leads to estimation of the first few impulse response coefficients of P_{d1} and not H_1, as is traditionally believed.

4. SIMULTANEOUS ANALYSIS

The variance of other outputs can increase, when the variance reduction of ith output is attempted through tuning of K_i. This effect is not taken into account by loop-by-loop analysis. To overcome this drawback, we derive an MV benchmark for simultaneous performance assessment of all loops in this section.

4.1 Conventional approach

A multivariable process can be represented as

$$G = D^{-1}\bar{G} \quad (24)$$

such that $\bar{G}$ and D^{-1} contain the invertible and non-invertible parts of G, respectively. We consider that $D(q)$ is a unitary interactor matrix, *i.e.* $D^T(q^{-1})\,D(q) = I$ (Huang and Shah, 1999). Let

$$q^{-d}DH = F + q^{-d}R \quad (25)$$

where d denotes the order of the interactor matrix. Then (Harris et al., 1996; Huang et al., 1997),

$$J_{\text{full}} = \min_K E[\text{tr}(y\,y^T)] = \|F\|_2^2 \quad (26)$$

The MV benchmark for simultaneous analysis is defined similar to (3). The bound on achievable output variance in (26), however, does not take the diagonal structure of the controller into account and thus may classify well-performing decentralized controllers as poorly performing. In the subsequent discussion, we present a lower bound on the achievable output variance for systems under decentralized control.

4.2 MV benchmark for Decentralized controllers

For regulatory control, we have $u = -K\,y$. Thus, the closed loop transfer function between a and y can be written as

$$y = S\,a; \quad S = (I + G\,K)^{-1}\,H \quad (27)$$

Since $E[a(t)a^T(t)] = I$,

$$J_{\text{decen}} = \min_K E[\text{tr}(y\,y^T)] = \min_K \|S\|_2^2 \quad (28)$$

When diagonal structure is imposed on K, the optimization problem in (28) becomes non-convex. A key observation to overcome this difficulty is that

$$\|S\|_2^2 = \sum_{i=0}^{\infty} \mathrm{tr}\,(S_i^T\, S_i) \geq \sum_{i=0}^{n} \mathrm{tr}\,(S_i^T\, S_i) \tag{29}$$

for any finite n. Thus, a lower bound on J_{decen} can be found by minimizing $\sum_{i=0}^{n} \mathrm{tr}\,(S_i^T\, S_i)$. Here, S_i represents the ith impulse response coefficient of S.

When the closed-loop system is stable, (27) can be expanded using Taylor Series expansion to get

$$S = \left[\sum_{i=0}^{\infty} (-1)^i (G\,K)^i\right] H \tag{30}$$

For given n, we define the following $nn_y \times nn_y$-dimensional Hankel matrices

$$G_H = \begin{bmatrix} G_0 & 0 & \cdots & 0 \\ G_1 & G_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G_{n-1} & \cdots & G_1 & G_0 \end{bmatrix} \tag{31}$$

$$K_H = \begin{bmatrix} K_0 & 0 & \cdots & 0 \\ K_1 & K_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ K_{n-1} & \cdots & K_1 & K_0 \end{bmatrix} \tag{32}$$

and the following $nn_y \times nn_d$-dimensional matrices

$$H_v = \begin{bmatrix} H_0^T & H_1^T & \cdots & H_{n-1}^T \end{bmatrix}^T \tag{33}$$

$$S_v = \begin{bmatrix} S_0^T & S_1^T & \cdots & S_{n-1}^T \end{bmatrix}^T \tag{34}$$

Based on (32)-(34), S_v can be compactly written as

$$S_v = \left[\sum_{i=0}^{n} (-1)^i (G_H\, K_H)^i\right] H_v \tag{35}$$

Now, a lower bound on J_{decen} can be found by solving

$$J_{\mathrm{decen}} = \min_K \sum_{i=0}^{\infty} \mathrm{tr}\,(S_i^T\, S_i) \geq \min_K\, \mathrm{tr}\,(S_v^T\, S_v) \tag{36}$$

for any finite n. Based on (35), we note that S_v and thus $\mathrm{tr}\,(S_v^T\, S_v)$ depend polynomially on the controller parameters. Thus, the optimization problem in (36) can be seen as finding the global minimal value of a polynomial. For this purpose, we use sums of squares (SOS) programming (Parillo, 2000) in this paper. SOS programming transforms the polynomial minimization problem to a semi-definite program, which is solved using Sedumi (Sturm, 1999) interfaced with Matlab through Yalmip (Löfberg, 2004) in this paper.

SOS programming does not necessarily provide the minimum value of the polynomial, but guarantees a global lower bound (Parillo, 2000). For any controller K, however, since

$$\sum_{i=0}^{n+1} \mathrm{tr}\,(S_i^T\, S_i) \geq \sum_{i=0}^{n} \mathrm{tr}\,(S_i^T\, S_i) \tag{37}$$

tight lower bound on J_{decen} can be found by increasing n sequentially until convergence. In comparison with Kariwala (2007), where only the first $(2d-1)$ impulse response coefficients of S are used to find a lower bound on J_{decen}, the use of SOS programming provides a tighter lower bound, whenever $n > (2d-1)$. A similar approach involving SOS programming has been used earlier by Sendjaja and Kariwala (2009) to characterize the achievable output variance of SISO systems under PID control.

Example 5. We consider the following 2×2 process, where

$$G = \begin{bmatrix} \dfrac{-0.1q^{-2}}{(1-0.1q^{-1})\,(1-0.2q^{-1})} & \dfrac{-0.25q^{-1}\,(1-0.3q^{-1})}{(1-0.1q^{-1})\,(1-0.2q^{-1})} \\ \dfrac{0.5q^{-1}\,(1+0.9q^{-1})}{(1-0.1q^{-1})\,(1-0.2q^{-1})} & \dfrac{-0.1q^{-2}}{(1-0.1q^{-1})\,(1-0.2q^{-1})} \end{bmatrix} \tag{38}$$

and $H = 1/(1-q^{-1})\, I$ (Kariwala, 2007). Then, $D = q\,I$, $F = I$ and $J_{\mathrm{full}} = \|F\|_2^2 = 2$.

By considering the contribution of first $(2d-1)$ impulse response coefficients of S towards $\|S\|_2^2$, Kariwala (2007) found $J_{\mathrm{decen}} \geq 4$. Using the SOS programming approach, however, we find $J_{\mathrm{decen}} \geq 8.023$. Using non-convex optimization with multiple randomized initial guesses, Kariwala (2007) showed that the exact value of J_{decen} is approximately 8.16. This example amply demonstrates the usefulness of the SOS programming approach for finding tight lower bound on least achievable value of output variance for systems under decentralized control. The reader should also note that for this process, the MV benchmark found using conventional approach will be approximately 4 times lower than the MV benchmark found by accounting for the controller structure. Thus, the conventional approach for performance assessment of decentralized controllers may incorrectly classify well-performing controllers as poorly performing.

Remark 6. Unlike loop-by-loop analysis (see Remark 4), it is difficult to estimate η_{decen} directly from closed-loop data. When G is known (or has been identified using open or closed-loop identification experiments), H can be estimated by pre-whitening the pseudo-signal $(y - Gu)$. Then, SOS programming can be used to identify a lower bound on J_{decen} and η_{decen} based on identified model. We point out that the knowledge of G is also required by other available approaches for performance assessment of decentralized controllers (Ko and Edgar, 1998; Jain and Lakshminarayanan, 2007). In practice, the task of identifying G should be undertaken, only when J_{full} differs significantly from the observed output variance.

4.3 MV benchmark for Decentralized PID Controllers

In industrial practice, the individual sub-controllers of the decentralized controller are often fixed to be of PID-type. Clearly, the presence of additional controller structure can further limit the least achievable variance of outputs. Next, we show that the SOS programming approach can be easily extended to derive a tight lower bound on J_{decen} for decentralized PID controllers.

We note that the decentralized PID controller can be expressed as

$$K_{PID} = \frac{1}{\Delta} \sum_{i=0}^{2} C_i\, q^{-1} \tag{39}$$

where C_i has diagonal structure and $\Delta = 1 - q^{-1}$ is the integrator. For

$$\hat{G} = \frac{1}{\Delta} G \tag{40}$$

let us define the Hankel matrices $\hat{G}_H$ and C_H, which have the same structure as G_H and K_H defined in (32). Using similar approach, as used in Section 4.2, it can be shown that for any finite n

$$S_v = \left[\sum_{i=0}^{n} (-1)^i (\hat{G}_H\, C_H)^i\right] H_v \tag{41}$$

Thus, a lower bound on J_{decen} for decentralized PID controller can be found by minimizing $\operatorname{tr}(S_v^T\, S_v)$ using SOS programming as before.

Example 7. We revisit Example 5. When individual sub-controllers are restricted to be of PI-type, the lower bound on J_{decen} increases to 9.980 (approximately 25% higher than unrestricted decentralized controller). The following sub-optimal controller is designed using trial and error

$$K_{\text{PI}} = \operatorname{diag}\left(\frac{-0.629 + 0.474\, q^{-1}}{\Delta}, \frac{-2.844 + 1.862\, q^{-1}}{\Delta}\right) \tag{42}$$

which provides $E[\operatorname{tr}(y\, y^T)] = 10.087$. When PID controllers are used, the lower bound on J_{decen} is 9.250 (approximately 15% higher than unrestricted decentralized controller). The following sub-optimal controller

$$K_{\text{PID}} = \operatorname{diag}\left(\frac{-0.884 + 0.924\, q^{-1} - 0.205\, q^{-2}}{\Delta}, \frac{-3.210 + 2.764\, q^{-1} - 0.869\, q^{-2}}{\Delta}\right) \tag{43}$$

provides $E[\operatorname{tr}(y\, y^T)] = 9.421$. For both cases (PI and PID control), the lower bounds are close to the upper bounds, which demonstrates that SOS programming technique can be used to find tight bounds on J_{decen} for decentralized PID controllers efficiently and reliably.

5. CONCLUSIONS AND OPEN ISSUES

The use of decentralized or multi-loop controllers is common in process industries. In this paper, we have shown that the use of existing MV benchmarks for SISO and MIMO systems for performance assessment of decentralized controllers may lead to incorrect conclusions regarding the opportunities for variance reduction through controller retuning. We proposed modified MV benchmarks for loop-by-loop analysis of the decentralized controller, which can be directly estimated from closed-loop data. An MV benchmark for simultaneous analysis of the decentralized controller is also proposed through the novel use of SOS programming, which guarantees a lower bound on the least achievable output variance. In summary, this paper takes a major step towards systematic performance assessment of decentralized controllers.

A limitation of the use of SOS programming approach is that the knowledge of process model is required. Furthermore, SOS programming requires solving large semidefinite programs, whose size and solution time increase rapidly with process dimensions. We are currently exploring the use of alternate approaches to handle the computational complexity of the SOS programming approach.

REFERENCES

Åström, K.J. (1970). *Introduction to Stochastic Control Theory.* Academic Press, London.

Harris, T.J. (1989). Assessment of control loop performance. *Can. J. Chem. Eng.*, 67(5), 856–861.

Harris, T.J., Boudreau, F., and Macgregor, J.F. (1996). Performance assessment of multivariable feedback controllers. *Automatica*, 32(11), 1505–1518.

Huang, B. and Shah, S.L. (1999). *Performance Assessment of Control Loops: Theory and Applications.* Springer-Verlag, London.

Huang, B., Shah, S.L., and Kwok, E.K. (1997). Good, bad or optimal? Performance assessment of multivariate processes. *Automatica*, 33(6), 1175–1183.

Jain, M. and Lakshminarayanan, S. (2007). Estimating performance enhancement with alternate control strategies for multiloop control system. *Chem. Eng. Sci.*, 62, 4644–4658.

Jelali, M. (2006). An overview of control performance assessment technology and industrial applications. *Control Eng. Pract.*, 14(5), 441–466.

Kariwala, V. (2007). Fundamental limitation on achievable decentralized performance. *Automatica*, 43(10), 1849–1854.

Kariwala, V., Forbes, J.F., and Meadows, E.S. (2005). Minimum variance benchmark for decentralized controllers. In *Proc. American Control Conference*, 1437–1442. Portland, OR, USA.

Ko, B. and Edgar, T.F. (1998). Multiloop PID control performance monitoring. In *Proc. The 8th Congress of Aston Pacific Confederation of Chemical Engineering.* Seoul, Korea.

Löfberg, J. (2004). YALMIP : A toolbox for modeling and optimization in MATLAB. In *Proc. the CACSD Conference.* Taipei, Taiwan.

Parillo, P.A. (2000). *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization.* Ph.D. thesis, California Institute of Technology, Pasadena, California, USA.

Qin, S.J. (1998). Controller performance monitoring - A review and assessment. *Comput. Chem. Eng.*, 23, 173–186.

Rotkowitz, M. and Lall, S. (2006). A characterization of convex problems in decentralized control. *IEEE T. Automat. Contr.*, 51(2), 274–286.

Sendjaja, A.Y. and Kariwala, V. (2009). Achievable PID performance using sums of squares programming. *J. Proc. Contr.*, In press.

Skogestad, S. (2003). Simple analytic rules for model reduction and PID controller tuning. *J. Proc. Contr.*, 13(4), 291–309.

Skogestad, S. and Postlethwaite, I. (2005). *Multivariable Feedback Control: Analysis and Design.* John Wiley & Sons, Chichester, UK, 2nd edition.

Sourlas, D.D. and Manousiouthakis, V. (1995). Best achievable decentralized performance. *IEEE T. Automat. Contr.*, 40(11), 1858–1871.

Sturm, J. (1999). Using SeDuMi 1.02, A Matlab toolbox for optimization over symmetric cones. *Optim. Method Softw.*, 11(1), 625–653.

Wood, R.K. and Berry, M.W. (1973). Terminal composition control of a binary distillation column. *Chem. Eng. Sci.*, 28(9), 1707–1717.

Yuz, J.I. and Goodwin, G.C. (2003). Loop performance assessment for decentralized control of stable linear systems. *Eur. J. Control*, 9(1), 116–130.