Observer Design for Systems with Continuous and Discrete Measurements ⋆

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Abstract: Classical observers are constructed on the basis of the nature of the measurement signals, namely, a continuous observer requires continuous output measurements. In this work, a novel observer which estimates continuous states when continuous and discrete measurements are available is presented. By resetting the initial condition of the observer at each sample instant, the convergence of the continuous states is guaranteed. The application to the the estimation of substrate and biomass concentrations in an anaerobic wastewater treatment process in which continuous and discrete measurements usually appear, shows the feasibility of the proposed scheme.

Keywords: jump observer, anaerobic digestion, discrete measurements

1. INTRODUCTION

Because of the increasing complexity and necessity for safety of industrial processes, efficient monitoring, decision and control systems are becoming more an more important. This is particularly true in the case of bioprocesses where the state of the live organisms of the system must be closely monitored. Extensive surveys have been published on this topic (Dochain, 2008). Furthermore, the last two decades have seen an increasing interest in improving the operation of bioprocesses by applying advanced control schemes. In particular, biological waste treatment process, more efficient that the traditional physicochemical methods but at the same time more complex, call for a consistent good performance,which leads to a needing for more efficient instrumentation, control and automation.

To apply any control strategy it is necessary to measure the process main variables, this can be performed placing sensors (**?**), however, although in many cases continuous measurements are easily available, for example the temperature or pH, due to economical reasons or consuming time techniques, other key variables can be only measured intermittently, or even not measured at all. For this reason the non measurable state variables should be estimated from available measurements (Meleiro and Filho, 2000). To deal with these problems, many solutions have been proposed in the past such as the well known classical Kalman filters and Luenberger observers (Ray, 1980) in both, continuous and discrete approaches. On of the reasons for the popularity of these estimators is that they are easy to implement since the algorithm can be derived directly from the state space model. However, these state observers can not be easily implemented when both continuous and discrete information must be considered. In this direction Scali et al. (1997) have proposed an extended Kalman filter which update some observer parameters each time that the sampled date is available. Using Lyapunov functions , Liu et al. (2008) and Muñoz de la Peña and Christofides (2008) have designed controllers that involve continuous and discrete retarded measurements. Nguang and Shi (2003) also use discrete measurements to design continuous fuzzy control algorithms. Based on this idea in this work it is proposed a continuous observer to be continuously updated from the continuous measurements and also retune the states at each instant when the discrete measurement are available.

This work is organized as follows. A review of jump observers is presented in section 2, then in section 3 the observational problem is formulated, while the proposed solution is developed in section 4. In section 5 we analyze the dynamic behavior through numerical simulations for an anaerobic digestion system. Finally we close the paper with some concluding remarks.

2. BASIC FACTS OF JUMP OBSERVERS

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad \forall t \in [0, \infty) \tag{1}$$

$$y(k\delta) = Cx(k\delta) \qquad k = 1, 2, 3, \ldots, \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^q$ are the state, input and output vectors, respectively. In this case the outputs are obtained at each sampling time δ.

The usual way to estimate the unknown states of system (1) from output (2) consists in discretizing the system

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and design a discrete observer. However, the observer thus obtained provides only information at each sampling period. Additionally, to obtain a discrete version of (1) it is necessary to have a well defined input in order to place the appropriate holder (for example a zero holder or a exponential holder), hence unexpected input variations during intersampling periods may produce discrete observer failures (García-Sandoval, 2006). For this reason, an interesting problem would be to construct an observer given by

$$\dot{z}(t) = Az(t) + Bu(t) \qquad \forall t \neq k\delta \tag{3}$$

$$z\left(k\delta^{+}\right) = z(k\delta) - G\left[y(k\delta) - Cz(k\delta)\right] \quad t = k\delta \tag{4}$$

where $z \in \mathbb{R}^n$ are the observer states and $z\left(k\delta^{+}\right)$ denotes the updated observer states at each sampling instant. This is a continuous observer which updates its states at each sampling instant. The next lemma establishes conditions for the existence of such observer.

Lemma 1. Consider system (1)-(2) and suppose the pair $\left(e^{A\delta}, C\right)$ is observable, then an observer of the form (3)-(4) with the matrix gain G such that matrix $(I + GC)\, e^{A\delta}$ is Schur, guaranteeing that $\lim_{t\to\infty}\left[x(t) - z(t)\right] = 0$.

Proof. See Appendix.

Remark 2. The main feature of observer (3)-(4) remains in the fact that the intersampling state information is available at any time and it is not necessary to have a pre-established dynamic behavior for the input. Equation (3) can be seen as a continuous open loop observer in the intersampling period and whose states, according to (4), are reseted each sampling period.

3. PROBLEM FORMULATION

Consider the dynamic system

$$\dot{x}(t) = f(x(t), u(t)) \quad \forall t \in [0, \infty) \tag{5a}$$

$$y_1(t) = C_1 x(t) \qquad \forall t \in [0, \infty) \tag{5b}$$

$$y_2(k\delta) = C_2 x(k\delta) \qquad k = 1, 2, 3, \ldots \tag{5c}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y_1 \in \mathbb{R}^{q_1}$, $y_2 \in \mathbb{R}^{q_2}$ are the state, input and output vectors for the dynamic system, respectively. The outputs are divided into continuous (y_1), and discrete (y_2) with sampling time δ. For this system it is desirable to design a continuous observer which uses both discrete and continuous measurements, in order to have continuous information about the full vector state. The following assumption is instrumental for the observer design.

Assumption 3. Defining

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0,u=0} \quad \text{and} \quad B = \left.\frac{\partial f}{\partial u}\right|_{x=0,u=0}$$

as the linear matrices for system (5), it is assumed that the pair (A, C), with

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

is observable but, the pairs (A, C_1) and (A, C_2) related with continuous and discrete measurements, are not necessarily completely observable. That is, the observability matrix of these pairs may not have full rank.

In the following section it is presented a continuous observer for system (5), which is the main result of this work.

4. OBSERVER DESIGN

Assume that there is a transformation $T \in \mathbb{R}^{n\times n}$, such that the linear approximation of system (5a)-(5b) becomes

$$\begin{aligned} \dot{z} &= \bar{A}z(t) + \bar{B}u(t) \\ y_1 &= \bar{C}_1 z(t) \end{aligned} \tag{6}$$

where

$$z = Tx = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad \bar{A} = TAT^{-1} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

$$\bar{B} = TB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \bar{C}_1 = C_1 T^{-1} = (C_{11}\ 0)$$

$z_1 \in \mathbb{R}^{n_1}$, $z_2 \in \mathbb{R}^{n_2}$, and the pair (A_{11}, C_{11}) is completely observable. In this case a partial observer for z_1 can be designed in such way that given a matrix G_{11}, $(A_{11} - G_{11}C_{11})$ is Hurwitz. Applying the inverse transformation, the proposed observer is,

$$\dot{\zeta}(t) = f(\zeta(t), u(t)) - G_1\left(C_1\zeta(t) - y_1(t)\right)$$

where

$$G_1 = T^{-1}\begin{pmatrix} G_{11} \\ 0 \end{pmatrix}.$$

This is a partial observer which only make use of continuous measurements (5b), however, using discrete measurements it is possible to design a jump observer as described in section 2, which include both continuous and discrete measurement. The following theorem states this result.

Theorem 4. Consider the system (5), which has a set of continuous measurements (5b) and a set of discrete measurements (5c) with sampling time δ. Furthermore consider that there is a transformation $T \in \mathbb{R}^{n\times n}$ which transforms the linear approximation of system (5a)-(5b) to its observable canonical form (6), while the matrix $G_1 = T^{-1}\left(G_{11}^T\ 0\right)^T$, is calculated in such a way that $(A_{11} - G_{11}C_{11})$ is Hurwitz and the matrix G_2 is such that $(I + G_2 C)\, A_d$ is Schur, with $A_d = e^{(A - G_1 C_1)\delta}$. Then, an observer for system (5), which takes continuous measurements and is also updated each sampling period is given by

$$\begin{aligned} \dot{\zeta}(t) &= f(\zeta(t), u(t)) && \forall t \neq k\delta \\ &\quad - G_1\left(C_1\zeta(t) - y_1(t)\right), \end{aligned} \tag{7a}$$

$$\begin{aligned} \zeta\left(k\delta^{+}\right) &= \zeta(k\delta) && t = k\delta, \\ &\quad + G_2\left(C\zeta(k\delta) - y(k\delta)\right), && k = 1, 2, 3, \ldots \end{aligned} \tag{7b}$$

where $\zeta \in \mathbb{R}^n$ are the observer states and $\zeta\left(k\delta^{+}\right)$ are its updated values at each sampling time and

$$y(k\delta) = \begin{pmatrix} y_1(k\delta) \\ y_2(k\delta) \end{pmatrix}.$$

This observer guarantees that, in a neighborhood of the origin, the error between the system and the observer states tends asymptotically to zero, i.e. $\lim_{t\to\infty}\left[x(t) - \zeta(t)\right] = 0$.

Proof. First, consider the linear approximations of both, system (5a) and observer (7a)

$$\dot{x}(t) = Ax(t) + Bu(t) \quad \forall t \in [0, \infty) \tag{8}$$

$$\dot{\zeta}(t) = (A - G_1C_1)\zeta(t) + Bu(t) + G_1y_1(t) \tag{9}$$

additionally, consider that there is a matrix T that transforms the system (8) and its output (5b) to its observable canonical form (6), i.e.

$$z = Tx = \mathrm{col}(z_1, z_2),$$
$$\xi = T\zeta = \mathrm{col}(\xi_1, \xi_2),$$

where z_1 and ξ_1 are the observable modes of x and ζ. Then for the observable subsystems of z and ξ defining the error $e_1(t) = z_1(t) - \xi_1(t)$, whose dynamic is

$$e_1(t) = (A_{11} - G_{11}C_{11})e_1(t).$$

Since G_{11} is such that $(A_{11} - G_{11}C_{11})$ is Hurwitz, $e_1(t)$ tends asymptotically to zero. On the other hand, using discrete measurements, $y(k\delta)$, a jump observer (7) which allows the updating of the continuous dynamic observer states in every sampling period it is designed, taking advantage of the discrete information. Defining now the error

$$\eta(t) = x(t) - \zeta(t)$$
$$\eta(k\delta^+) = x(k\delta) - \eta(k\delta^+)$$

its linear dynamic approximation around $\eta = 0$ is

$$\dot{\eta}(t) = (A - G_1C_1)\eta(t) \qquad \forall t \neq k\delta$$
$$\eta(k\delta^+) = (I + G_2C)\eta(k\delta) \qquad t = k\delta, \quad k = 1, 2, 3, \ldots$$

As described in Lemma 1, these dynamics are stable if the pair (A_d, C) with $A_d = e^{(A-G_1C_1)\delta}$ is observable and the gain G_2 is such that the matrix $(I + G_2C)A_d$ is Schur, thereby ensuring that $\lim_{k\to\infty}[x(k\delta) - \zeta(k\delta)] = 0$ and thus $\lim_{t\to\infty}[x(t) - \zeta(t)] = 0$, which proves the theorem.

Observer (7) can be seen as a hybrid observer since incorporates continuous dynamics (7a) and a discrete event (7b) which modifies the continuous part. It should be also noted that the calculation of the observer part for continuous measurements is independent of the discrete observer part, however, the total discrete observer depends on the gain G_1.

5. STUDY CASE

Last years, the environmental laws have been tightened and it has became mandatory treating wastewater from industries as well households (?Huntington, 1998). Because of this, the wastewater treatment control processes have received great importance, especially anaerobic processes are being widely considered as an alternative for the treatment of wastewater because it produces smaller quantities of organic matter and also yields a high-energy gas (Méndez-Acosta et al., 2008). To achieve the control of these processes, state observers are frequently used, however for economical reasons some key variables can just be measured using long sampling times, while others may be measured more often. For this reason, a jump observer is proposed as presented in theorem 4.

There exists many dynamic models to describe anaerobic process (?Batstone et al., 2002; Bernard et al., 2006). However, to apply the proposed observer, a macroscopic model of the anaerobic process developed and validated by Bernard et al. (2001) it is considered,

$$\dot{X}_1 = (\mu_1(S_1) - \alpha D)X_1 \tag{10a}$$
$$\dot{S}_1 = -k_1\mu_1(S_1)X_1 + (S_{1in} - S_1)D \tag{10b}$$
$$\dot{X}_2 = (\mu_2(S_2) - \alpha D)X_2 \tag{10c}$$
$$\dot{S}_2 = -k_3\mu_2(S_2)X_2 + k_2\mu_1(S_1)X_1 + (S_{2in} - S_2)D \tag{10d}$$

where X_1, X_2, S_1, S_2, are respectively the concentrations of acidogenic bacteria, methanogenic bacteria, Chemical Oxygen Demand (COD) and Volatile Fatty Acids (VFA), D is the dilution rate, defined by the ratio $D = Q/V$, where Q is the feeding flow and V the digester volume, S_{1in} and S_{2in} are respectively the concentrations of influent organic substrate and of influent VFA. The k_is are pseudo-stoichiometric coefficients associated to the bioreactions. Parameter $\alpha \in (0, 1]$ represents the fraction of the biomass which is not retained in the digester (Hess and Bernard, 2008). The bacterial growth rates $\mu_1(S_1)$ and $\mu_2(S_2)$, are nonlinear functions given respectively by the Monod and Haldane kinetics (Henze and Harremoes, 1983)

$$\mu_1(S_1) = \mu_{\max 1}\frac{S_1}{S_1 + K_{S1}}$$
$$\mu_2(S_2) = \mu_{\max 2}\frac{S_2}{S_2 + K_{S2} + (S_2/K_{I2})^2}$$

where $\mu_{1\max}$, K_{S1}, $\mu_{2\max}$, K_{S2} and K_{I2} are the maximum bacterial growth rate and the half-saturation constant associated to the substrate S_1, the maximum bacterial growth rate in the absence of inhibition, and the saturation and inhibition constants associated to substrate S_2, respectively. The values of parameters and the input concentrations used for simulations are listed in Tables 1 and 2.

If we consider that VFA concentration (S_2) is a continuous measurement while the COD concentration (S_1) can just be periodically acquired (in fact in real operations, VFA concentration can be obtained up to every hour or less (Méndez-Acosta et al., 2008), hence it can be considered continuous compared with the resident time and the COD concentration that could be measured even just once a

Table 1. Model Parameters (Alcaraz-González et al., 2003)

Parameter	Value
μ_1	$1.2\,\mathrm{d}^{-1}$
$\mu_{\max 2}$	$0.69\,\mathrm{d}^{-1}$
K_{S1}	$4.95\,\mathrm{kg\ COD/m^3}$
K_{S2}	$9.28\,\mathrm{mol\ VFA/m^3}$
K_{I2}	$20\,\mathrm{mol\ VFA/m^3}$
k_1	6.6 kg COD/ kg X_1
k_2	7.8 mol VFA/ kgX_1
k_3	611.2 mol VFA/ kgX_2
α	0.5 (addimentional)

Table 2. Input Concentrations

Substrate	Value
S_{1in}	20 Kg COD/m^3
S_{2in}	100 mol VFA/m^3

day), the jump observer developed in the previous section can be then applied to the dynamic system (10) writing it in the form

$$\dot{x}(t) = \begin{pmatrix} \mu_1(S_1)X_1 \\ -k_1\mu_1(S_1)X_1 \\ \mu_2(S_2)X_2 \\ -k_3\mu_2(S_2)X_2 + k_2\mu_1(S_1)X_1 \end{pmatrix} + \begin{pmatrix} -\alpha X_1 \\ S_{1in} - S_1 \\ -\alpha X_2 \\ S_{2in} - S_2 \end{pmatrix} u(t) \tag{11}$$

$$y_1(t) = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} x(t)$$
$$y_2(k\delta) = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} x(k\delta)$$

where

$$x(t) = \begin{pmatrix} X_1 \\ S_1 \\ X_2 \\ S_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and $u(t) = D(t)$. System (11) can be represented as

$$\dot{x}(t) = f(x) + g(x)u(t). \tag{12}$$

To calculate the jump observer (7) is necessary to linearize the system (12) around a neighborhood of equilibrium points (Hess and Bernard, 2008; Méndez-Acosta et al., 2008) so the system has the form

$$\dot{x}(t) = Ax(t) + Bu(t) + \hat{f}(x,u)$$

where

$$A = \left[\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}u\right]_{x=0,u=0}, \quad B = g(x)|_{x=0}$$

are the linear approximation matrices around the steady state [see (Hess and Bernard, 2008) for a detailed steady state analysis]. In this case, the observability matrices for pairs (A, C_1), (A, C_2) and (A, C) have ranks 4, 2 and 4, respectively, i.e. using S_2 it is possible to estimate the four states, while using S_1 it is just possible to estimate the acidogenic part of the system. This is obvious since system (10) has a cascade dynamic form between acidogenic and methanogenic dynamics.

Considering a sampling time (δ) equal to 1 and parameters listed in Tables 1 and 2, it is easy to verify that A, B and A_d take the values

$$A = \begin{pmatrix} 0 & 0.7900 & 0 & 0 \\ -1.8756 & -5.7825 & 0 & 0 \\ 0 & 0 & 0 & 0.0015 \\ 2.2166 & 6.1622 & -173.6936 & -1.4701 \end{pmatrix},$$

$$B = \begin{pmatrix} -2.7976 \\ 18.4640 \\ -0.1413 \\ 90.0000 \end{pmatrix},$$

$$A_d = \begin{pmatrix} 0.8033 & 0.1145 & 0 & 0 \\ -0.2719 & -0.0349 & 0 & 0 \\ 0.0003 & 0.0008 & 0.9186 & 0.0007 \\ 0.1556 & 0.3069 & -87.2825 & 0.1798 \end{pmatrix},$$

Using LQR techniques to calculate observer gains, observer (7) is designed in order to fulfill theorem 4, obtaining

Table 3. Initial conditions for simulations runs.

State variable	$X_1(0)$ $\mathrm{kg/m^3}$	$S_1(0)$ $\mathrm{kg/m^3}$	$X_2(0)$ $\mathrm{kg/m^3}$	$S_2(0)$ $\mathrm{mol/m^3}$
Plant	1.433	0.1	0.2	0.4
Observer	0.5	0.3	0.1	1

$$G_1 = \begin{pmatrix} -0.0005 \\ 0.0115 \\ -0.9985 \\ 17.2429 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0.0007 & 0.3923 \\ -0.0002 & -0.1330 \\ 0 & 0.0003 \\ 0 & -0.0002 \end{pmatrix}.$$

5.1 Simulation Results

In order to illustrate the performance of observer (7), some numerical simulations were carried out. Initial conditions and input concentrations for these simulations are listed in Table 3 and 2, while dilution rate was considered as a time varying sinusoidal signal around the nominal value. To verify if the incorporation of the discrete measurement to the continuous observer reduces convergence time, hybrid observer (7) was compared with a continuous observer identical to (7a) without the use of (7b) (or equivalently, for this observer G_2 was settled equal to zero). Figure 1 shows the dynamic behavior of hybrid observer, the discrete actualization is clearly visible in Figure 1a where at time $t = 1\,\mathrm{d}$ there is a jump on the acidogenic biomass estimation. As can be seen, observer states converges after approximately tree days. In contrast, the continuous observer (see Figure 2) converges in approximately twelve days, i.e. four times slower than the hybrid observer. Comparing both observers it easy to see that the hybrid observer obtained a faster convergence rate.

6. CONCLUSIONS

An nonlinear observer which updates the states using continuous and discrete measurements was presented. Despite this is a local observer, since observer gain matrices were calculated using the linear approximation of the original nonlinear system, its application to an anaerobic digestion model presents an excellent performance and stability, obtaining an improvement in convergence rate in comparison with an observer which only uses the continuous information. As future work, the authors are considering to extend this theory to the case where there exists parametric variations in the original plant, as well as the use of these observers to the control of systems with continuous and discrete measurements.

(Chapter head:)*

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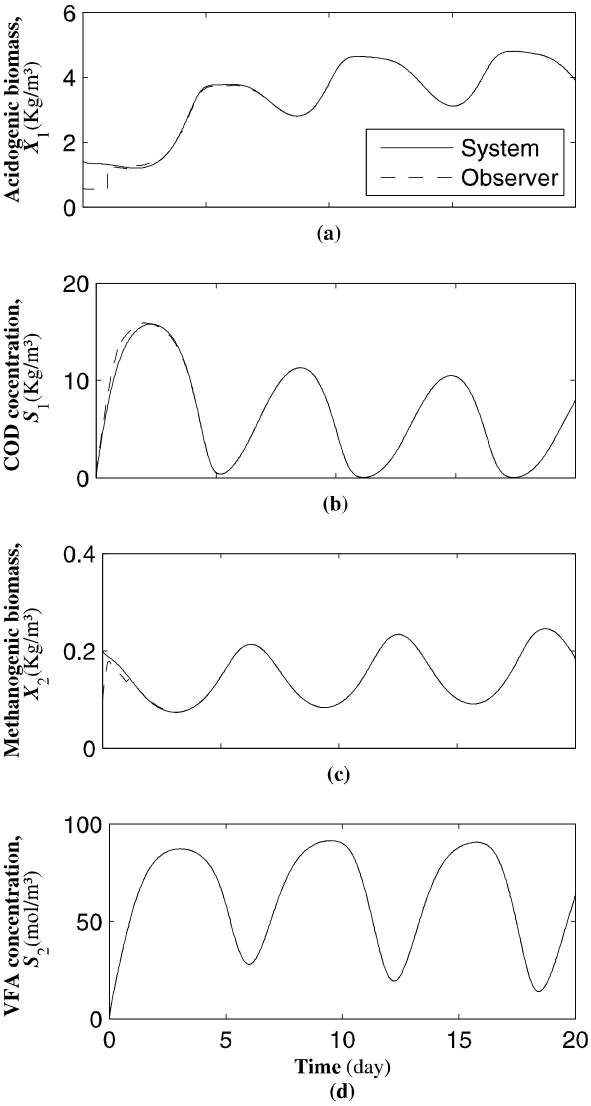

Fig. 1. Hybrid observer simulation.

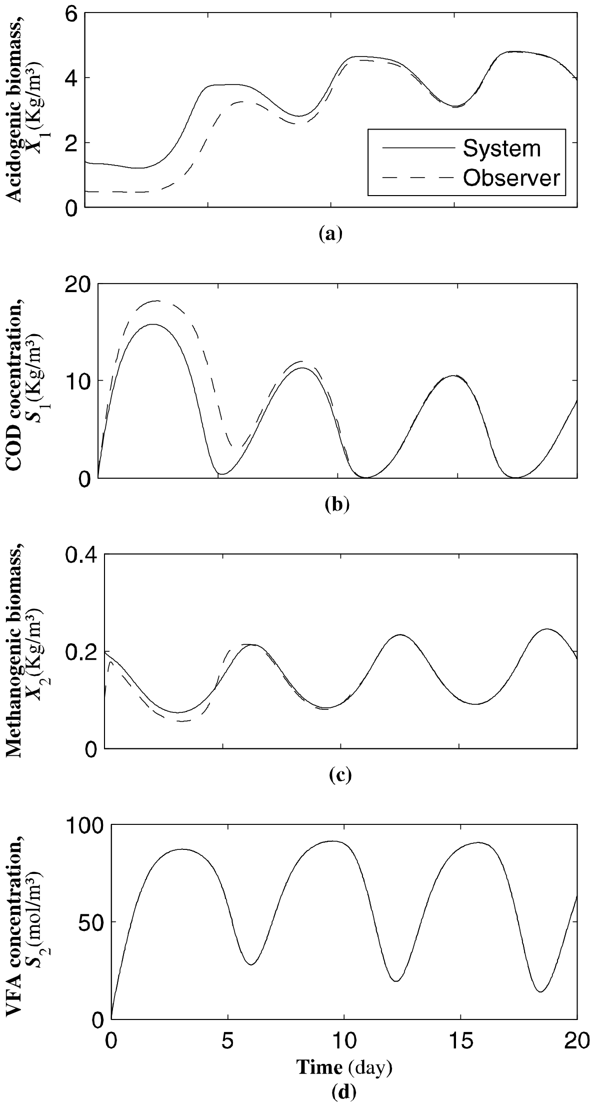

Fig. 2. Continuous observer simulation.

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Appendix A. APPENDIX

Proof. [Lemma 1] Let us define

$$\xi(t) = x(t) - z(t), \quad \text{and } \xi\left(k\delta^+\right) = x(k\delta) - z\left(k\delta^+\right),$$

where $\xi(t)$ represents the continuous error and $\xi\left(k\delta^+\right)$ is the updated error for each sampling period. Note that $x\left(k\delta^+\right) = x(k\delta)$ since system (1) is continuous. Now

$$\dot{\xi}(t) = A\xi(t) \qquad \forall t \neq k\delta \tag{A.1}$$

$$\xi\left(k\delta^+\right) = (I + GC)\,\xi(k\delta) \qquad t = k\delta. \tag{A.2}$$

Solving (A.1) for $t \in \left[k\delta^+, (k+1)\,\delta\right]$, it follows that

$$\xi(k+1) = A_d \xi\left(k\delta^+\right), \tag{A.3}$$

where $A_d = e^{A\delta}$. From (A.2) and (A.3) it is obtained

$$\begin{aligned} \xi\left((k+1)\,\delta^+\right) &= (I + GC)\,\xi(k+1) \\ &= (I + GC)\,A_d\xi\left(k\delta^+\right), \end{aligned}$$

and thus, if the pair (A_d, CA_d) is observable, then a matrix G can be calculated such that $A_d + GCA_d$ is Schur and the error $\xi\left(k\delta^+\right)$ will converge to zero, hence $\lim_{k\to\infty}\left[x(k\delta) - z\left(k\delta^+\right)\right] = 0$; then for $k\delta < t \leq (k+1)\,\delta$ the solution $z(t)$ converges to $x(t)$, that is $\lim\limits_{t\to\infty}\left[x(t) - z(t)\right] = 0$. On the other hand, to prove that the pair (A_d, CA_d) is observable if the pair (A_d, C) is observable, consider its observability matrix

$$\mathcal{O} = \begin{pmatrix} CA_d \\ CA_d^2 \\ \vdots \\ CA_d^n \end{pmatrix},$$

where $A_d \in \mathbb{R}^{n\times n}$, then using the Hamilton-Cailey theorem (Kailath, 1980)

$$A_d^n = a_0 I + a_1 A_d + \cdots + a_{n-1} A_d^{n-1},$$

the observability matrix becomes

$$\mathcal{O} = \begin{pmatrix} CA_d \\ CA_d^2 \\ \vdots \\ a_0 C + a_1 CA_d + \cdots + a_{n-1} CA_d^{n-1} \end{pmatrix}.$$

Since A_d is obtained through a discretization of matrix A then $a_0 \neq 0$ and $\mathcal{O}$ has full rank if the pair (A_d, C) is observable.