Robust Adaptive MPC for Systems with Exogeneous Disturbances ⋆

V. Adetola * **M. Guay** *

* *Department of Chemical Engineering, Queen's University, Kingston, Ontario, Canada (e-mail: martin.guay@chee.queensu.ca)*

Abstract: In this paper, we consider the problem of Adaptive model predictive control subject to exogenous disturbances. Using a novel set-based adaptive estimation, the problem of robust adaptive MPC is proposed and solved for a class of linearly parameterized uncertain nonlinear systems subject to state and input constraints. Two formulations of the adaptive MPC routine are proposed. A minmax approach is first considered. A Lipschitz-based formulation, amenable to real-time computations, is then proposed. A chemical reactor simulation example is presented that demonstrates the effectivenessof the technique.

Keywords: Adaptive control, Robust MPC, Nonlinear MPC

1. INTRODUCTION

Most physical systems possess consists of parametric and non-parametric uncertainties and the system dynamics can be influenced by exogeneous disturbances as well. Examples in chemical engineering include reaction rates, activation energies, fouling factors, and microbial growth rates. Since parametric uncertainty may degrade the performance of MPC, mechanisms to update the unknown or uncertain parameters are desirable in application. One possibility would be to use state measurements to update the model parameters off-line. A more attractive possibility is to apply adaptive extensions of MPC in which parameter estimation and control are performed online. In this paper, we extend an adaptive MPC framework to nonlinear systems with both constant parametric uncertainty and additive exogenous disturbances.

The literature contains very few results on the design of adaptive nonlinear MPC Adetola and Guay (2004); Mayne and Michalska (1993). Existing design techniques are restricted to systems that are linear in the unknown (constant) parameters and do not involve state constraints. Although MPC exhibits some degree of robustness to uncertainties, in reality, the degree of robustness provided by nominal models or certainty equivalent models may not be sufficient in practical applications. Parameter estimation error must be accounted for in the computation of the control law.

This paper is inspired by DeHaan and Guay (2007); DeHaan et al. (2007). While the focus in DeHaan and Guay (2007); DeHaan et al. (2007) is on the use of adaptation to reduce the conservatism of robust MPC controller, this study addresses the problem of adaptive MPC and incorporates robust features to guarantee closed-loop stability and constraint satisfaction. Simplicity is achieved here-in by generating a parameter estimator for the unknown parameter vector and parameterizing the control policy in terms of these estimates rather than adapting a parameter uncertainty set directly.

First, a min-max feedback nonlinear MPC scheme is combined with an adaptation mechanism. The parameter estimation routine are used to update the parameter uncertainty set, at certain time instants, in a manner that guarantees non-expansion of the set leading to a gradual reduction in the conservativeness or computational demands of the algorithms. The min-max formulation explicitly accounts for the effect of future parameter estimation and automatically injects some useful excitation into the closed-loop system to aid in parameter identification.

Second, the technique is extended to a less computationally demanding robust MPC algorithm. The nominal model rather than the unknown bounded system state is controlled, subject to conditions that ensure that given constraints are satisfied for all possible uncertainties. State prediction error bound is determined based on assumed Lipschitz continuity of the model. Using a nominal model prediction, it is impossible to predict the actual future behavior of the parameter estimation error as was possible in the min-max framework. It is shown how the future model improvement over the prediction horizon can be considered by developing a worst-case upper bound on the future parameter estimation error. The conservativeness of the algorithm reduces as the error bound decreases monotonically over time.

The paper is as follows. The problem description is given in section 2. The parameter estimation routine is presented in section 3.Two approaches to robust adpative model predictive control are detailed in section 4. This is followed by a simulation example in section 5 and brief conclusions in section 6.

2. PROBLEM SET-UP

Consider the uncertain nonlinear system

$$\dot{x} = f(x,u) + g(x,u)\theta + \vartheta \triangleq \mathcal{F}(x,u,\theta,\vartheta) \tag{1}$$

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where the disturbance $\vartheta \in \mathcal{D} \subset \mathbb{R}^{n_d}$ is assumed to satisfy a known upper bound $\|\vartheta(t)\| \le M_\vartheta < \infty$. The objective of the study is to (robustly) stabilize the plant to some target set $\Xi \subset \mathbb{R}^{n_x}$ while satisfying the pointwise constraints $x \in \mathbb{X} \in \mathbb{R}^{n_x}$ and $u \in \mathbb{U} \in \mathbb{R}^{n_u}$. The target set is a compact set, contains the origin and is robustly invariant under no control. It is assumed that θ is uniquely identifiable and lie within an initially known compact set $\Theta^0 = B(\theta_0, z_\theta)$ where θ_0 is a nominal parameter value, z_θ is the radius of the parameter uncertainty set.

3. PARAMETER AND UNCERTAINTY SET ESTIMATION

3.1 Parameter Adaptation

Let the estimator model for (1) be selected as

$$\dot{\hat{x}} = f(x,u) + g(x,u)\hat{\theta} + k_w\, e + w\dot{\hat{\theta}}, \quad k_w > 0 \tag{2}$$

$$\dot{w} = g(x,u) - k_w\, w, \qquad w(t_0) = 0. \tag{3}$$

resulting in state prediction error $e = x - \hat{x}$ and auxiliary variable $\eta = e - w\tilde{\theta}$ dynamics:

$$\dot{e} = g(x,u)\tilde{\theta} - k_w\, e - w\dot{\hat{\theta}} + \vartheta \qquad e(t_0) = x(t_0) - \hat{x}(t_0) \tag{4}$$

$$\dot{\eta} = -k_w\, \eta + \vartheta, \qquad \eta(t_0) = e(t_0). \tag{5}$$

Since ϑ is not known, an estimate of η is generated from

$$\dot{\hat{\eta}} = -k_w\, \hat{\eta}, \qquad \hat{\eta}(t_0) = e(t_0). \tag{6}$$

with resulting estimation error $\tilde{\eta} = \eta - \hat{\eta}$ dynamics

$$\dot{\tilde{\eta}} = -k_w\, \tilde{\eta} + \vartheta, \qquad \tilde{\eta}(t_0) = 0. \tag{7}$$

Let $\Sigma \in \mathbb{R}^{n_\theta \times n_\theta}$ be generated from

$$\dot{\Sigma} = w^T w, \qquad \Sigma(t_0) = \alpha\, I \succ 0, \tag{8}$$

based on equations (2), (3) and (6), thye preferred parameter update law is given by

$$\dot{\Sigma}^{-1} = -\Sigma^{-1} w^T w \Sigma^{-1}, \qquad \Sigma^{-1}(t_0) = \frac{1}{\alpha} I \tag{9a}$$

$$\dot{\hat{\theta}} = \mathrm{Proj}\left\{\gamma\, \Sigma^{-1} w^T (e - \hat{\eta}),\ \hat{\theta}\right\}, \qquad \hat{\theta}(t_0) = \theta^0 \in \Theta^0 \tag{9b}$$

where $\gamma = \gamma^T > 0$ and $\mathrm{Proj}\{\phi,\ \hat{\theta}\}$ denotes a Lipschitz projection operator such that

$$-\mathrm{Proj}\{\phi,\ \hat{\theta}\}^T \tilde{\theta} \le -\phi^T \tilde{\theta}, \tag{10}$$

$$\hat{\theta}(t_0) \in \Theta^0 \ \Rightarrow\ \hat{\theta}(t) \in \Theta^0_\epsilon, \ \ \forall\, t \ge t_0. \tag{11}$$

where $\Theta^0_\epsilon \triangleq B(\theta^0, z^0_\theta + \epsilon)$, $\epsilon > 0$. More details on parameter projection can be found in Krstic et al. (1995). To proof the following lemma, we need the following result

Lemma 1. Desoer and Vidyasagar (1975) Consider the system

$$\dot{x}(t) = Ax(t) + u(t) \tag{12}$$

Suppose the equilibrium state $x_e = 0$ of the homogeneous equation is exponentially stable,

(1) if $u \in \mathrm{Ł}_p$ for $1 < p < \infty$, then $x \in \mathrm{Ł}_p$ and
(2) if $u \in \mathrm{Ł}_p$ for $p = 1$ or 2, then $x \to 0$ as $t \to \infty$.

Lemma 2. The identifier (9) is such that the estimation error $\tilde{\theta} = \theta - \hat{\theta}$ is bounded. Moreover, if

$$\vartheta \in \mathcal{L}_2 \qquad \text{or} \qquad \int_{t_0}^{\infty} \left[\|\tilde{\eta}\|^2 - \underline{\gamma}\, \|e - \hat{\eta}\|^2\right] d\tau < +\infty \tag{13}$$

with $\underline{\gamma} = \lambda_{\min}(\gamma)$ and the strong condition

$$\lim_{t\to\infty} \lambda_{\min}\big(\Sigma\big) = \infty \tag{14}$$

is satisfied, then $\tilde{\theta}$ converges to zero asymptotically.

Proof: Let $V_{\tilde{\theta}} = \tilde{\theta}^T \Sigma\, \tilde{\theta}$, it follows from (9) and the relationship $w\tilde{\theta} = e - \hat{\eta} - \tilde{\eta}$ that

$$\begin{aligned}\dot{V}_{\tilde{\theta}} &\le -2\underline{\gamma}\, \tilde{\theta}^T w^T (e - \hat{\eta}) + \tilde{\theta}^T w^T w \tilde{\theta} \\ &= -\underline{\gamma}\, (e - \hat{\eta})^T (e - \hat{\eta}) + \|\tilde{\eta}\|^2,\end{aligned} \tag{15}$$

implying that $\tilde{\theta}$ is bounded. Moreover, it follows from (15) that

$$V_{\tilde{\theta}}(t) = V_{\tilde{\theta}}(t_0) + \int_{t_0}^{t} \dot{V}_{\tilde{\theta}}(\tau) d\tau \tag{16}$$

$$\le V_{\tilde{\theta}}(t_0) - \underline{\gamma} \int_{t_0}^{t} \|e - \hat{\eta}\|^2\, d\tau + \int_{t_0}^{t} \|\tilde{\eta}\|^2\, d\tau \tag{17}$$

Considering the dynamics of (7), if $\vartheta \in \mathcal{L}_2$, then $\tilde{\eta} \in \mathcal{L}_2$ (Lemma 1). Hence, the right hand side of (17) is finite in view of (13), and by (14) we have $\lim_{t\to\infty} \tilde{\theta}(t) = 0$ ■

3.2 Set Adaptation

An update law that measures the worst-case progress of the parameter identifier in the presence of disturbance is given by:

$$z_\theta = \sqrt{\frac{V_{z\theta}}{\lambda_{\min}(\Sigma)}} \tag{18a}$$

$$V_{z\theta}(t_0) = \lambda_{\max}\Big(\Sigma(t_0)\Big)\, (z^0_\theta)^2 \tag{18b}$$

$$\dot{V}_{z\theta} = -\underline{\gamma}\, (e - \hat{\eta})^T (e - \hat{\eta}) + \left(\frac{M_\vartheta}{k_w}\right)^2. \tag{18c}$$

Using the parameter estimator (9) and its error bound z_θ (18), the uncertain ball $\Theta \triangleq B(\hat{\theta}, z_\theta)$ is adapted online according to the following algorithm:

Algorithm 1. Beginning from time $t_{i-1} = t_0$, the parameter and set adaptation is implemented iteratively as follows:

1 **Initialize** $z_\theta(t_{i-1}) = z^0_\theta$, $\hat{\theta}(t_{i-1}) = \hat{\theta}^0$ and $\Theta(t_{i-1}) = B(\hat{\theta}(t_{i-1}), z_\theta(t_{i-1}))$.

2 At time t_i, using equations (9) and (18) **perform** the update

$$\Big(\hat{\theta},\ \Theta\Big) = \begin{cases} \Big(\hat{\theta}(t_i),\ \Theta(t_i)\Big), & \text{if } z_\theta(t_i) \le z_\theta(t_{i-1}) \\ & \quad -\|\hat{\theta}(t_i) - \hat{\theta}(t_{i-1})\| \\ \Big(\hat{\theta}(t_{i-1}),\ \Theta(t_{i-1})\Big), & \text{otherwise} \end{cases} \tag{19}$$

3 **Iterate back** to step 2, **incrementing** $i = i + 1$.

The algorithm ensure that Θ is only updated when z_θ value has decreased by an amount which guarantees a contraction of the set. Moreover z_θ evolution as given in (18) ensures non-exclusion of θ as shown below.

Lemma 3. The evolution of $\Theta = B(\hat{\theta}, z_\theta)$ under (9), (18) and algorithm 1 is such that

i) $\Theta(t_2) \subseteq \Theta(t_1),\ t_0 \le t_1 \le t_2$
ii) $\theta \in \Theta(t_0) \Rightarrow \theta \in \Theta(t),\ \forall t \ge t_0$

Proof:

i) If $\Theta(t_{i+1}) \nsubseteq \Theta(t_i)$, then
$$\sup_{s\in\Theta(t_{i+1})} \|s - \hat{\theta}(t_i)\| \ge z_\theta(t_i). \tag{20}$$
However, it follows from triangle inequality and algorithm 1 that Θ, at update times, obeys
$$\begin{aligned}&\sup_{s\in\,\Theta(t_{i+1})} \|s - \hat{\theta}(t_i)\| \\ &\quad\le \sup_{s\in\,\Theta(t_{i+1})} \|s - \hat{\theta}(t_{i+1})\| + \|\hat{\theta}(t_{i+1}) - \hat{\theta}(t_i)\| \\ &\quad\le z_\theta(t_{i+1}) + \|\hat{\theta}(t_{i+1}) - \hat{\theta}(t_i)\| \le z_\theta(t_i),\end{aligned}$$
which contradicts (20). Hence, Θ update guarantees $\Theta(t_{i+1}) \subseteq \Theta(t_i)$ and the strict contraction claim follows from the fact that Θ is held constant over update intervals $\tau \in (t_i, t_{i+1})$.

ii) We know that $V_{\tilde{\theta}}(t_0) \le V_{z\theta}(t_0)$ (by definition) and it follows from (15) and (18c) that $\dot{V}_{\tilde{\theta}}(t) \le \dot{V}_{z_\theta}(t)$. Hence, by the comparison lemma, we have
$$V_{\tilde{\theta}}(t) \le V_{z\theta}(t), \quad \forall t \ge t_0. \tag{21}$$
and since $V_{\tilde{\theta}} = \tilde{\theta}^T \Sigma\, \tilde{\theta}$, it follows that
$$\|\tilde{\theta}(t)\|^2 \le \frac{V_{z\theta}(t)}{\lambda_{\min}(\Sigma(t))} = z_\theta^2(t), \quad \forall t \ge t_0. \tag{22}$$
Hence, if $\theta \in \Theta(t_0)$, then $\theta \in B(\hat{\theta}(t), z_\theta(t)),\ \forall t \ge t_0$.

■

4. ROBUST ADAPTIVE MPC

4.1 A Min-max Approach

The formulation of the min-max MPC consists of maximizing a cost function with respect to $\theta \in \Theta$, $\vartheta \in \mathcal{D}$ and minimizing over feedback control policies κ. The robust receding horizon control law is
$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) \triangleq \kappa^*(0, x, \hat{\theta}, z_\theta) \tag{23a}$$
$$\kappa^* \triangleq \arg \min_{\kappa(\cdot,\cdot,\cdot,\cdot)} J(x, \hat{\theta}, z_\theta, \kappa) \tag{23b}$$
where
$$J(x, \hat{\theta}, z_\theta, \kappa) \triangleq \max_{\theta\in\Theta,\ \vartheta\in\mathcal{D}} \int_0^T L(x^p, u^p) d\tau + W(x^p(T), \tilde{\theta}^p(T)) \tag{24a}$$
s.t. $\forall \tau \in [0,\, T]$
$$\dot{x}^p = f(x^p, u^p) + g(x^p, u^p)\theta + \vartheta, \ \ x^p(0) = x \tag{24b}$$
$$\dot{w^p} = g^T(x^p, u^p) - k_w\, w^p, \ \ w^p(0) = w \tag{24c}$$
$$(\dot{\Sigma}^{-1})^p = -(\Sigma^{-1})^p w^T w (\Sigma^{-1})^p, \quad (\Sigma^{-1})^p(0) = \Sigma^{-1} \tag{24d}$$
$$\dot{\hat{\theta}}^p = \text{Proj}\left\{\gamma\,(\Sigma^{-1})^p w^T (e - \hat{\eta}),\ \hat{\theta}\right\}$$
$$\tilde{\theta}^p = \theta - \hat{\theta}^p, \quad \hat{\theta}^p(0) = \hat{\theta} \tag{24e}$$
$$u^p(\tau) \triangleq \kappa(\tau, x^p(\tau), \hat{\theta}^p(\tau)) \in \mathbb{U} \tag{24f}$$
$$x^p(\tau) \in \mathbb{X}, \quad x^p(T) \in \mathbb{X}_f(\tilde{\theta}^p(T)) \tag{24g}$$

The effect of future parameter adaptation is also accounted for in this formulation. The conservativeness of the algorithm is reduced by parameterizing both W and $\mathbb{X}_f$ as functions of $\tilde{\theta}(T)$. While it is possible for the set Θ to contract upon θ over time, the robustness feature due to $\vartheta \in \mathcal{D}$ will still remain.

Algorithm 2. The MPC algorithm performs as follows: At sampling instant t_i

(1) **Measure** the current state of the plant $x(t)$ and obtain the current value of matrices w and Σ^{-1} from equations (3) and (9a) respectively
(2) Obtain the current value of parameter estimates $\hat{\theta}$ and uncertainty bound z_θ from (9b) and (18) respectively
If $\ z_\theta(t_i) \le z_\theta(t_{i-1}) - \|\hat{\theta}(t_i) - \hat{\theta}(t_{i-1})\|$
$$\hat{\theta} = \hat{\theta}(t_i), \quad z_\theta = z_\theta(t_i)$$
,
Else
$$\hat{\theta} = \hat{\theta}(t_{i-1}), \quad z_\theta = z_\theta(t_{i-1})$$
End
(3) **Solve** the optimization problem (23) and apply the resulting feedback control law to the plant until the next sampling instant
(4) **Increment** $i = i+1$. Repeat the procedure from step 1 for the next sampling instant.

4.2 Lipschitz-based Approach

In this section, we present a Lipschitz-based method whereby the nominal model rather than the unknown bounded system state is controlled, subject to conditions that ensure that given constraints are satisfied for all possible uncertainties. State prediction error bound is determined based on the Lipschitz continuity of the model. A knowledge of appropriate Lipschitz bounds for the x-dependence of the dynamics $f(x,u)$ and $g(x,u)$ are assumed as follows:

Assumption 4. A set of functions $\mathcal{L}_j : \mathbb{X} \times \mathbb{U} \to \mathbb{R}^+$, $j \in \{f, g\}$ are known which satisfy
$$\mathcal{L}_j(\mathbb{X}, u) \ge \min\left\{\mathcal{L}_j \;\middle|\; \sup_{x_1,x_2\in\mathbb{X}}\Big(\|j(x_1,u) - j(x_2,u)\| - \mathcal{L}_j\|x_1 - x_2\|\Big) \le 0\right\},$$
where for $j \equiv g$ is interpreted as an induced norm since $g(x,u)$ is a matrix.

Assuming a knowledge of the Lipschitz bounds for the x-dependence of the dynamics $f(x,u)$ and $g(x,u)$ as given in Assumption 4 and let $\Pi = z_\theta + \|\hat{\theta}\|$, a worst-case deviation $z_x^p \ge \max_{\theta\in\Theta} \|x - x^p\|$ can be generated from
$$\dot{z}_x^p = (\mathcal{L}_f + \mathcal{L}_g \Pi) z_x^p + \|g(x^p, u)\| z_\theta + M_\vartheta, \quad z_x^p(t_0) = 0. \tag{26}$$
Using this error bound, the robust Lipschitz-based MPC is given by
$$u = \kappa_{mpc}(x, \hat{\theta}, z_\theta) = u^*(0) \tag{27a}$$
$$u^*(.) \triangleq \arg \min_{u^p_{[0,T]}} J(x, \hat{\theta}, z_\theta, u^p) \tag{27b}$$
where

$$J(x,\hat{\theta},z_\theta,u^p) = \int_0^T L(x^p,u^p)d\tau + W(x^p(T),z_\theta^p) \quad (28a)$$

$s.t.\ \forall\tau\in[0,\,T]$

$$\dot{x}^p = f(x^p,u^p) + g(x^p,u^p)\hat{\theta}, \quad x^p(0)=x \quad (28b)$$

$$\dot{z}_x^p = (\mathcal{L}_f + \mathcal{L}_g\Pi)z_x^p + \|g^p\|z_\theta + M_\vartheta,\ z_x^p(0)=0 \quad (28c)$$

$$X^p(\tau) \triangleq B(x^p(\tau),z_x^p(\tau)) \subseteq \mathbb{X}, \quad u^p(\tau)\in\mathbb{U} \quad (28d)$$

$$X^p(T) \subseteq \mathbb{X}_f(z_\theta^p) \quad (28e)$$

The effect of the disturbance is built into the uncertainty cone $B(x^p(\tau),z_x^p(\tau))$ via (28c). Since the uncertainty bound is no more monotonically decreasing in this case, the uncertainty radius z_θ which appears in (28c) and in the terminal expressions of (28a) and (28e) are held constant over the prediction horizon. However, the fact that they are updated at sampling instants when z_θ shrinks reduces the conservatism of the robust MPC and enlarges the terminal domain that would otherwise have been designed based on a large initial uncertainty $z_\theta(t_0)$.

Algorithm 3. The Lipschitz-based MPC algorithm performs as follows: At sampling instant t_i

(1) **Measure** the current state of the plant $x = x(t_i)$
(2) **Obtain** the current value of the parameter estimates $\hat{\theta}$ and uncertainty bound z_θ from equations (9) and (18) respectively,
  If $\quad z_\theta(t_i) \le z_\theta(t_{i-1})$
$$\hat{\theta} = \hat{\theta}(t_i), \quad z_\theta = z_\theta(t_i)$$
  Else
$$\hat{\theta} = \hat{\theta}(t_{i-1}), \quad z_\theta = z_\theta(t_{i-1})$$
  End
(3) **Solve** the optimization problem (27) and apply the resulting feedback control law to the plant until the next sampling instant
(4) **Increment** i:=i+1; repeat the procedure from step 1 for the next sampling instant.

5. CLOSED-LOOP ROBUST STABILITY

Robust stabilization to the target set Ξ is guaranteed by appropriate selection of the design parameters W and X_f. The robust stability conditions require the satisfaction of the following criteria.

Criterion 5. The terminal penalty function $W : \mathbb{X}_f \times \tilde{\Theta}^0 \to [0,\,+\infty]$ and the terminal constraint function $\mathbb{X}_f : \tilde{\Theta}^0 \to \mathbb{X}$ are such that for each $(\theta,\,\hat{\theta},\,\tilde{\theta}) \in (\Theta^0\times\Theta^0\times\tilde{\Theta}^0_\epsilon)$, there exists a feedback $k_f(.,\hat{\theta}) : \mathbb{X}_f \to \mathbb{U}$ satisfying

(1) $0 \in \mathbb{X}_f(\tilde{\theta}) \subseteq \mathbb{X}, \quad \mathbb{X}_f(\tilde{\theta})$ closed
(2) $k_f(x,\hat{\theta}) \in \mathbb{U},\ \forall x\in\mathbb{X}_f(\tilde{\theta})$
(3) $W(x,\tilde{\theta})$ is continuous with respect to $x\in\mathbb{R}^{n_x}$
(4) $\forall\, x\in\mathbb{X}_f(\tilde{\theta})\backslash\Xi$, $\mathbb{X}_f(\tilde{\theta})$ is strongly positively invariant under $k_f(x,\hat{\theta})$ with respect to $\dot{x}\in f(x,k_f(x,\hat{\theta})) + g(x,k_f(x,\hat{\theta}))\Theta + \mathcal{D}$
(5) $L(x,k_f(x,\hat{\theta})) + \frac{\partial W}{\partial x}\mathcal{F}(x,k_f(x,\hat{\theta}),\theta,\vartheta) \le 0,\ \forall x\in\mathbb{X}_f(\tilde{\theta})\backslash\Xi$.

Criterion 6. For any $\tilde{\theta}_1,\ \tilde{\theta}_2\in\tilde{\Theta}^0$ $s.t.$ $\|\tilde{\theta}_2\|\le\|\tilde{\theta}_1\|$,

(1) $W(x,\tilde{\theta}_2) \le W(x,\tilde{\theta}_1), \ \forall x\in\mathbb{X}_f(\tilde{\theta}_1)$
(2) $\mathbb{X}_f(\tilde{\theta}_2) \supseteq \mathbb{X}_f(\tilde{\theta}_1)$

The revised condition C5 require W to be a local robust CLF for the uncertain system 1 with respect to $\theta\in\Theta$ and $\vartheta\in\mathcal{D}$.

5.1 Main Results

Theorem 7. Let $X_{d0} \triangleq X_{d0}(\Theta^0) \subseteq \mathbb{X}$ denote the set of initial states with uncertainty Θ^0 for which (23) has a solution. Assuming criteria 5 and 6 are satisfied, then the closed-loop system state x, given by (1,9,18,23), originating from any $x_0\in X_{d0}$ feasibly approaches the target set Ξ as $t\to+\infty$.

Proof: *Feasibility*: The closed-loop stability is based upon the feasibility of the control action at each sample time. Assuming, at time t, that an optimal solution $u^p_{[0,T]}$ to the optimization problem (23) exist and is found. Let Θ^p denote the estimated uncertainty set at time t and Θ^v denote the set at time $t+\delta$ that would result with the feedback implementation of $u_{[t,t+\delta]} = u^p_{[0,\delta]}$. Also, let x^p represents the worst case state trajectory originating from $x^p(0)=x(t)$ and x^v represents the trajectory originating from $x^v(0) = x+\delta v$ under the same feasible control input $u^v_{[\delta,T]} = u^p_{[\delta,T]}$. Moreover, let $X^a_{\Theta^b} \triangleq \{x^a|\,\dot{x}^a\in\mathcal{F}(x^a,u^p,\Theta^b)\triangleq f(x^a,u^p)+g(x^a,u^p)\Theta^b\}$.

Since the $u^p_{[0,T]}$ is optimal with respect to the worst case uncertainty scenario, it suffice to say that $u^p_{[0,T]}$ drives any trajectory $x^p\in X^p_{\Theta^p}$ into the terminal region $\mathbb{X}^p_f$. Since Θ is non-expanding over time, we have $\Theta^v\subseteq\Theta^p$ implying $x^v\in X^p_{\Theta^v}\subseteq X^p_{\Theta^p}$. The terminal region $\mathbb{X}^p_f$ is strongly positively invariant for the nonlinear system (1) under the feedback $k_f(.,.)$, the input constraint is satisfied in $\mathbb{X}^p_f$ and $\mathbb{X}^v_f\supseteq\mathbb{X}^p_f$ by criteria 2.2, 2.4 and 3.2 respectively. Hence, the input $u=[u^p_{[\delta,T]},k_{f[T,T+\delta]}]$ is a feasible solution of (23) at time $t+\delta$ and by induction, the optimization problem is feasible for all $t\ge0$.

Stability: The stability of the closed-loop system is established by proving strict decrease of the optimal cost $J^*(x,\hat{\theta},z_\theta) \triangleq J(x,\hat{\theta},z_\theta,\kappa^*)$. Let the trajectories $(x^p,\,\hat{\theta}^p,\,\tilde{\theta}^p,z_\theta^p)$ and control u^p correspond to any worst case minimizing solution of $J^*(x,\hat{\theta},z_\theta)$. If $x^p_{[0,T]}$ were extended to $\tau\in[0,T+\delta]$ by implementing the feedback $u(\tau)=k_f(x^p(\tau),\hat{\theta}^p(\tau))$ on $\tau\in[T,T+\delta]$, then criterion 5(5) guarantees the inequality

$$\int_T^{T+\delta} L(x^p,k_f(x^p,\hat{\theta}^p))d\tau + W(x^p_{T+\delta},\tilde{\theta}^p_T) - W(x^p_T,\tilde{\theta}^p_T) \le 0 \quad (29)$$

where in (29) and in the remainder of the proof, $x^p_\sigma \triangleq x^p(\sigma)$, $\tilde{\theta}^p_\sigma \triangleq \tilde{\theta}^p(\sigma)$, for $\sigma = T,\,T+\delta$.

The optimal cost

$$J^*(x,\hat{\theta},z_\theta) = \int_0^T L(x^p,u^p)d\tau + W(x^p_T,\tilde{\theta}^p_T)$$

$$\ge \int_0^T L(x^p,u^p)d\tau + W(x^p_T,\tilde{\theta}^p_T) \quad (30)$$

$$+\int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p))d\tau + W(x^p_{T+\delta}, \tilde{\theta}^p_T) - W(x^p_T, \tilde{\theta}^p_T) \quad (31)$$

$$\geq \int_0^\delta L(x^p, u^p)d\tau + \int_\delta^T L(x^p, u^p)d\tau \quad (32)$$

$$+\int_T^{T+\delta} L(x^p, k_f(x^p, \hat{\theta}^p))d\tau + W(x^p_{T+\delta}, \tilde{\theta}^p_{T+\delta}) \quad (33)$$

$$\geq \int_0^\delta L(x^p, u^p)d\tau + J^*(x(\delta), \hat{\theta}(\delta), z_\theta(\delta)) \quad (34)$$

Then, it follows from (34) that

$$J^*(x(\delta), \hat{\theta}(\delta), z_\theta(\delta)) - J^*(x, \hat{\theta}, z_\theta) \leq -\int_0^\delta L(x^p, u^p)d\tau \leq -\int_0^\delta \mu_L(\|x, u\|)d\tau. \quad (35)$$

where μ_L is a class $\mathcal{K}_\infty$ function. Hence $x(t) \to 0$ asymptotically.

Remark 8. In the above proof,

- (31) is obtained using inequality (29)
- (33) follows from criterion 5.1 and the fact that $\|\tilde{\theta}\|$ is non-increasing
- (34) follows by noting that the last 3 terms in (33) is a (potentially) suboptimal cost on the interval $[\delta, T+\delta]$ starting from the point $(x^p(\delta), \hat{\theta}^p(\delta))$ with associated uncertainty set $B(\hat{\theta}^p(\delta), z^p_\theta(\delta))$.

The closed-loop stability is established by the feasibility of the control action at each sample time and the strict decrease of the optimal cost J^*. The proof follows from the fact that the control law is optimal with respect to the worst case uncertainty $(\theta,\ \vartheta) \in (\Theta,\ \mathcal{D})$ scenario and the terminal region $\mathbb{X}^p_f$ is strongly positively invariant for (1) under the (local) feedback $k_f(.,.)$. ■

Theorem 9. Let $X'_{d0} \triangleq X'_{d0}(\Theta^0) \subseteq \mathbb{X}$ denote the set of initial states for which (27) has a solution. Assuming Assumption 4 and Criteria 5 and 6 are satisfied, then the origin of the closed-loop system given by (1,9,18,27) is feasibly asymptotically stabilized from any $x_0 \in X'_{d0}$ to the target set Ξ.

The proof of the Lipschitz-based control law follows from that of theorem 7.

6. SIMULATION EXAMPLE

To illustrate the effectiveness of the proposed design, we consider the regulation of the CSTR subject to an additional disturbance on the temperature dynamic:

$$\dot{C}_A = \frac{q}{V}(C_{Ain} - C_A) - k_0 \exp\left(\frac{-E}{R\,T_r}\right) C_A$$

$$\dot{T}_r = \frac{q}{V}(T_{in} - T_r) - \frac{\Delta H}{\rho\, c_p} k_0 \exp\left(\frac{-E}{R\,T_r}\right) C_A + \frac{UA}{\rho\, c_p\, V}(T_c - T_r) + \vartheta$$

where $\vartheta(t)$ is an unknown function of time. We also assume that the reaction kinetic constant k_0 and ΔH are only nominally known.

It is assumed that reaction kinetic constant k_0 and heat of reaction ΔH are only nominally known and parameterized as $k_0 = \theta_1 \times 10^{10}\,\text{min}^{-1}$ and $\Delta H\, k_0 = -\theta_2 \times 10^{15}$ J/mol min with the parameters satisfying $0.1 \leq \theta_1 \leq 10$ and $0.1 \leq \theta_2 \leq 10$. The objective is to adaptively regulate the unstable equilibrium $C^{eq}_A = 0.5$ mol/l, $T^{eq}_r = 350$ K, $T^{eq}_c = 300$ K while satisfying the constraints $0 \leq C_A \leq 1$, $280 \leq T_r \leq 370$ and $280 \leq T_c \leq 370$. The nominal operating conditions, which corresponds to the given unstable equilibrium are taken from Magni et al. (2001): q=100 l/min, V=100 l, ρ=1000 g/l, c_p=0.239 J/g K, E/R = 8750 K, $UA = 5{\times}10^4$ J/min K, $C_{Ain} = 1$ mol/l and $T_{in} = 350$ K.

The control objective is to robustly regulate the reactor temperature and concentration to the (open loop) unstable equilibrium $C^{eq}_A = 0.5$ mol/l, $T^{eq}_r = 350$ K, $T^{eq}_c = 300$ K by manipulating the temperature of the coolant stream T_c.

Defining $x = [\frac{C_A - C^{eq}_A}{0.5}, \frac{T_r - T^{eq}_r}{20}]'$, $u = \frac{T_c - T^{eq}_c}{20}$, the stage cost $L(x, u)$ was selected as a quadratic function of its arguments:

$$L(x, u) = x^T\, Q_x\, x + u^T\, R_u\, u \quad (36a)$$

$$Q_x = \begin{bmatrix} 0.5 & 0 \\ 0 & 1.1429 \end{bmatrix} \qquad R_u = 1.333. \quad (36b)$$

The terminal penalty function used is a quadratic parameter-dependent Lyapunov function $W(x, \theta) = x^T P(\theta) x$ for the linearized system. Denoting the closed-loop system under a local robust stabilizing controller $u = k_f(\theta)\, x$ as $\dot{x} = A_{cl}(\theta) x$. The matrix $P(\theta) := P_0 + \theta_1 P_1 + \theta_2 P_2 + \ldots \theta_{n\theta} P_{n\theta}$ was selected to satisfy the Lyapunov system of LMIs

$$P(\theta) > 0$$
$$A_{cl}(\theta)^T P(\theta) + P(\theta) A_{cl}(\theta) < 0$$

for all admissible values of θ. Since θ lie between known extrema values, the task of finding $P(\theta)$ reduces to solving a finite set of linear matrix inequalities by introducing additional constraints Gahinet et al. (1996). For the initial nominal estimate $\theta^0 = 5.05$ and $z^0_\theta = 4.95$, the matrix $P(\theta^0)$ obtained is

$$P(\theta^0) = \begin{bmatrix} 0.6089 & 0.1134 \\ 0.1134 & 4.9122 \end{bmatrix} \quad (37)$$

and the corresponding terminal region is

$$\mathbb{X}_f = \{x : x^T P(\theta^0)\, x \leq 0.25\}. \quad (38)$$

For simulation purposes, the disturbance is selected as a fluctuation of the inlet temperate $\vartheta(t) = 0.01\, T_{in} \sin(3\text{t})$ and the true values of the unknown parameters were also chosen as $k_0 = 7.2 \times 10^{10}\text{min}^{-1}$ and ΔH = - 5.0×10^4 J/mol. The stage cost (36), terminal penalty (37) and terminal region (38) were used. The Lipschitz-based approach was used for the controller calculations and the result was implemented according to Algorithm 3. As depicted in Figures 1 to 3, the robust adaptive MPC drives the system to a neighborhood of the equilibrium while satisfying the imposed constraints and achieves parameter convergence. Figure 4 shows that the uncertainty bound z_θ also reduces over time.

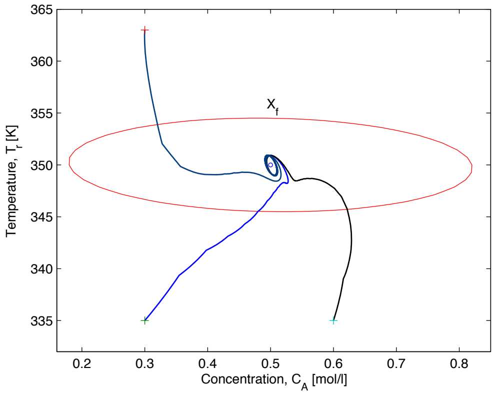

Fig. 1. Closed-loop reactor trajectories under additive disturbance $\vartheta(t)$

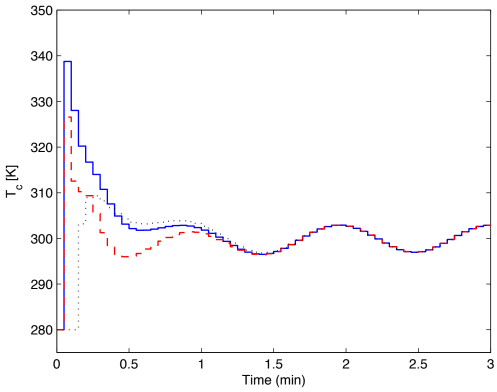

Fig. 2. Closed-loop input profiles for states starting at different initial conditions $(C_A(0),\ T_r(0))$: $(0.3,\ 335)$ is solid line, $(0.6,\ 335)$ is dashed line and $(0.3,\ 363)$ is the dotted line

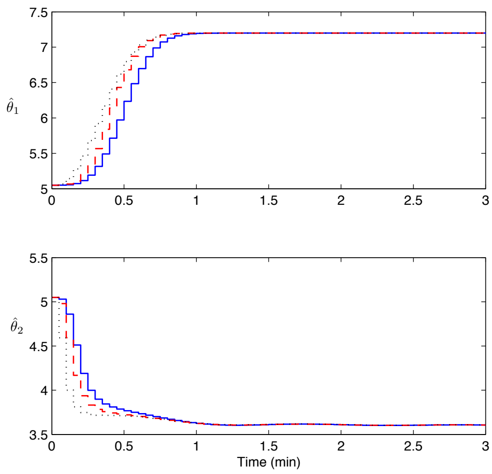

Fig. 3. Closed-loop parameter estimates profile for states starting at different initial conditions $(C_A(0),\ T_r(0))$: $(0.3,\ 335)$ is solid line, $(0.6,\ 335)$ is dashed line and $(0.3,\ 363)$ is the dotted line

7. CONCLUSIONS

The adaptive MPC design technique is extended to constrained nonlinear systems with both parametric and time

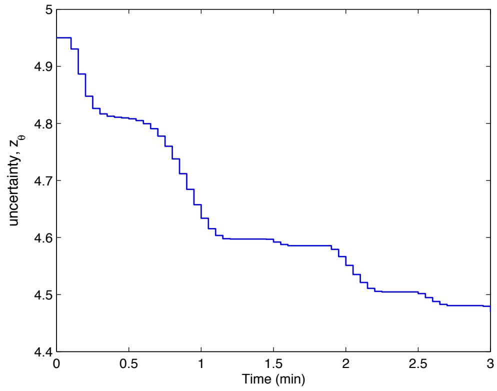

Fig. 4. Closed-loop uncertainty bound trajectories for initial condition $(C_A,\ T_r) = (0.3,\ 335)$

varying disturbances. The proposed robust controller updates the plant model online when model improvement is guaranteed. The embedded adaptation mechanism enables us to construct less conservative terminal design parameters based upon subsets of the original parametric uncertainty. While the introduced conservatism/computation complexity due to the parametric uncertainty reduces over time, the portion due to the disturbance $\vartheta \in \mathcal{D}$ remains active for all time.

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