Explicit Robust Model Predictive Control ⋆

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Abstract: Explicit robust multi–parametric feedback control laws are designed for constrained dynamic systems involving uncertainty in the left-hand side(LHS) of the underlying MPC optimization model. Our proposed procedure features: (i) a robust reformulation/optimization step, (ii) a dynamic programming framework for the model predictive control (MPC) problem formulation, and (iii) a multi-parametric programming solution step.

Keywords: Robust model predictive control, multi-parametric programming, dynamic programming, multi–parametric control, explicit control.

1. INTRODUCTION

Robust model predictive control (Robust MPC) is an important class of constrained, model–based control methods that can explicitly account for the presence of modeling uncertainties in the controlled process, which has received significant attention in control systems research–an indicative list of related publications is given in (Bemporad and Morari, 1999; Mayne et al., 2000; Sakizlis et al., 2004; Wang and Rawlings, 2004; Pistikopoulos et al., 2007a) and references within. On the other hand, explicit MPC, which has also received equal attention recently (Pistikopoulos et al., 2002, 2007a), is a control method where the online MPC optimization problem is solved off–line with multi–parametric programming methods to obtain the optimal control actions as a set of functions of the system states. The MPC controller can then be implemented online as a set of simple feedback control laws based on function evaluations instead of using online optimization with complex and increased computational demands.

Despite these significant advances, explicit, robust MPC is still an important area of research. It is evident from the relevant literature (Bemporad et al., 2003; Wang and Rawlings, 2004; Pistikopoulos et al., 2007a) that, even for the case of linear MPC, the underlying optimization model of the MPC is nonlinear due to the uncertainties appearing both in the left–hand side and right–hand side of the optimization constraints (Borrelli, 2003; Pistikopoulos et al., 2007a). This imposes difficulties for the application of the existing multi–parametric programming techniques and special treatment is required to ensure that the constraints are always satisfied (Bemporad et al., 2003; Kouramas et al., 2009).

Explicit robust MPC was investigated in Sakizlis et al. (2004) for the case of linear dynamic systems with additive state disturbances (right–hand side uncertainty in the optimization model). A dynamic programming based method, for linear dynamic systems with linear objective costs and uncertainties in left–hand side of the optimization model was studied in Bemporad et al. (2003). Furthermore, an explicit robust MPC with a quadratic objective and left–hand side uncertainties, based on robust optimization methods (Ben-Tal and Nemirovski, 2000; Lin et al., 2004), was presented in Kouramas et al. (2009) where the MPC optimization is treated as a robust multi–parametric optimization problem. Explicit robust MPC problems with quadratic costs have not yet been fully studied since the underlying multi–parametric optimization problem becomes nonlinear due to the uncertain coefficients in the constraints (Kouramas et al., 2009). On the other hand, employing dynamic programming methods for even the simple case of explicit MPC (with no uncertainties) results either into solving a demanding global optimization problem (Faísca et al., 2008) at each stage of the dynamic programming procedure or overlapping critical regions in the explicit solution.

This work presents a novel method for *Explicit Robust Model Predictive Control* based on dynamic programming methods (Bellman (2003); Faísca et al. (2008)) and robust optimization techniques (Ben-Tal and Nemirovski, 2000; Lin et al., 2004) that (i) allows the use of quadratic objective functions, (ii) accounts for the uncertainties in the left–hand side of the underlying MPC optimization problem, and (iii) overcomes the limitations of previous methods and the need for global optimization at each stage of the dynamic programming.

We focus on the following explicit robust MPC problem

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$$
\begin{aligned}
V^*(x) = &\min_U J(U,x) \\
= &\min_U \sum_{k=0}^{N-1} \{x_k^T Q x_k + u_k^T R u_k\} + x_N^T P x_N \qquad (1)\\
&\text{s.t. } x_{k+1} = Ax_k + Bu_k,\ \forall\, \Delta A \in \mathcal{A},\ \Delta B \in \mathcal{B}\\
&\quad Cx_k \le d,\ \ k = 0,1,\ldots,N\\
&\quad Mu_k \le \mu,\ \ k = 0,1,\ldots,N-1\\
&\quad Tx_N \le \tau\\
&\quad x = x_0
\end{aligned}
$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the system input and N the prediction horizon. We assume that the underlying system is uncertain in that the system matrices are described as

$$
\begin{aligned}
&x_{k+1} = Ax_k + Bu_k,\ A = A_0 + \Delta A,\ B = B_0 + \Delta B \qquad (2)\\
&\Delta A \in \mathcal{A} = \{\Delta A \in \mathbb{R}^{n\times n} | -\varepsilon_a|A_0| \le \Delta A \le \varepsilon_a|A_0|\}\\
&\Delta B \in \mathcal{B} = \{\Delta B \in \mathbb{R}^{n\times n} | -\varepsilon_\beta|B_0| \le \Delta B \le \varepsilon_\beta|B_0|\}
\end{aligned}
$$

where A_0, B_0 are of known constant values but the values of matrices $\Delta A, \Delta B$ are not known but are bounded as given in (2) and $\epsilon_a, \epsilon_\beta \in [0,1)$. The system states and inputs are also subject to the following linear constraints

$$x \in \mathcal{X} = \{x \in \mathbb{R}^n | Cx \le d\} \qquad (3)$$

$$u \in \mathcal{U} = \{u \in \mathbb{R}^m | Mu \le \mu\} \qquad (4)$$

where the sets $\mathcal{X}, \mathcal{U}$ are assumed to be compact, non-empty polytopic sets that include the origin in their interior and with $C \in \mathbb{R}^{n_c \times n}$, $d \in \mathbb{R}^{n_c}$, $M \in \mathbb{R}^{m_M \times m}$ and $\mu \in \mathbb{R}^{m_M}$. The proposed approach and the underlying mathematical framework for solving (1) will be discussed in detail in the following sections.

2. EXPLICIT ROBUST MODEL PREDICTIVE CONTROL

The proposed approach is realized in three key steps:

(1) dynamic programming: the MPC optimization is recast in a multi-stage optimization setting,
(2) robust reformulation: the constraints at each stage are reformulated to account for the worst-case uncertainty, and
(3) multi-parametric programming: each one of the reformulated stages is solved as multi-parametric programming problems where the optimization variables are the incumbent control inputs, given the optimal solutions of the previous steps.

These steps are described in detail in the following.

2.1 Dynamic Programming – Multi-stage optimization

The robust MPC problem (1) can be expressed as a multi–stage optimization problem since it involves a discrete-time dynamic system and a stage–additive quadratic objective function. The same procedure was applied for the nominal system case (where $\varepsilon_a, \varepsilon_\beta = 0$) in Faísca et al. (2008) Dynamic programming techniques (Bellman, 2003) can be applied to decompose (1) into a set of *stage-wise* problems of smaller dimensions, significantly reducing the complexity of the initial problem (Bellman (2003) and Faísca et al. (2008)) - at each stage k the following optimization problem is considered

$$
\begin{aligned}
V_k(x_k) = &\min_{u_k \in \mathcal{U}} J_k(u_k, x_k)\\
= &\min_{u_k \in \mathcal{U}} \sum_{i=k}^{N-1} \{x_i^T Q x_i + u_i^T R u_i\} + x_N^T R x_N \qquad (5)\\
&\text{s.t. } x_{i+1} = Ax_i + Bu_i,\ i = k,\ldots,N\\
&\quad Cx_k \le d,\ \ Cx_{k+1} \le d,\ \ Mu_k \le \mu,\\
&\quad \forall\, \Delta A \in \mathcal{A},\ \Delta B \in \mathcal{B}
\end{aligned}
$$

The optimization is taken only on the current stage input u_k and only the constraints on x_k and x_{k+1} have to be considered. The main idea is to solve the single–stage optimization problem (5) as a robust mp–QP problem and obtain the control variable u_k at each stage as an explicit function of current state x_k

$$u_k = f_k^*(x_k) \qquad (6)$$

or

$$u_k = K_k^i x_k + c_k^i \ \ \textbf{if}\ \ x_k \in CR_k^i\ ,\ \ i = 1,\ldots,L_k$$

A method for solving (5) as a robust mp–QP problem and deriving (6) is presented in the following sections. The proposed procedure for solving (1) as a multi-stage problem is the following: starting from time $k = N-1$, problem (5) is solved iteratively at each time k until $k = 0$ where the procedure stops. At the initial stage $k = N-1$ the extra terminal constraint $Tx_N \le \tau$ should also be added in (5).

In order to ensure that a feasible solution u_k exists for all $k = 0,1,\ldots N-1$ an extra feasibility constraint is introduced in each of the single stage problems (5)

$$x_{k+1} \in \mathcal{X}^{k+1}\ ,\ \ \mathcal{X}^{k+1} = \cup_{i=1}^{L_{k+1}} CR_{k+1}^i \qquad (7)$$

where $\mathcal{X}^{k+1}$ is the union of all critical regions of the explicit solution $u_{k+1} = f_{k+1}^*(x_{k+1})$ from the previous stage $k+1$ i.e. $\mathcal{X}^{k+1}$ is the set of states x_{k+1} for which the optimization problem at the stage $k+1$ has a feasible solution. Since the set of all critical regions is a convex polyhedral set (Pistikopoulos et al., 2002), the set $\mathcal{X}^{k+1}$ is given by a set of linear inequalities

$$\mathcal{X}^{k+1} = \{x \in \mathbb{R}^n | H^{k+1} x \le h^{k+1}\} \qquad (8)$$

Adding the constraints (8) in (5) will ensure that the future state x_{k+1} lies in the set $\mathcal{X}^{k+1}$ and hence one of the critical regions CR_{k+1}^i, and therefore a feasible control input $u_{k+1} = f_{k+1}^*(x_{k+1})$ at time $k+1$ can be obtained. For simplicity the inequalities $Cx_{k+1} \le d$ and (8) will be replaced by the single inequality

$$\mathcal{G}^k x_{k+1} \le b^k$$

where $\mathcal{G}^k = [C^T \ \ {H^{k+1}}^T]^T$ and $b^k = [d^T \ \ {h^{k+1}}^T]^T$.

We will now proceed to describe how to reformulate (5) to a robust mp–QP problem. Considering u_k as the optimization variable and $\theta_k = \left[x_k^T\, u_{k+1}^T \ldots u_{N-1}^T\right]^T$ as the vector of parameters, and by incorporating the system dynamics $x_{k+1} = Ax_k + Bu_k$ into the objective and constraints, one obtains the following multi-parametric optimization problem

$$
\begin{aligned}
V_k(x_k) = &\min J_k(u_k, \theta_k)\\
= &\min_{u_k \in \mathcal{U}} \left\{\frac{1}{2} u_k^T H u_k + \theta_k^T F u_k\right\} + \theta_k^T Y \theta_k \qquad (9)\\
&\text{s.t. } \mathcal{G}^k A x_k + \mathcal{G}^k B u_k \le b^k,\ \ Cx_k \le d,\ \ Mu_k \le \mu\\
&\quad \forall\, \Delta A \in \mathcal{A},\ \Delta B \in \mathcal{B}
\end{aligned}
$$

where the matrices H, F, Y are functions of the matrices A, B, Q and R. When there is no uncertainty in the underlying system dynamics $\varepsilon_a = \varepsilon_\beta = 0$, (5) is a simple mp–QP problem and can be solved with the known mp–QP method (Pistikopoulos et al., 2007b). However, in the presence of uncertainty (when $\varepsilon_a, \varepsilon_\beta \in [0, 1)$ are non-zeros) special treatment of (9) is required to reformulate it into mp–QP problem.

Remark 1. In conventional dynamic programming, the optimal value $u_{k+1} = f^*_{k+1}(x_{k+1})$ would have been incorporated into the formulation of (14) to create an optimization problem where only u_k is the optimization variable and x_k the parameter. However, even for the simple case with no uncertainties, this would have resulted into a nonlinear multi–parametric programming problem (since $u_{k+1} = f^*_{k+1}(x_{k+1})$ is a piecewise affine function) that would need to employ global optimization methods to be solved (Borrelli, 2003; Faísca et al., 2008). Our approach is based on the work of Faísca et al. (2008) for the case of explicit MPC with no uncertainties, where this issue is overcome by substituting previous solutions u_{k+1} in the current solution u_k after the multi–parametric programming has been solved.

2.2 Robustification Step

The main issue for applying multi-parametric optimization techniques for the solution of (9) is the presence of the uncertain matrices A, B in the objective and the inequalities of (9). The objective function can be set to penalize only the behaviour of the nominal system $x_{k+1} = A_0 x_k + B_0 u_k$, that is to say the objective function in (9) is formed by replacing $x_{k+1} = A_0 x_k + B_0 u_k$ in the objective (5) and H, F, Y are constant matrices. However, it is very important to guarantee the feasibility of the constraints in the presence of the uncertainty. Problem (9) can then be recast as

$$V_k(x_k) = \min_{u_k \in \mathcal{U}} \left\{ \frac{1}{2} u_k^T H u_k + \theta_k^T F u_k \right\} + \theta_k^T Y \theta_k \qquad (10)$$
$$\text{s.t. } \mathcal{G}^k A_0 x_k + \mathcal{G}^k \Delta A x_k + \mathcal{G}^k B_0 u_k + \mathcal{G}^k \Delta B u_k \le b^k$$
$$C x_k \le d, \;\; M u_k \le \mu, \;\; \forall\, \Delta A \in \mathcal{A}, \; \Delta B \in \mathcal{B}$$

It is obvious from (10) that due to variations of $\Delta A, \Delta B$ constraint violations might occur. Solving (10) is a robust multi-parametric optimization problem where u_k is the optimization variable and θ_k is the vector of parameters. The objective is to find a solution $u^*_k(\theta_k)$ which can guarantee constraint satisfaction for all admissible values of the uncertainty i.e. for all $\Delta A \in \mathcal{A}$ and $\Delta B \in \mathcal{B}$.

Definition 2.1. A solution $u^*_k(\theta_k)$ of robust mp-QP problem (10) is a *robust* or *reliable* solution if it is feasible for (10) both for the nominal system ($A = A_0$, $B = B_0$) and the uncertain system i.e. if it is feasible for all admissible values of the uncertainty i.e. for all $\Delta A \in \mathcal{A}$ and $\Delta B \in \mathcal{B}$.

In order to avoid constraint violations, the constraints have to be *immunized* against the model uncertainty (see Ben-Tal and Nemirovski (2000) and Lin et al. (2004)). In order to account for the uncertainty in (10), the inequality constraints of (10) are replaced by the following two inequalities

$$\mathcal{G}^k A_0 x_k + \mathcal{G}^k B_0 u_k \le d \qquad (11)$$

$$\mathcal{G}^k A_0 x_k + \varepsilon_a |\mathcal{G}^k||A_0||x_k| + \mathcal{G}^k B_0 u_k$$
$$+ \varepsilon_\beta |\mathcal{G}^k||B_0||u_k| \le b^k + \delta \max\{1, |d|\} \qquad (12)$$

The first inequality ensures that the problem is feasible for the nominal system case while the second inequality represents the realisation of the constraint for the worst-case value of the uncertainty. The newly introduced variable δ is a measure of the infeasibility tolerance for the constraint in the problem i.e. how much the constraint can be relaxed to ensure a feasible solution. If no infeasibility is allowed then $\delta = 0$.

Replacing the new constraints (11)–(12) into (10) results into a multi–parametric nonlinear programming problem. To overcome this, (12) is replaced by the following linear inequalities

$$\mathcal{G}^k A_0 x_k + \varepsilon_a |\mathcal{G}^k||A_0| z_k + \mathcal{G}^k B_0 u_k$$
$$+ \varepsilon_\beta |\mathcal{G}^k||B_0| \omega_k \le b^k + \delta \max\{1, |d|\} \qquad (13)$$
$$- z_k \le x_k \le z_k, \;\; -\omega_k \le u_k \le \omega_k, \;\; z_k, \omega_k \ge 0$$

It is obvious that if a pair x_k, u_k satisfies (13) then, since $|x_k| \le z_k$ and $|u_k| \le \omega_k$, it also satisfies (12). By replacing (13) in (10) the new robust mp–QP formulation is obtained for each stage

$$V_k(x_k) = \min_{u_k, z_k, \omega_k} \left\{ \frac{1}{2} u_k^T H u_k + \theta_k^T F u_k \right\} + \theta_k^T Y \theta_k \qquad (14)$$
$$\text{s.t. } \mathcal{G}^k A_0 x_k + \mathcal{G}^k B_0 u_k \le b^k$$
$$\mathcal{G}^k A_0 x_k + \varepsilon_a |\mathcal{G}^k||A_0| z_k + \mathcal{G}^k B_0 u_k$$
$$+ \varepsilon_\beta |\mathcal{G}^k||B_0| \omega_k \le b^k + \delta \max\{1, |d|\}$$
$$- z_k \le x_k \le z_k, \;\; -\omega_k \le u_k \le \omega_k, \;\; z_k, \omega_k \ge 0$$
$$C x_k \le d, \;\; M u_k \le \mu$$

where now the parameters are θ_k, the optimization variable is $\pi_k = \left[u_k^T, z_k^T, \omega_k^T\right]^T$, the objective function is a quadratic function and the constraints are all linear inequalities. The new formulation (14) is an mp–QP problem and can be solved by employing the mp–QP methods of Pistikopoulos et al. (2002) and Pistikopoulos et al. (2007b) which is discussed next.

2.3 Multi–Parametric Quadratic Programming

In order to solve (14) as an mp–QP problem, the following three steps have to be followed

Step 1. The Karush–Kuhn–Tucker (KKT) conditions are first applied for problem (14) (see Bazaraa and Shetty (1979)):

$$\nabla \mathcal{L}(\pi_k, \lambda, \theta_k) = 0, \;\; \lambda_i \psi_i(\pi_k, \theta_k) = 0, \;\; \forall \;\; i = 1, \ldots, p$$
$$\mathcal{L} = J_k(\pi_k, \theta_k) + \sum_{i=1}^{p} \lambda_i \psi_i(\pi_k, \theta_k) \qquad (15)$$

where $J_k(\pi_k, \theta_k)$ is the objective function of (14), $\psi(\pi_k, \theta_k) \le 0$ is the vector of the inequality constraints in (14) and λ is the vector of the Lagrange multipliers.

Step 2. The basic sensitivity theorem (Fiacco (1976)) is then applied to the KKT conditions (15). For simplicity we set $\theta = \theta_k$ and $\pi = \pi_k$.

Theorem 2. Let θ_0 be a vector of parameter values and $(\pi_0, \lambda_0, \mu_0)$ a KKT triple corresponding to (15), where

λ_0 is nonnegative and π_0 is feasible in (14). Also assume that (i) strict complementary slackness (SCS) holds, (ii) the binding constraint gradients are linearly independent (LICQ: Linear Independence Constraint Qualification), and (iii) the second-order sufficiency conditions (SOSC) hold. Then, in neighbourhood of θ_0, there exists a unique, once continuously differentiable function, $z(\theta) = [\pi(\theta), \lambda(\theta)]$, satisfying (15) with $z(\theta_0) = [\pi(\theta_0), \lambda(\theta_0)]$ where $\pi(\theta)$ is a unique isolated minimiser for (14), and

$$\begin{pmatrix} d\pi(\theta_0)/d\theta \\ d\lambda(\theta_0)/d\theta \end{pmatrix} = -(M_0)^{-1} N_0, \tag{16}$$

where, M_0 and N_0 are the Jacobians of system (15) with respect to z and θ (Fiacco, 1983, pp. 80–81), (Pistikopoulos et al., 2002).

Step 3. A general analytic expression for π_k is then derived by applying the following corollary of Dua et al. (2002)

Corollary 3. First-order estimation of $\pi(\theta)$, $\lambda(\theta)$, near $\theta = \theta_0$ (Fiacco, 1983): Under the assumptions of Theorem 2, a first-order approximation of $[\pi(\theta), \lambda(\theta)]$ in a neighbourhood of θ_0 is,

$$\begin{bmatrix} \pi(\theta) \\ \lambda(\theta) \end{bmatrix} = \begin{bmatrix} \pi_0 \\ \lambda_0 \end{bmatrix} + (M_0)^{-1} \cdot N_0 \cdot \theta + o(||\theta||), \tag{17}$$

where $(\pi_0, \lambda_0) = [\pi(\theta_0), \lambda(\theta_0)]$, $M_0 = M(\theta_0)$, $N_0 = N(\theta_0)$, and $\phi(\theta) = o(||\theta||)$ means that $\phi(\theta)/||\theta|| \to 0$ as $\theta \to \theta_0$.

The critical region (set of θ) where (17) remains optimal can then be obtained as follows (Dua et al., 2002). If $\breve{\psi}$ corresponds to the non-active constraints, and $\tilde{\lambda}$ corresponds to the active constraints then each critical region is defined by

$$\begin{array}{ll} \breve{\psi}(u(\theta_k), \theta_k) \leq 0 & \text{(Feasibility conditions)}, \\ \tilde{\lambda}(\theta_k) \geq 0 & \text{(Optimality conditions)}. \end{array} \tag{18}$$

It is obvious from step 1.–3. and corollary 3 that the the explicit solution π_k^* of (14) is given by a conditional piecewise linear function (Dua et al. (2002) and Pistikopoulos et al. (2007a)) i.e. $\pi_k = f_k^*(\theta_k)$. Consequently, the control u_k is also obtained as an explicit function of the parameter θ_k as follows

$$u_k = f_k^*(\theta_k) = f_k^*(x_k, u_{k+1}, \ldots, u_{N-1}) \tag{19}$$

or

$$u_k = K_k^i \theta_k + c_k^i, \textbf{ if } \theta_k \in CR_k^i, \; i = 1, \ldots, L_k \tag{20}$$

where K_k^i, c_k^i are matrices and vectors of appropriate dimensions and the critical regions $CR_k^i \subset \mathbb{R}^n$ are sets defined by (18). The same procedure repeats iteratively, starting at $k = N-1$ and stopping at $k = 0$ and hence the full profile of control policies $u_k(\theta_k)$, $k = 0, 1, \ldots, N-1$ is derived.

Although u_k is a function of θ_k, the objective is to obtain u_k as an explicit control function of the incumbent state x_k thus obtaining a feedback control strategy. We can overcome this issue by following an approach similar to Faísca et al. (2008) for the nominal explicit MPC case. As the procedure is repeated repetitively and backwards from $k = N-1$ to $k = 0$, the control inputs $u_{k+1}, \ldots, u_{N-1}$ before stage k are obtained as in (19)

$$\begin{aligned} u_{k+1} &= f_{k+1}^*(x_{k+1}, u_{k+2}, \ldots, u_{N-1}) \\ &\vdots \\ u_{N-1} &= f_{N-1}^*(x_{N-1}) \end{aligned} \tag{21}$$

All the above control inputs are piecewise linear functions of their arguments. Note also that since the control inputs $u_{k+1}, \ldots, u_{N-1}$ are functions of the future states $x_{k+1}, \ldots, x_{N-1}$ they are also functions of the incumbent input u_k and state x_k. By incorporating the previous solutions (21) into (19) and by performing algebraic manipulation we obtain the explicit control law $u_k = f_k^*(x_k)$ (see for more details Faísca et al. (2008)). The final critical regions of $u_k = f_k^*(x_k)$ are defined as a union of the inequalities (of the critical regions) of (19) and of each of the critical regions of (21). This results in (i) realisable feasible sets of inequalities describing the feasible critical regions of $u_k = f_k^*(x_k)$ and (ii) empty sets of inequalities where no feasible solution exists. Feasibility tests, as the ones presented in Faísca et al. (2008), are finally performed, during the substitution of (21) into (19), to obtain the final feasible critical regions.

2.4 Algorithm for Robust mp–MPC

The dynamic programming based procedure that was described above is summarized in table 1. The Algorithm starts at $k = N-1$ and iterates through Steps 2 and 3 until $k = 0$. At the k^{th} stage of the algorithm, problem (14) is solved following the analysis in sections 2.1–2.3. Each of the inputs u_k is obtained as an explicit function of the corresponding state x_k i.e. $u_k = f_k(x_k)$ where $f_k(x_k)$ is a piecewise linear function similar to (6). At the termination of the algorithm a sequence of admissible control policies is obtained $u_0^* = f_0^*(x_0)$, $u_1^* = f_1^*(x_1)$, $\ldots$, $u_{N-1}^* = f_{N-1}^*(x_{N-1})$. Each of these control policies are *reliable* (or *robust*) control policies for each of the stage problems (14). Since each control policy also guarantees that the state and input constraints $x_k \in \mathcal{X}$ and $u_k \in \mathcal{U}$ at each stage are satisfied, then the control sequence $U = \{u_0^*, u_1^* \ldots u_{N-1}^*\}$ is also a robust solution for the initial robust mp-MPC problem (1). The following lemma can then be stated

Lemma 4. The control sequence $U = \{u_0^*, u_1^* \ldots u_{N-1}^*\}$, where u_k^*, $k = 0, 1, \ldots, N-1$ are the optimal control policies obtained by solving (14) iteratively using the algorithm in table 1, is a robust (or reliable solution) of (1).

Table 1. Algorithm for Robust Multi–Parametric MPC

Step 1.	Set $k = N-1$: solve the mp-QP problem (14) with x_{N-1} being the parameters and obtain $u_{N-1}^* = f_{N-1}^*(x_{N-1})$.
Step 2.	Set k to the current stage: solve the k^{th} stage–wise mp–QP problem (14) with $x_k, u_k, \ldots, u_{N-1}$ being the parameters and obtain $u_k^* = f_k^*(x_k, u_k, \ldots, u_{N-1})$.
Step 3.	Obtain the control law $u_k = f_k(x_k)$ by comparing the sets of solutions (19) and (21).
Step 4.	Set $k = k-1$: if $k = 0$ stop, else go to Step 2.

The main advantage of the proposed algorithm is that it can handle robust Model Predictive Control problems with quadratic objectives in the presence of uncertainties in the

LHS of the underlying optimization model at each stage of the proposed dynamic programming procedure. This is achieved by treating the optimization problem for each stage of the procedure as a convex robust mp–QP problem (14) with linear constraints, avoiding the nonlinearities introduced by the presence of uncertainty in (11).

The introduction of the two new variables z, ω also results in an increase of the number of constraints in the optimization as it can be seen from (13). The total number of optimization variables in the resulting robust mp–QP problem (14) is $2m + n$, while the total number of linear inequalities is $2n_c + m_M + n + m$. One can notice that both the number of optimization variables and inequalities for problem (14), after the robustification step, is linear with the number of system states and inputs. Thus the complexity of the mp–QP problem is not significantly increased. Finally, the number of parameters of the mp–QP (14) at each stage is equal to $n + (N - k - 1)m$, hence it increases as k decreases. This will have an important effect on the number of critical regions at each stage and eventually in the overall number of critical regions.

3. EXAMPLE

Consider the following robust MPC example where

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \varepsilon_a = \varepsilon_\beta = 0.2$$

$$\begin{bmatrix} -10 \\ -10 \end{bmatrix} \le x_k \le \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad -1 \le u_k \le 1$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 1, \quad P = \begin{bmatrix} 2.6005 & 2.081 \\ 2.081 & 3.3306 \end{bmatrix}, \quad N = 3$$

The target set in this example is considered to be simply the set of state constraints while $\delta = 0$ is set equal to zero. The algorithm, presented in Table 1, is applied and the results can be seen in Figures 1, 2, 3. In the first iteration of the algorithm the robust multi-parametric programming problem (14) for $k = 2$ is solved, where the parameter is $\theta_2 = x_2$. The critical regions of the explicit solution $u_2 = f_2(x_2)$ are shown in Figure 1. Then, the procedure is repeated for the stages $k = 1$ and $k = 0$ to obtain the explicit controls $u_1 = f_1(x_1)$ and $u_0 = f_0(x_0)$. The critical regions for these stages are shown in Figures 2 and 3 respectively. One can notice that the area of the critical regions at each stage k decreases as k decreases. This happens since the set of states which can be driven to the target set (which here is the set of constraints) reduces as k reduces. Also, the number of critical regions increases since at each stage the number of parameters increases. Two different simulations for two different initial states of the explicit robust MPC control is shown in Figure 4 where the system matrices A, B are perturbed around their nominal values. Finally, table 1 shows some of the critical regions and corresponding control functions for the explicit solution at stage 0.

4. CONCLUDING REMARKS

A new algorithm for robust multi–parametric MPC was presented when uncertainty is introduced in the LHS of the underlying MPC optimization model. Based on dynamic programming and robust optimization, the algorithm obtains the control input explicitly as function of the states by solving a set of convex mp–QP problems and avoid the need for employing multi–parametric global optimization. Current work is focusing on the generalisation of the presented results to the following problems: (i) explicit robust MPC of constrained dynamic systems with uncertainty

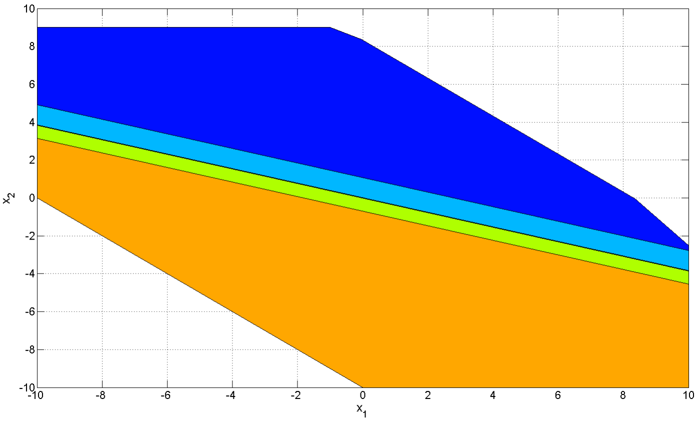

Fig. 1. Critical regions of the explicit robust MPC for stage 2, $u_2 = f_2(x_2)$

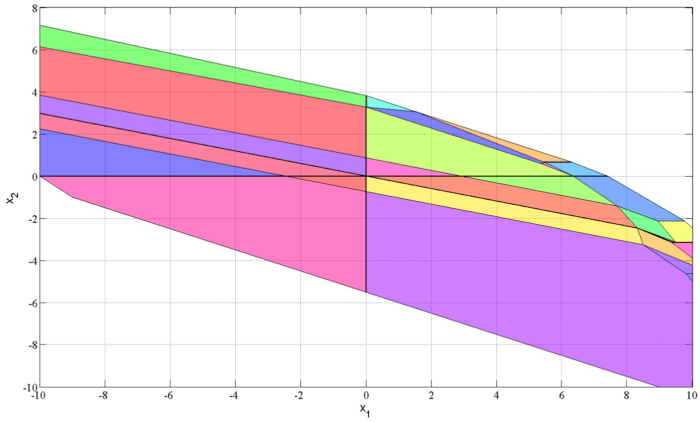

Fig. 2. Critical regions of the explicit robust MPC for stage 1, $u_1 = f_1(x_1)$

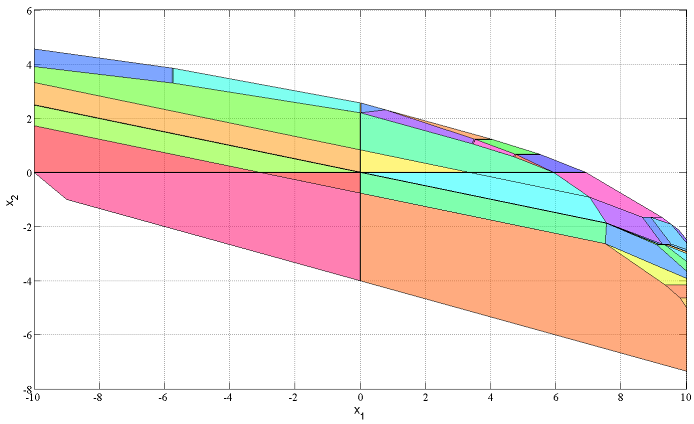

Fig. 3. Critical regions of the explicit robust MPC for stage 0, $u_0 = f_0(x_0)$

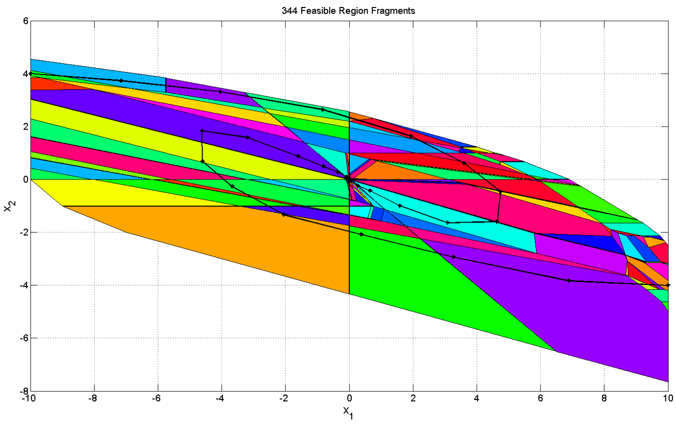

Fig. 4. Simulation of the uncertain system state trajectory with explicit robust MPC.

Table 2. Critical Regions and the corresponding Control Laws for stage 0

Critical Regions No.	Control Law	Critical Regions
1	$u = 1$	$\begin{bmatrix} 0.2174 & 1 \\ 0 & 1 \\ -0.5 & -1 \\ -0.3333 & -1 \\ 1 & 0 \end{bmatrix} x \leq \begin{bmatrix} -1.748 \\ -1.01 \\ 5.5 \\ 4.3333 \\ -0.01 \end{bmatrix}$
2	$u = 1$	$\begin{bmatrix} 0.2174 & 1 \\ -0.3333 & -1 \\ -1 & 0 \\ 1 & 0 \end{bmatrix} x \leq \begin{bmatrix} -1.7480 \\ 4.3333 \\ 0.01 \\ 0.0050 \end{bmatrix}$
3	$u = -1$	$\begin{bmatrix} -0.2513 & -1 \\ 0.9 & 1 \\ 0 & 1 \\ 0 & -1 \\ -0.7821 & -1 \\ -1 & -0.9524 \end{bmatrix} x \leq \begin{bmatrix} -0.1791 \\ 6.65 \\ -1.6562 \\ 2.1250 \\ -4.5301 \\ -6.3333 \end{bmatrix}$
4	$u = -0.5662x_1 - 1.3573x_2 + 1.0378$	$\begin{bmatrix} -1 & -0.2757 \\ 0.4172 & 1 \\ 1 & 0.631 \\ -0.4172 & -1 \\ -1 & -0.555 \end{bmatrix} x \leq \begin{bmatrix} -7.8743 \\ 1.5014 \\ 7.3523 \\ -0.7720 \\ -7.0852 \end{bmatrix}$
5	$u = -0.4701x_1 - 1.3476x_2 - 0.0001$	$\begin{bmatrix} 1 & 0.9042 \\ -1 & -1 \\ -1 & -0.7395 \\ -1 & -0.1585 \\ 1 & 1 \end{bmatrix} x \leq \begin{bmatrix} 0.079 \\ 0.01 \\ -0.0211 \\ -0.0579 \\ 0.005 \end{bmatrix}$

and additive disturbance, both in the LHS and RHS of the underlying multi–parametric optimization model (Sakizlis et al. (2004)), (ii) explicit robust MPC of hybrid systems – based on multi–parametric Mixed Integer Linear Programming (Faísca et al. (2009)) and (iii) multi–parametric Global Optimisation (Dua et al. (2004))..

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