Robust Nonlinear Model Predictive Control using Volterra Models and the Structured Singular Value (μ)

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Abstract: A methodology is proposed for designing robust nonlinear model predictive controllers based on a Volterra series model with uncertain coefficients. The objective function of the on-line optimization is formulated in terms of a Structured Singular Value (μ). The proposed formulation considers a penalty on the manipulated variables actions and manipulated variables and terminal condition constraints.

Keywords: Model predictive control, nonlinear control, robust control, structured singular value.

1. INTRODUCTION

Linear model predictive control is a widely accepted control strategy in the chemical industry. Many theoretical studies and industrial applications of linear MPC have been reported elsewhere (Qin and Badgwell, 2003). On the other hand, the nonlinear behaviour of chemical processes has motivated researchers and practitioners to consider predictive control strategies based on nonlinear process models referred to as nonlinear model predictive controllers (NMPC) (Findeisen and Allgöwer, 2002).

Some of the challenging requirements related to the industrial implementation of NMPC are: (1) a reliable nonlinear model of the process is needed that can be effectively used for real time control and (2) ensuring robustness to model error.

Both first principles as well as empirical input-output models have been used in the past for nonlinear predictive control strategies. Although first principles models have the advantage of formally satisfying basic energy and mass balances of the process, they are often too complex for real time control and their structure is generally not amenable for formal robust analysis and design. NMPC strategies based on empirical models such as Hammerstein and Volterra series (Hérnandez and Arkun, 1993; Maner et al., 1996; Parker and Doyle III, 2001) have been reported but their robustness with respect to model error have not been thoroughly studied.

The need to address robustness arises from the fact that the models used for predictive control are never exact. Although a good amount of research work has been conducted on robustness of linear predictive controllers, the robustness of nonlinear predictive controllers has not been extensively studied. The lack of robustness guarantees is currently perceived as one of the key obstacles for wide industrial acceptance of NMPC strategies (Nagy and Braatz, 2003a).

The current work investigates the design of a robust NMPC algorithm based on an empirical Volterra series model. Volterra series models have been shown to efficiently describe general nonlinear systems (Schetzen, 1980). A key idea for this study is that based on the Volterra model it is possible to formulate the robust predictive control problem as a μ-Structured Singular Value test that can be used on-line to calculate optimal control actions. The μ (Structured Singular Value, Doyle, 1982) norm is used, at each sampling instant, to calculate a bound on the norm of a vector containing both output and input predictions along a predefined prediction time horizon in the presence of disturbances and uncertainty in the Volterra model coefficients. Then, the calculated bound is minimized with respect to the optimal control actions to be sent to the process.

The paper is organized as follows. In section 2 the formulation of the μ test and the optimization problem based on the calculated bound is presented. In the same section terminal conditions to enforce stability as well as conditions to enforce manipulated variable constraints are also presented. Section 3 presents two case studies and conclusions are presented in Section 4. Mathematical details are presented in Appendix A.

2. METHODOLOGY

2.1 Model Predictive Control

MPC minimizes a cost function that considers the future errors whit respect to the manipulated variables. For simplicity of notation a single input single output (SISO) case is shown but the formulation can be easily extended to the multivariable case. Considering that y^{pr} is the predicted value of the controlled variable and y^{sp} is the controlled variable set point, a vector of predictions can be written as follows:

$$\mathbf{Y} = \begin{bmatrix} y^{pr}(k_0) + d(k_0) - y^{sp}(k_0) \\ \vdots \\ y^{pr}(k_0+p) + d(k_0+p) - y^{sp}(k_0+p) \end{bmatrix} \quad (1)$$

where k_0 is the initial sampling instant, p is the prediction horizon and d is a measured disturbance. The objective function of the controller proposed in this work minimizes the maximum absolute value of each element of the **Y** vector with respect to the manipulated variables u as follows:

$$\min_{\text{wrt } u(k_0),\ldots,u(k_0+m)} \|\mathbf{Y}\|_\infty \tag{2}$$

where m is the control horizon. In principle norms other than the infinity norm of the output may be considered in the formulation but are beyond the scope of the current study. It will be also shown in subsections 2.4 to 2.6 that the vector **Y** in (2) may be augmented by additional variables, other than predicted outputs, to enforce a terminal condition and manipulated variables constraints. The following subsection discusses the Volterra models used to calculate the prediction **Y**.

2.2 Volterra series

The general structure of a Volterra series model is given as follows:

$$y^{pr}(k_0) = \sum_{\sigma_1=1}^{\infty}\sum_{\sigma_i=1}^{\infty} h_i(\sigma_1,\ldots,\sigma_i)u(k_0-\sigma_1)u(k_0-\sigma_i) \tag{3}$$

where u is the manipulated variable, y is the controlled variable and h_i are the coefficients of the Volterra series. For practical purposes the series is truncated and the resulting expression is

$$y^{pr}(k_0) = h_0 + \sum_{n=1}^{N}\sum_{i_1=1}^{M}\cdots\sum_{i_n=1}^{M}\left(h_n(i_1,\ldots,i_n)\times u(k_0-i_i)u(k_0-i_n)\right) \tag{4}$$

without loss of generality it can be consider that h_0=0. For example, for N=2, the value of the controlled variable is

$$y^{pr}(k_0) = \sum_{n=1}^{M} h_n^L u(k_0-n) + \sum_{i=1}^{M}\sum_{j=i}^{M} h_{i,j}^{NL} u(k_0-i)u(k_0-j) \tag{5}$$

where M is the memory of the system. The linear h_n^L, and nonlinear $h_{i,j}^{NL}$, Volterra series coefficients can be obtained by least squares regression using process input-output data by imposing an appropriate input sequence. For a system with polynomial degree N, it has been shown that is necessary to use a N+1 level pseudorandom multilevel sequence (Nowak and Van Veen, 1994). Confidence intervals for the coefficients, to be used in the calculations as uncertainty bounds associated to these coefficients, can be obtained using least squares regression.

2.3 Calculation of the worst predicted output

The worst predicted output calculation can be performed by a Structured Singular Value (SSV) test (Nagy and Braatz, 2003b). The main motivation to use the SSV test is that it allows finding the worst $\|\mathbf{Y}\|_\infty$ when uncertainty in the Volterra coefficients is considered. Accordingly, (5) can be modified to include parameter uncertainty as follows where h_n^L and $h_{i,j}^{NL}$ are the nominal value of the coefficients and δh_n^L and $\delta h_{i,j}^{NL}$ are the uncertainty associated to the coefficients:

$$y^{pr}(k_0) = \sum_{n=1}^{M}\left[h_n^L + \delta h_n^L\right]u(k_0-n) + \sum_{i=1}^{M}\sum_{j=i}^{M}\left[h_{i,j}^{NL} + \delta h_{i,j}^{NL}\right]u(k_0-i)u(k_0-j) + w(k_0) \tag{6}$$

w is a feedback term that considers the current difference between the actual process output and the predicted output:

$$w(k_0) = y^{real}(k_0-1) - y^{pr}(k_0-1) \tag{7}$$

By selecting an appropriate interconnection matrix **M** and uncertainty block structure Δ, the worst value of a variable in the presence of model error can be calculated by the following SSV test (Braatz et al., 1994; Nagy and Braatz, 2003b)

$$\max_{\text{wrt } \delta h_i^L, \delta h_{i,j}^{NL}, w} \|\mathrm{Y}\|_\infty \geq k_{ssv} \Leftrightarrow \mu_\Delta(\mathrm{M}) \geq k_{ssv} \tag{8}$$

Thus, a bound on the worst deviation of $\|\mathbf{Y}\|_\infty$, i.e. the norm of the prediction vector can be obtained by the following skew μ problem:

$$\max_{\text{wrt } \delta h_i^L, \delta h_{i,j}^{L}, w} \|\mathbf{Y}\|_\infty = \max_{\text{wrt } k_{ssv}}(k_{ssv}) \quad \text{st } \mu_\Delta(\mathbf{M}) \geq k_{ssv} \tag{9}$$

A key idea in (9) is that the feedback term in (7) is also treated as an uncertainty and the maximization in (9) is carried out with respect to both this feedback error and the uncertainties in coefficients. Accordingly, the uncertainty block Δ is as follows:

$$\mathbf{\Delta} = \mathrm{diag}(\mathbf{\Delta}_1, \mathbf{\Delta}_2, \mathbf{\Delta}_3) \tag{10}$$

where Δ_3 is a complex scalar square matrix of dimensions p×p related to performance and Δ_1 and Δ_2 are real scalar square matrices related to the uncertainty in feedback and Volterra series coefficients respectively with the following dimensions:

$$\mathbf{\Delta}_1 = \left(p + 2\sum_{i=1}^{p} i + \sum_{i=1}^{p-1}\sum_{j=1}^{i} j\right)\times\left(p + 2\sum_{i=1}^{p} i + \sum_{i=1}^{p-1}\sum_{j=1}^{i} j\right) \tag{11}$$

$$\mathbf{\Delta}_2 = \mathrm{diag}\left[\mathbf{\Delta}_{2_1} \quad \ldots \quad \mathbf{\Delta}_{2_p}\right] \tag{12}$$

$$\mathbf{\Delta}_{2_i} = \left[p\times p \quad \ldots \quad 1\times 1\right]^{\mathrm{T}} \; i \leq 2 \tag{13}$$

$$\mathbf{\Delta}_{2_i} = \left[(p+2-i)\times(p+2-i) \quad \ldots \quad 1\times 1\right]^{\mathrm{T}} \; i > 2 \tag{14}$$

The problem stated in (8) and (9) can be used within the predictive control problem defined in (2) as follows:

$$\min_{u(k_0),\ldots,u(k_0+m)} \left[\max(k_{ssv})\right] \quad \text{wrt } k_{ssv} \quad \text{st } \mu_\Delta(\mathbf{M}) \geq k_{ssv} \tag{15}$$

The vector **Y** can be modified, as mentioned in section 2.1 to include additional terms as follows: (1) a penalty term to prevent an excessive movement of the manipulated variables, (2) manipulated variables to enforce constraints and (3) a terminal condition to ensure convergence. These terms are explained in the following subsections.

2.4 Manipulated variables movements penalization

Define:

$$\mathbf{y}^{\Delta u} = \begin{bmatrix} W_1^{\Delta u}\left[u(k_0) - u(k_0-1)\right] \\ \vdots \\ W_m^{\Delta u}\left[u(k_0+m) - u(k_0+m-1)\right] \end{bmatrix} \tag{16}$$

Redefining: $\mathbf{Y}=[\, \mathbf{y}^{pr} \;\; \mathbf{y}^{\Delta u}\,]^{\mathrm{T}}$ it is ensured by (9) that the elements of $\mathbf{y}^{\Delta u}$ satisfy $\max(y_i^{\Delta u}) \leq k_{ssv}$ for $i=1,\ldots,m$. Thus, the maximum weighted manipulated variable movement is bounded at each sampling instant by k_{ssv}.

2.5 Manipulated variables constraints

Define:

$$\mathbf{y}^{uc} = \left[k_{ssv}\frac{u(k_0)}{u_{\max}(k_0)} \quad \cdots \quad k_{ssv}\frac{u(k_0+m)}{u_{\max}(k_0+m)} \right]^{\mathrm{T}} \tag{17}$$

Redefining: $\mathbf{Y}=[\, \mathbf{y}^{pr} \;\; \mathbf{y}^{\Delta u} \;\; \mathbf{y}^{uc}\,]^{\mathrm{T}}$ it is ensured by (9): that the elements of $\mathbf{y}^{uc}$ satisfy $\max(k_{ssv}u(i)/u_{\max}(i)) \leq k_{ssv}$ for $i=k_0,\ldots,k_0+m$ which can be simplified to: $\max(u(i)) \leq u_{\max}(i)$ for $i=k_0,\ldots,k_0+m$. Thus, the manipulated variables are bounded at each sampling instant by $u_{\max}(i)$ for $i=k_0,\ldots,k_0+m$.

2.6 MPC terminal condition

A terminal condition is used to ensure that at steady state the predicted output stays within a neighborhood ε near the origin (Chen and Allgöwer, 1998). Although not shown here for brevity, it can be shown that the use of the terminal condition together with the manipulated variables constraints ensures stability providing that the terminal condition is feasible with respect to constraints. Define:

$$\mathbf{y}^{tc} = \frac{k_{ssv}}{\varepsilon}\left[u(k_0+m)\sum_{i=1}^{p}\left(h_i^L + \delta h_i^L\right)\right] + \frac{k_{ssv}}{\varepsilon}\left[u^2(k_0+m)\sum_{i=1}^{p}\sum_{j=i}^{p}\left(h_{i,j}^{NL} + \delta h_{i,j}^{NL}\right)\right] \tag{18}$$

ε can be selected by the user but a smaller value results in more conservative control. Redefining: $\mathbf{Y}=[\mathbf{y}^{pr} \; \mathbf{y}^{\Delta u} \; \mathbf{y}^{uc} \; \mathbf{y}^{tc}]^{\mathrm{T}}$ it is ensured by (9) that

$$\max\begin{bmatrix} \dfrac{k_{ssv}}{\varepsilon}\left(u(k_0+m)\displaystyle\sum_{i=1}^{p}\left(h_i^L + \delta h_i^L\right)\right) + \\ \dfrac{k_{ssv}}{\varepsilon}\left(u^2(k_0+m)\displaystyle\sum_{i=1}^{p}\sum_{j=i}^{p}\left(h_{i,j}^{NL} + \delta h_{i,j}^{NL}\right)\right) \end{bmatrix} \leq k_{ssv} \tag{19}$$

which can be simplified to:

$$\max\begin{bmatrix} u(k_0+m)\displaystyle\sum_{i=1}^{p}\left(h_i^L + \delta h_i^L\right) + \\ u^2(k_0+m)\displaystyle\sum_{i=1}^{p}\sum_{j=i}^{p}\left(h_{i,j}^{NL} + \delta h_{i,j}^{NL}\right) \end{bmatrix} \leq \varepsilon \tag{20}$$

Details on the construction of **M** for a Volterra series model NMPC strategy considering the additional terms of subsections 2.4, 2.5 and 2.6 can be found on Appendix A at the end of this paper.

3. CASE STUDIES

Different case studies are presented to show the more important features of the proposed algorithm. For simplicity a SISO case is presented where an approximated Volterra model describing the effect of coolant temperature on reactor concentration for a CSTR is as follows (Gao, 2004):

$$\begin{aligned} y(k_0) = \; & h_1^L u(k_0) + h_2^L u(k_0-1) + h_3^L u(k_0-2) + \\ & h_{1,1}^{NL} u^2(k_0) + h_{1,2}^{NL} u(k_0) u(k_0-1) + \\ & h_{1,3}^{NL} u(k_0) u(k_0-2) + h_{2,2}^{NL} u^2(k_0-1) + \\ & h_{2,3}^{NL} u(k_0-1) u(k_0-2) + h_{3,3}^{NL} u^2(k_0-2) \end{aligned} \tag{21}$$

$h_1^{\mathrm{L}} = 0.2835$, $h_2^{\mathrm{L}} = 0.1445$, $h_3^{\mathrm{L}} = 0.0594$, $h_{1,1}^{\mathrm{NL}} = -0.0072$, $h_{1,2}^{\mathrm{NL}} = -0.049$, $h_{1,3}^{\mathrm{NL}} = -0.0281$, $h_{2,2}^{\mathrm{NL}} = -0.0379$, $h_{2,3}^{\mathrm{NL}} = -0.017$, $h_{3,3}^{\mathrm{NL}} = -0.0081$. The MPC prediction and control horizons are p=3 and m=2.

The first study is intended to illustrate the possibility of tuning the proposed algorithm through the value of $W_1^{\Delta u}$, i.e. the weight used to penalize manipulated variables from sampling instant (k_0–1) to sampling instant (k_0). The response of the process to a pulse disturbance is studied and the set point is equal to zero. The disturbance is as follows: from $0<k_0\leq 20$ d=5, then from $20<k_0\leq 40$ d=0. For this case it is considered that there is no uncertainty in the Volterra model coefficients.

Figures 1 and 2 show that the weight imposed on the movement on the manipulated variables can be effectively used to tune the closed loop response. To illustrate the significance of the nonlinear terms, a simulation is carried on with a controller based solely on the linear part of the Volterra model. The results (dotted line in Figures 1 and 2) illustrate that the nonlinear model based controller provides as expected, a better and more consistent performance than the linear model based one.

To illustrate the constraint handling capabilities of the algorithm, the response to a pulse disturbance is studied. The disturbance is: for $0<k_0\leq 5$ d=5, for $5<k_0\leq 10$ d=50 and for

10< k_0 ≤15 d=5. The value of the manipulated variable is restricted to $|u(k_0)| \leq 5.5$ and $W_1^{\Delta u}$=2. For this case it was considered that the Volterra series coefficients are known accurately, i.e. there is no model uncertainty. It can be seen from Figure 3 that the controller keeps the value of the manipulated variable within the allowed limits.

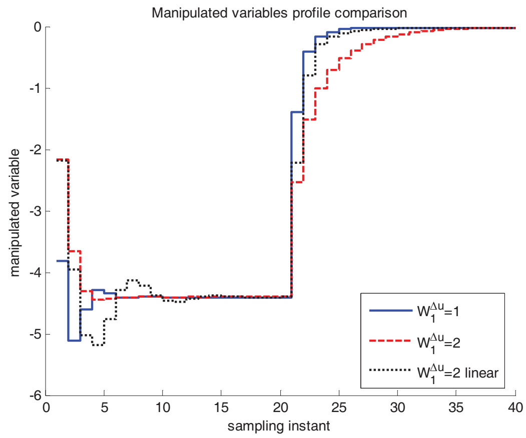

Fig. 1. Manipulated variable profile for case study 1.

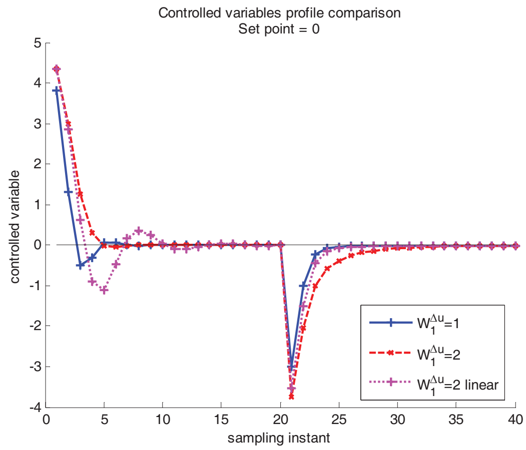

Fig. 2. Controlled variable profile for case study 1.

A key feature of the proposed MPC is that it allows considering that the Volterra series coefficients are not exactly known. To illustrate this feature of the algorithm it is assumed that certain coefficients are uncertain as follows: $h_1^{L} = 0.2551 \pm 0.0383$, $h_{2,2}^{NL} = -0.0360 \pm 0.0072$ and $h_{3,3}^{NL} = -0.0089 \pm 0.0018$. In this case the Volterra series coefficients of the plant are the same as those used for case study 1 and 2. Furthermore, the disturbance affecting the process is the same as that of case study 1. Figures 4 and 5 show the manipulated and controlled variable profile when $W_1^{\Delta u}$=2. The response obtained with the uncertain model MPC is more oscillatory but still acceptable. The figures show that the control variable converges to a value very close to zero and the manipulated variables are kept within limits. The small offset observed in the manipulated variable with respect to u=0 arises from the requirement of the terminal condition in the presence of model uncertainty.

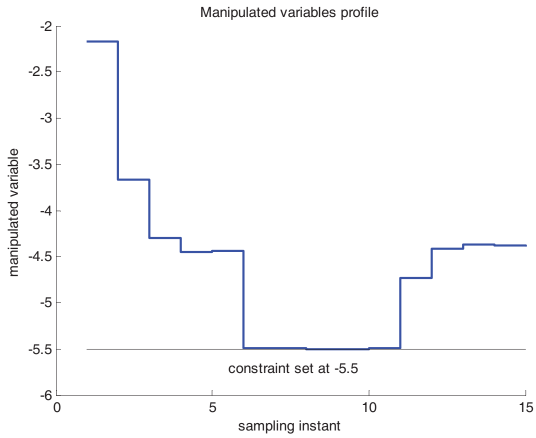

Fig. 3. Manipulated variable profile for case study 2.

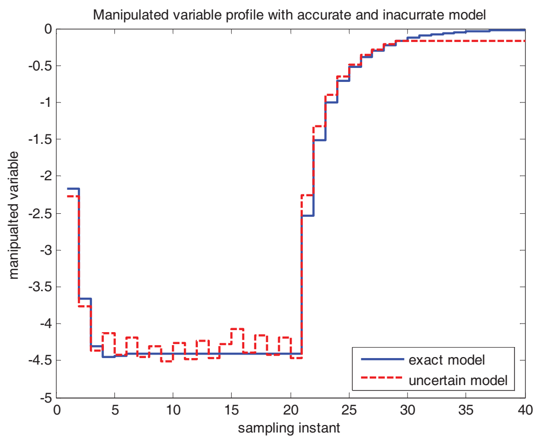

Fig. 4. Manipulated variable profile for case study 3.

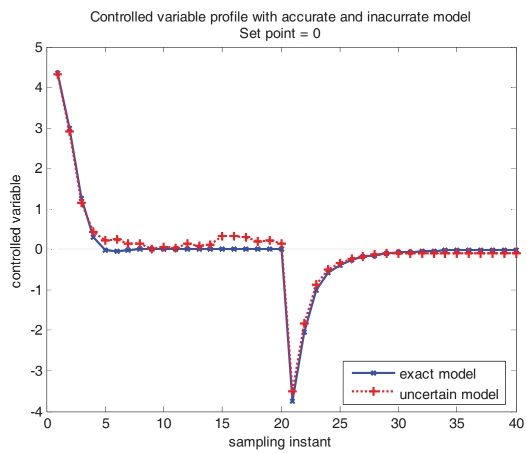

Fig. 5. Controlled variable profile for case study 3.

4. CONCLUSIONS

A novel robust NMPC controller based on a Volterra model was presented. The methodology uses μ analysis to calculate, for an uncertain plant model, the worst possible norm of a vector of inputs and outputs. The interconnection matrix can include terms to account for manipulated variables movement weighting, manipulated variables constraints and robust stability properties enforced through a terminal condition. The application of this technique to MIMO problems is currently being investigated.

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Appendix A. CONSTRUCTION OF THE MATRIX M

The use of an appropriate interconnection matrix **M** allows quantifying the effect that an input has on the system's output in the presence of uncertainty through a linear fractional transformation (LFT). If **M** is built according to the following structure

$$\mathbf{M} = \left[\begin{array}{c|c} \mathbf{M}_{11}^{\mathrm{LFT}} & \mathbf{M}_{12}^{\mathrm{LFT}} \\ \hline \mathbf{M}_{21}^{\mathrm{LFT}} & \mathbf{M}_{22}^{\mathrm{LFT}} \end{array}\right] \quad (22)$$

the effect that the input has on the output in the presence of uncertainty is:

$$\mathbf{Y}(k) = \left[\mathbf{M}_{21}^{\mathrm{LFT}} \boldsymbol{\Delta} \left[\mathbf{I} - \mathbf{M}_{11}^{\mathrm{LFT}} \boldsymbol{\Delta}\right]^{-1} \mathbf{M}_{12}^{\mathrm{LFT}} + \mathbf{M}_{22}^{\mathrm{LFT}}\right] \begin{bmatrix} \boldsymbol{w}(k) \\ 0 \end{bmatrix} \quad (23)$$

where $\boldsymbol{w}(k) = [w(k), \ldots, w(k)]^{\mathrm{T}}_{p}$. The interconnection matrix **M** that considers manipulated variables movement penalization, manipulated variables constraints and terminal condition has the following structure:

$$\mathbf{M} = \left[\begin{array}{ccc|c} \mathbf{M}_{11} & \mathbf{M}_{12} & \mathbf{M}_{13} & \mathbf{M}_{14} \\ \mathbf{M}_{21} & \mathbf{M}_{22} & \mathbf{M}_{23} & \mathbf{M}_{24} \\ \mathbf{M}_{31} & \mathbf{M}_{32} & \mathbf{M}_{33} & \mathbf{M}_{34} \\ \hline \mathbf{M}_{41} & \mathbf{M}_{42} & \mathbf{M}_{43} & \mathbf{M}_{44} \end{array}\right] \quad (24)$$

In (24) $\mathbf{M}_{11}$, $\mathbf{M}_{12}$, $\mathbf{M}_{13}$, $\mathbf{M}_{21}$, $\mathbf{M}_{22}$, $\mathbf{M}_{23}$, $\mathbf{M}_{31}$, $\mathbf{M}_{33}$, $\mathbf{M}_{34}$ and $\mathbf{M}_{44}$ are matrices of appropriate dimensions that have all its elements equal to zero. $\mathbf{M}_{14}$, $\mathbf{M}_{24}$, $\mathbf{M}_{32}$, $\mathbf{M}_{41}$, $\mathbf{M}_{42}$ and $\mathbf{M}_{43}$ are defined as:

$$\mathbf{M}_{14} = \mathrm{diag}\begin{bmatrix}\mathbf{D} & \mathbf{A}\end{bmatrix} \quad (25)$$

$$\mathbf{M}_{24} = \mathrm{diag}\begin{bmatrix}\begin{bmatrix}\mathbf{U}^{L} \\ \mathbf{U}^{CP}\end{bmatrix} & \mathbf{B}\end{bmatrix} \quad (26)$$

$$\mathbf{M}_{32} = \mathbf{E} \quad (27)$$

$$\mathbf{M}_{41} = \mathrm{diag}\begin{bmatrix}\mathbf{I}_{p} & \mathbf{C}\end{bmatrix} \quad (28)$$

$$\mathbf{M}_{42} = \begin{bmatrix}\mathbf{H}^{L} & 0 & \mathbf{H}^{CP} \\ 0 & \mathbf{F} & 0\end{bmatrix} \quad (29)$$

$$\mathbf{M}_{43} = \begin{bmatrix}\mathbf{V}^{Lac} & \mathbf{V}^{NLac} & \mathbf{V}^{CPac}\end{bmatrix} \quad (30)$$

The rest of the matrices are defined as follows:

$$\mathbf{A} = \mathrm{diag}\left[\mathbf{A}_{11}, \mathbf{A}_{12}, \mathbf{A}_{13}\right] \quad (31)$$

$$\mathbf{A}_{11} = k_{ssv} d(k_{0} + p) \quad (32)$$

$$\mathbf{A}_{12} = k_{ssv}\text{diag}\left[\begin{bmatrix} u(k_0-1) \\ u(k_0) \end{bmatrix} \quad \ldots \quad \begin{bmatrix} u(k_0+m-1) \\ u(k_0+m) \end{bmatrix}\right] \tag{33}$$

$$\mathbf{A}_{13} = k_{ssv}\text{diag}[u(k_0),\ldots,u(k_0+m)] \tag{34}$$

$$\mathbf{D} = k_{ssv}\text{diag}\begin{bmatrix} d(k_0)+y^{sp}(k_0) \\ \vdots \\ d(k_0+p)+y^{sp}(k_0+p) \end{bmatrix} \tag{35}$$

$$\mathbf{U}^L = k_{ssv}\left[\mathbf{U}_1^L \quad \ldots \quad \mathbf{U}_l^L \quad \ldots \quad \mathbf{U}_p^L\right]^{\mathrm{T}} \tag{36}$$

$$\mathbf{U}_l^L = \begin{bmatrix} 0_{((p+1-l)\times(l-1))} & u(x)\mathbf{I}_{p+1-l} \\ 0_{((p+1-l)\times(l-1))} & u(x)^2\mathbf{I}_{p+1-l} \end{bmatrix} \tag{37}$$

$$\mathbf{U}^{CP} = k_{ssv}\left[\mathbf{U}_{1,1}^{CP} \quad \mathbf{U}_{1,2}^{CP} \quad \ldots \quad \mathbf{U}_{i,j-1}^{CP} \quad \mathbf{U}_{i,j}^{CP}\right]$$
$$i = 1,\ldots,p-1,\ j = 1,\ldots,p-i \tag{38}$$

$$\mathbf{U}_{i,j}^{CP} = \left[\mathbf{U}_{i,j\,1,1}^{CP} \quad \mathbf{U}_{i,j\,1,2}^{CP}\right] \tag{39}$$

$$\mathbf{U}_{i,j\,1,1}^{CP} = 0_{((p+1-i-j)\times(i+j-1))} \tag{40}$$

$$\mathbf{U}_{i,j\,1,2}^{CP} = u(i+j)u(j)\mathbf{I}_{p+1-i-j} \tag{41}$$

$$\mathbf{B} = \left[k_{ssv}u(k_0+m) \quad k_{ssv}u^2(k_0+m)\right]^{\mathrm{T}} \tag{42}$$

$$\mathbf{C} = \begin{bmatrix} \dfrac{k_{ssv}}{\varepsilon}, \text{diag}\left[\left[W_i^{\Delta u} \quad -W_i^{\Delta u}\right], \dfrac{k_{ssv}}{u_{\max}(i)}\right] \end{bmatrix}$$
$$i = 1,\ldots,m \tag{43}$$

$$\mathbf{H}^L = \left[\mathbf{H}_1^L,\ldots,\mathbf{H}_l^L,\ldots,\mathbf{H}_p^L\right] \tag{44}$$

$$\mathbf{H}_l^L = \begin{bmatrix} \mathbf{H}_{l\,1,1}^L & \mathbf{H}_{l\,1,2}^L \\ \mathbf{H}_{l\,2,1}^L & \mathbf{H}_{l\,2,2}^L \end{bmatrix} \tag{45}$$

$$\mathbf{H}_{l\,1,1}^L = \mathbf{H}_{l\,1,2}^L = 0_{((l-1)\times(p+1-l))} \tag{46}$$

$$\mathbf{H}_{l\,2,1}^L = \text{diag}\left(h_1^L,\ldots,h_{p+1-l}^L\right) \tag{47}$$

$$\mathbf{H}_{l\,2,2}^L = \text{diag}\left(h_{1,1}^{NL},\ldots,h_{p+1-l,p+1-l}^{NL}\right) \tag{48}$$

$$\mathbf{H}^{CP} = \left[\mathbf{H}_{1,1}^{CP}, \mathbf{H}_{1,2}^{CP},\ldots,\mathbf{H}_{i,j-1}^{CP}, \mathbf{H}_{i,j}^{CP}\right] \begin{matrix} i = 1,\ldots,p-1 \\ j = 1,\ldots,p-i \end{matrix} \tag{49}$$

$$\mathbf{H}_{i,j}^{CP} = \begin{bmatrix} 0_{((i+j-1)\times(p+1-i-j))} \\ \text{diag}\left(h_{1,i+1}^{NL},\ldots,h_{p+1-i-j,p+1-j}^{NL}\right) \end{bmatrix} \tag{50}$$

$$\mathbf{F} = \begin{bmatrix} \dfrac{k_{ssv}}{\varepsilon}\displaystyle\sum_{i=1}^{p} h_i^L & \dfrac{k_{ssv}}{\varepsilon}\displaystyle\sum_{i=1}^{p}\sum_{j=i}^{p} h_{i,j}^{NL} \end{bmatrix} \tag{51}$$

$$\mathbf{V}^{Lac} = \left[\mathbf{V}_1^{Lac},\ldots,\mathbf{V}_l^{Lac},\ldots,\mathbf{V}_p^{Lac}\right] \tag{52}$$

$$\mathbf{V}_l^{Lac} = \left[0_{((l-1)\times(p+2-l))} \quad \delta h_x^L \mathbf{I}_{p+2-l}\right]^{\mathrm{T}} \tag{53}$$

$$\mathbf{V}^{NLac} = \left[\mathbf{V}_1^{NLac},\ldots,\mathbf{V}_l^{NLac},\ldots,\mathbf{V}_p^{NLac}\right] \tag{54}$$

$$\mathbf{V}_l^{NLac} = \left[0_{((l-1)\times(p+2-l))} \quad \delta h_{x,x}^{NL} \mathbf{I}_{p+2-l}\right]^{\mathrm{T}} \tag{55}$$

$$\mathbf{V}^{CPac} = \left[\mathbf{V}_{1,1}^{CPac}, \mathbf{V}_{1,2}^{CPac},\ldots,\mathbf{V}_{i,j}^{CPac}\right] \begin{matrix} i = 1,\ldots,p-1 \\ j = 1,\ldots,p-i \end{matrix} \tag{56}$$

$$\mathbf{V}_{i,j}^{CPac} = \left[0_{((i+j-1)\times(p+3-i-j))} \quad \delta h_{i,j}^{NL} \mathbf{I}_{p+3-i-j}\right]^{\mathrm{T}} \tag{57}$$

In order to obtain the matrix **E** it is necessary to build a column vector that contains the Volterra series coefficients according to the following structure:

$$\mathbf{VE} = \left[\mathbf{VE}_1 \quad \ldots \quad \mathbf{VE}_{p+1}\right]^{\mathrm{T}} \tag{58}$$

The matrix **VE** has the following dimensions:

$$\mathbf{VE} = \left(p + p + \sum_{i=1}^{p-1} p - i\right) \times 1 \tag{59}$$

The rule to construct **VE** is as follows:

$$\mathbf{VE}_1 = \left[h_1^L \quad \ldots \quad h_p^L\right]^{\mathrm{T}} \tag{60}$$

$$\mathbf{VE}_2 = \left[h_{1,1}^{NL} \quad \ldots \quad h_{p,p}^{NL}\right]^{\mathrm{T}} \tag{61}$$

$$\mathbf{VE}_i = \left[h_{1,i-1}^{NL} \quad \ldots \quad h_{p+2-i,p}^{NL}\right]^{\mathrm{T}} \quad i \geq 3 \tag{62}$$

Finally the matrix **E** is constructed according to the following program:

$$\begin{aligned}
&\text{for } i = 1,\ldots,p+p+\sum_{q=1}^{p-1} p-q \\
&\text{for } ir = 1,\ldots,p \\
&\text{for } ic = 1,\ldots,p+p+\sum_{iq=1}^{p-1} p-iq \\
&\text{if}\left[\mathbf{H}^L\mathbf{H}^{CP}\right]_{ir,ic} = \mathbf{VE}_{i,1} \\
&index_{i,1} = ic \\
&\text{end} \\
&\text{end} \\
&\text{end} \\
&\text{end}
\end{aligned} \tag{63}$$

The elements of the matrix **E** are zero except the following:

$$\begin{aligned}
&\text{for } ir = 1,\ldots,2\sum_{i=1}^{p} i + \sum_{i=1}^{p-1}\sum_{j=1}^{i} j \\
&\mathbf{E}_{ir,index(ir,1)} = k_{ssv} \\
&\text{end}
\end{aligned} \tag{64}$$