Feedforward for stabilization

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Abstract: This paper demonstrates how feedforward control can assist in stabilizing unstable systems. Feedback control is necessary for stabilization, but feedforward can be used to avoid input constraints which would otherwise cause the system to go unstable. Thus, if disturbances can be measured, feedforward from disturbances can be a simple and low cost way of avoiding loss of stability due to input constraints.

Keywords: Feedforward, input constraints, stabilization

1. INTRODUCTION

Fundamental limitations in achievable control performance have received a lot of attention in the control literature. A number of important results in this area is covered in Skogestad and Postlethwaite (2005). One such fundamental limitation for unstable systems is that the range of actuation for the inputs must be sufficiently large to avoid saturation. If the inputs saturate, feedback is broken, and hence the stabilizing effect of the controller is lost. Ensuring that the inputs do not saturate is therefore important in order to guarantee closed loop stability, although an unstable system may remain stable despite the inputs being saturated for a limited period, as shown in Favez et al. (2006). If input saturation is avoided, local (linear) stability of the closed loop system is sufficient for stability.

Feedforward is normally used to improve control performance at high frequencies, beyond the achievable bandwidth for stable closed loop control. In this paper, feedforward is instead used to reduce the magnitude of the plant input moves, and therefore to avoid instability due to input constraints.

2. BACKGROUND

Consider a controlled system such as the one illustrated in Fig. 1. For the linear, unconstrained case with only feedback control ($K_f = 0$), we get

$$u = KSr - KSG_d d \tag{1}$$

where $S = (I + GK)^{-1}$. The dependence on the Laplace variable s is suppressed for notational convenience, whenever it is not needed for clarity.

Glover (1986) has shown that for unstable systems, the minimal achievable H_∞ norm of KS is given by

$$\|KS\|_\infty \geq 1/\underline{\sigma}_H(\mathcal{U}(G)^*) \tag{2}$$

where $\underline{\sigma}_H$ denotes the smallest Hankel singular value, and $\mathcal{U}(G)^*)$ denotes the anti-stable part of the plant G, with its unstable pole(s) mirrored into the left half plane.

Observe that for relationships like (2) to have any relevance for evaluating the likelihood of input saturation - with subsequent loss of stabilizing feedback - the plant model G needs to be appropriately scaled. Skogestad and Postlethwaite (2005) recommend scaling plant inputs such that $|u| < 1$ corresponds to inputs within the range of actuation, and scaling outputs such that $|y| < 1$ means that the control offset is acceptable. Similarly, the inputs of the disturbance model G_d should be scaled to get $|d| < 1$ for the expected range of disturbances, and outputs scaled in the same way as for G. In scaled variables, the references are then scaled to give $|r| < R(\omega)$ for the expected range of reference changes. Such scaling is implicitly assumed throughout this paper, and consequently the input saturation limits are assumed to be at ± 1.

Thus, with variables appropriately scaled, sinusoidal reference changes will not cause input saturation provided

$$\|KS\|_\infty < 1/R(\omega)\forall\omega \tag{3}$$

Although reference signals may contain more than a single frequency, and input saturation due to reference changes may therefore occur even if this relationship is fulfilled, this relationship is nevertheless useful in assessing whether

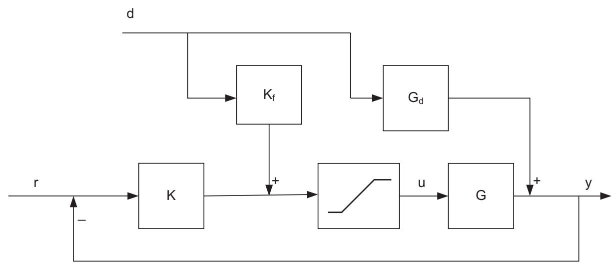

Fig. 1. Feedback and feedforward control, with limited input actuation range.

input saturation is a problem. However, it is also clear from the above that input saturation due to reference changes is not a fundamental problem - one may simply reduce the magnitude of the reference changes to avoid saturation.

On the other hand, it is typically not possible to control the magnitude of external disturbances d. Karivala et al. (2005) extended Glover's result to find that when using feedback only

$$\|KSG_d\|_\infty \geq 1/\underline{\sigma}_H(\mathcal{U}(G_{d,ms}^{-1}G)^*) \tag{4}$$

where $G_{d,ms}$ is the minimum phase and stable version of G_d, i.e., with both RHP poles and RHP zeros mirrored into the left half plane.

Accounting for the feedforward term K_f (but still assuming the saturation element to be inactive), we get

$$u = KSr + S_I(K_f - KG_d)d \tag{5}$$

where $S_I = (I + KG)^{-1}$. Note that $S = S_I$ for SISO systems, but this need not be the case for multivariable systems. From (5) we observe that introducing feedforward gives a new degree of freedom for minimizing input usage in the face of disturbances. Below, we will investigate in what situations this allows for a significant reduction of input usage, thus enabling closed loop stability.

3. STABLE DISTURBANCE MODELS

Consider the case where the plant is unstable from input to output, and hence requires feedback control for stabilization, but the unstable mode is not excited by the disturbances. This is motivated by the following example from Skogestad and Postlethwaite (2005):

Example:

$$G(s) = \frac{5}{(10s+1)(s-1)} \tag{6}$$

$$G_d(s) = \frac{k_d}{(s+1)(0.2s+1)} \tag{7}$$

The transfer functions are assumed to be appropriately scaled, as described above. From (4), we find that for $k_d > 0.54$, $\|KSG_d\|_\infty \geq 1$ for any feedback controller, and hence sinusoidal disturbances can drive the inputs to saturation. This is further illustrated in Skogestad and Postlethwaite (2005), where a stabilizing feedback controller is designed, but where saturation occurs for a step disturbance of magnitude 1 with $k_d = 0.5$. We seem to be in the paradoxical situation where control is not needed to counter the effect of disturbances (since a control offset of 1 is acceptable), but the controller needed to stabilize the system saturates due to the presence of the disturbance. Clearly, it would be better to do nothing to counteract the disturbance, but only manipulate the input to provide stabilization. However, a standard feedback controller does not distinguish the control offset caused by the (stable) disturbance from the offset caused by the unstable mode.

Equation (4) does not distinguish between stable and unstable disturbance models. For stable disturbance models, the feedforward controller K_f can be used to counter the effect of the disturbance on the input. That is, in stead of the conventional (ideal) feedforward

$$K_f = -G^{-1}G_d \tag{8}$$

which cancels the effect of the disturbance on the output [1], the ideal feedforward can from (5) be seen to be

$$K_f = KG_d \tag{9}$$

which cancels the effect of the disturbance on the input.

With this in mind, we revisit the example above, for the case with $k_d = 1$, meaning that feedback alone will not be able to maintain stability in the face of disturbances. The controller

$$K(s) = \frac{(10s+1)^2}{s(0.01s+1)} \tag{10}$$

will stabilize the unconstrained system. However, in Fig 2 we see that a unit step in the disturbance (applied at time $t = 1s$) will drive the input to saturation. Figure 3 shows that the system goes unstable as a result of the saturation. This is exactly as expected. The feedback controller $K(s)$ in (10) contains an integrator, and hence direct application of the ideal feedforward in (9) will mean that K_f will contain an integrator that is not stabilized by feedback. To avoid this problem, the controller is implemented as illustrated in Fig. 4, with the integrator in the block K_2. The corresponding feedforward is $K_f = K_1G_d$, with the overall feedback controller given by $K = K_2K_1$.

With this slight modification, we obtain the results in Figs. 5 and 6. The solid lines represents the 'ideal' feedforward control according to (9), whereas the dash-dot line is conventional feedforward according to (8). Clearly, the conventional feedforward does not avoid the input saturation. On the other hand, the modified feedforward according to (9) simply does nothing to counter the effect of the disturbance. Even though the control offset is acceptable according to the scaling used, most people would probably prefer the responses represented by the dashed line. This is obtained by augmenting the feedforward in (9) with a high pass filter, and results in offset-free control at steady state. Clearly, the pass band of the high pass filter should include frequencies significantly lower than that corresponding to the RHP pole(s).

4. UNSTABLE DISTURBANCE MODELS

It was shown above that it is simple to use feedforward from the disturbance to avoid input saturation and hence loss of stabilizing feedback, when the plant is unstable but the disturbance transfer function is stable. If the disturbance transfer function is unstable, the issue becomes more complicated.

Note, that for stabilization of the unstable disturbance transfer function to make sense, the unstable mode(s) must also be a part of the plant transfer function. That is, it must be possible to reformulate the plant and disturbance transfer functions as indicated in Fig. 7, with $G_3(s)$ a

[1] Neglecting for the moment the effects of possibly unknown initial conditions for the disturbance dynamics.

stable transfer function. In this case, the disturbance will obviously excite the unstable mode, and it therefore does not make sense to avoid the use of the manipulated input when a disturbance occurs.

Furthermore, the direct application of the 'ideal' feedforward in (9) would mean using an unstable feedforward element K_f, which would lead to an internally unstable control system. Instead, we would like to find the *stable* feedforward element K_f which minimizes the term $(K_f - KG_d)$ in (5). The term KG_d can be split into a stable and an anti-stable part. The stable part can be used directly in K_f, whereas we need a *stable approximation to the anti-stable part of* KG_d.

Approximation of an anti-stable transfer function by a stable transfer function (or *vice versa*) is known as a Nehari extension problem. That is, we want to find the optimal stable $Q(s)$ such that $\|Q(s)+R(s)\|_\infty$ is minimized, where $R(s)$ is ant-stable. A solution to this problem can be found in Glover (1984). In Glover (1984), it is also shown that the optimal error is given by $\|R^*\|_H$, where $\|\cdot\|_H$ denotes the Hankel norm, and R^* is the 'stable version of R', with the unstable poles mirrored into the left half plane. Thus, we would like to design a feedback controller K that not only stabilizes the plant, but also makes the Hankel norm of the unstable part of KG_d small. However, with

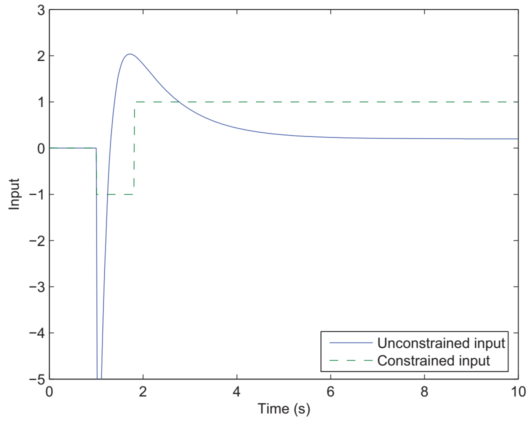

Fig. 2. Response in the input to a unit step in the disturbance as time $t = 1s$, using only feedback control.

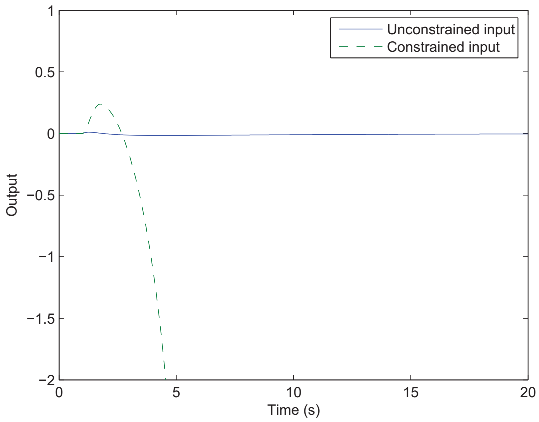

Fig. 3. Response in the output to a unit step in the disturbance as time $t = 1s$, using only feedback control.

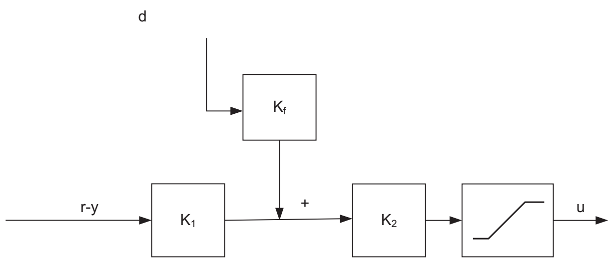

Fig. 4. Implementation of overall feedback/feedforward controller, with the integrator in the block K_2.

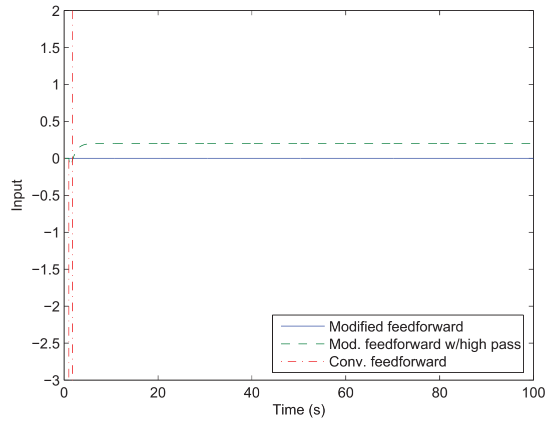

Fig. 5. Response in the input to a unit step in the disturbance as time $t = 1s$, using combined feedback and feedforward control.

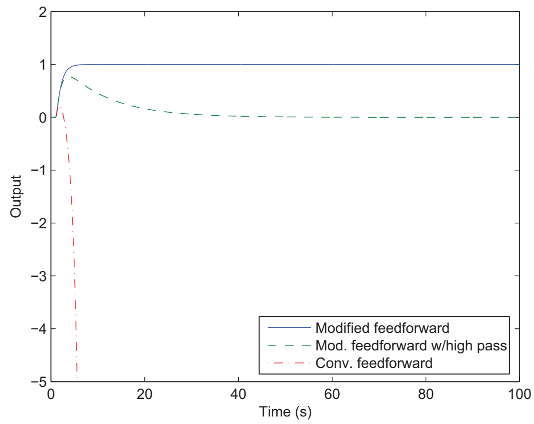

Fig. 6. Response in the output to a unit step in the disturbance as time $t = 1s$, using combined feedback and feedforward control.

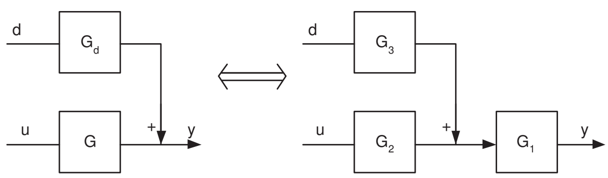

Fig. 7. Reformulation of plant and disturbance transfer functions.

a simple reformulation of the feedforward arrangement, this complication is easily avoided. This rearrangement is illustrated in Fig. 8, and may be regarded more as a 'reference governor' approach than feedforward in the ordinary sense.

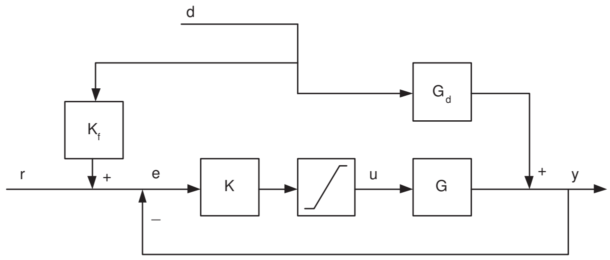

Fig. 8. 'Feedforward' arrangement for an unstable disturbance transfer function.

With the rearranged 'feedforward', the transfer function from the disturbance d to e, the input to the feedback controller K (assuming the saturation element is inactive), is given by

$$e = S(K_f - G_d)d \tag{11}$$

where $S = (I+GK)^{-1}$. Thus, for a given controller K, the controller input (and therefore also the controller output) will be small if the term $(K_f - G_d)$ is small. Next, the validity of the bound (4) and the design of the feedforward controller will be illustrated for two different cases:

- A disturbance transfer function G_d whose only non-minimum phase term is an unstable pole.
- A disturbance transfer function G_d with non-minimum phase terms in addition to the unstable pole.

The benefit of feedforward will be found to be different in these two cases. However, first the H_∞ problem formulation will be briefly explained. For the case with a stable disturbance transfer function, this was not needed, since the design of the feedforward controller was done separately from the design of the feedback controller.

4.1 H_∞ problem formulation

Using feedforward in combination with feedback means that we are using a controller with two degrees of freedom (2-DOF controller)). We wish to investigate the benefit of using a 2-DOF controller for stabilizing the system while minimizing the use of inputs in the face of measured disturbances. However, the resulting H_∞ control synthesis problem violates the standard assumptions. Assumptions A2 and A4 of Zhou et al. (1996), p. 450 are violated.

A small measurement noise n is therefore added, and the magnitude of that measurement noise is reduced until further reduction does not significantly affect the H_∞ norm achieved. The block diagram corresponding to the resulting controller synthesis problem is shown in 9.

4.2 The unstable pole as the only non-minimum phase term in G_d

The same plant transfer function as in (6) is used, whereas the disturbance transfer function is modified to

$$G_d(s) = \frac{k_d}{(s-1)(0.2s+1)} \tag{12}$$

The parameter value $k_d = 1$ is still used. First, a realization of $[G_d \quad G]$ with only one unstable mode is found. Then a 2-DOF controller is designed according to Fig. 9, and compared to a 1-DOF controller (feedback only) designed to minimize KSG_d. For both cases, a H_∞ norm of 1.83 is achieved. This also corresponds to the bound in (4). In this case, there is thus no advantage derived from using a 2-DOF controller with feedforward from disturbances [2].

However, if the feedback controller is designed for some other criterion than minimising $\|KSG_d\|_\infty$, there may be a possible advantage in designing the feedforward using the idea of approximating G_d with a stable transfer function. To illustrate, we first design a feedback controller for minimizing $\|KS\|_\infty$, achieving a H_∞ norm of 4.40 - which agrees with the bound in (2). Using this controller, we would also get $\|KSG_d\|_\infty = 4.40$. Instead, we augment the controller with feedforward as illustrated in Fig. 8. The transfer function G_d can be split into stable and unstable parts, giving

$$G_{d,stable} = \frac{-5k_d}{6(s+5)}$$

$$G_{d,unstable} = \frac{5k_d}{6(s-1)}$$

The task is this to find a stable approximation to $G_{d,unstable}$. The formulae in Glover (1984) for doing so are not directly applicable, since $G_{d,unstable}$ has only one state. However, it is easily verified that a stable approximation which achieves the minimum bound on the approximation error is given by

$$\tilde{G}_{d,unstable} = -\frac{5k_d}{12} \tag{13}$$

With the feedforward $K_f = G_{d,stable} + \tilde{G}_{d,unstable}$ used as illustrated in Fig. 8, and the feedback controller K which minimizes $\|KS\|_\infty$, we achieve an H_∞ norm of 1.83 from disturbance d to input u, while maintaining closed loop stability.

4.3 G_d with non-minimum phase terms in addition to the unstable pole

Consider next the case when the unstable disturbance transfer function is augmented with an all-pass term, giving

[2] And, in order to achieve $|u| < 1$ we would need $k_d < 0.54$, as in the original example in Skogestad and Postlethwaite (2005).

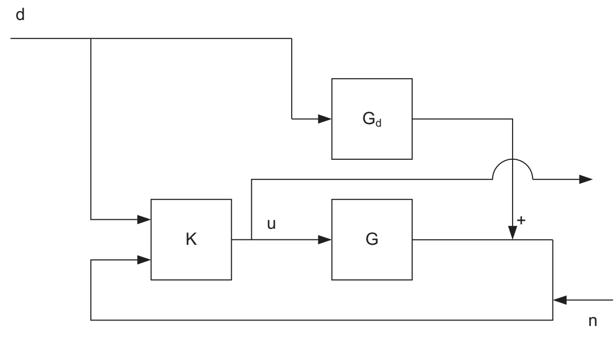

Fig. 9. H_∞ controller synthesis setup for 2-DOF controller.

$$G_d(s) = \frac{k_d(-10s+1)}{(s-1)(0.2s+1)(10s+1)} \tag{14}$$

The all-pass part of G_d cancels in the calculation of the bound in (4), and thus the minimum that can be achieved with feedback control alone is still $\|KSG_d\|_\infty = 1.83$. On the other hand, with a 2-DOF controller we achieve an H_∞ norm from d to u of 1.50. The same is achieved when designing a 1-DOF H_∞ controller minimizing $\|KS\|_\infty$ and subsequently adding feedforward to this controller in a manner similar to the preceding section.

Looking at the unstable part of $G_d(s)$ in (14), the reason for the improvement in input usage when adding the feedforward becomes apparent. One now finds that

$$G_{d,unstable} = \frac{-5k_d}{6(s-1)} \frac{9}{11}$$

The reduction in the unstable part of the disturbance transfer function by the factor 9/11 is a direct result of the all-pass term $(-10s+1)/(10s+1)$, since it modifies the residue at $s = 1$ by that same factor in the partial factor expansion of $G_d(s)$. Note that the improvement in H_∞ norm due to the introduction of feedforward also corresponds to the factor 9/11.

Stable all-pass terms will always reduce all residues in the RHP, and hence always reduce the size of the anti-stable part of $G_d(s)$ if there is a single unstable pole or a single pair of unstable complex conjugate poles. This covers a large fraction of the unstable system dynamics met in engineering practice. However, in general the unstable part of $G_d(s)$ may consist of several terms. The effect of all-pass terms will be different for the different unstable terms in $G_d(s)$, and it may therefore be possible for the unstable part of $G_d(s)$ to increase due to the presence of all-pass terms in the disturbance transfer function.

5. CONCLUSIONS

This paper illustrates how feedforward may be applied to reduce the input usage necessary for stabilizing unstable plants. If the disturbance transfer function is stable, one can thus easily remove the problem that disturbances may cause input saturation - with resulting loss of stabilizing feedback.

For the case of an unstable disturbance transfer function, it is clearly necessary to assume that the unstable mode is shared with the plant transfer function - otherwise it cannot be stabilized by feedback around the plant.

If the unstable pole is the only non-minimum phase term in the disturbance transfer function, feedforward has not been shown to improve on the optimal H_∞ norm achievable by feedback only. The bound on the H_∞ norm from d to u was found to apply for both 1-DOF and 2-DOF controllers in the example studied. However, if the 1-DOF controller is designed according to some other criterion than that of minimizing $\|KSG_d\|_\infty$, feedforward may be used to reduce the usage of inputs.

It is also found in an example that if the disturbance transfer function includes other non-minimum phase terms than the unstable pole, a 2-DOF controller can improve upon the optimal H_∞ norm achievable with feedback only. Feedforward may also in this case be added to a previously designed 1-DOF controller to reduce the usage of inputs.

Further work is necessary to quantify the optimal H_∞ norm from disturbance to plant input that is achievable when using a 2-DOF controller. Also, to simplify the analysis, the factor S has been ignored in (11), focusing instead on keeping $(K_f - G_d)$ small. Accounting for the factor S would lead to a *frequency weighted* Nehari extension problem. The possible benefit in accounting for this frequency weighting is not clear. In the examples studied, the optimal H_∞ norm for the 2-DOF problem has been achieved by appending feedforward (designed without accounting for the frequency weighting) to a 1-DOF controller design.

For the practising engineer, this paper points to the use of feedforward from disturbances to reduce input usage for stabilization. This may be an attractive alternative compared to alternative plant modifications in order to avoid input saturation (leading to loss of stabilizing feedback) in the face of disturbances.

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