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DESIGN OF ROBUST GAIN-SCHEDULED MPC CONTROLLERS FOR NONLINEAR PROCESSES

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Abstract: A methodology is proposed for the analysis and design of a robust gain-scheduled Model predictive Controllers (MPC) for nonlinear chemical processes. The stability and performance tests can be formulated as a finite set of linear matrix inequalities (LMI). A simulation study of a 2x2 system indicates that this approach can provide useful robust controllers, which guarantee closed-loop robust stability and performance.

Keywords: robust control, nonlinear systems, model predictive control.

1. INTRODUCTION

MPC is a widely accepted control algorithm in the chemical industry. In most of industrial applications the design of the controller is based on a nominal linear model of the process. Such control systems that provide optimal performance for a particular linear model may perform poorly when implemented on a physical nonlinear system (Zheng and Morari, 1993).

Due to the process nonlinearity, a system behaves differently at different operating conditions. Therefore, controllers that are based on one single linear model have to be tuned for robustness to model errors or uncertainty between the nominal model and the actual process behavior.

The basic philosophy in the literature for optimizing the performance of MPC-based design algorithms that explicitly account for model-plant error is to modify the on-line minimization problem to a min-max problem, where the worst-case value of the objective function is minimized over the set of plants that account for the nominal model and uncertainty (Campo and Morari, 1987; Zheng and Morari, 1993).

This approach is clearly computationally much more demanding than solving it for a nominal plant. To simplify the computational complexity, one must choose simplistic, albeit unrealistic, model uncertainty descriptions, e.g., fewer impulse response coefficients. Also, controllers that are tuned for robustness to model errors between a nominal linear model and the actual nonlinear process output tend to have been proposed lately to address the nonlinearity of the process and to improve the closed loop performance (Allgower, *et al*., 2000). However, it is difficult to guarantee stability and performance for these controllers and they generally require a nonlinear mechanistic model of the process that is often difficult to obtain.

In this work, an alternative gain-scheduled MPC design approach is proposed, which allows explicit consideration of the nonlinear behavior of the process. To design this controller, instead of using one step response model for output prediction, several linear step models will be identified for different regions defined based on the values of the manipulated variable *u*. Then, for each of these models, a linear MPC calculation can be conducted based on the current value of *u*. Thus, the controller is referred to as a gain-scheduled MPC because the

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MPC matrix gain changes as a function of the value of the manipulated variable u.

It is assumed in this work that a mechanistic model of the process is not available. Thus, for the purpose of controller design and robustness analysis, an empirical nonlinear state-affine model (Knapp and Budman, 2001) to be identified from experiments is used as follows:

$$
\begin{aligned}
&\mathbf{x}(t+1)=\mathbf{F}(\mathbf{u})\mathbf{x}(t)+\mathbf{G}(\mathbf{u})\mathbf{u}(t)\\
&\mathbf{F}(\delta)=\{\mathbf{F}_0+\sum_{i=1}^{K-1}\mathbf{F}_i u_{i,t}\}\\
&\mathbf{G}(\delta)=\{\mathbf{G}_1+\sum_{i=1}^{K-1}\mathbf{G}_{i+1} u_{i,t}\}\\
&\mathbf{y}(t)=\mathbf{H}_0\mathbf{x}(t)+\mathbf{W_f}d(t)\\
&d(t+1)=BWd(t)+(1-BW)\nu(t)
\end{aligned}
\qquad (1)
$$

For a process given by the state-affine model , it is valid to assume that in a small neighborhood of a pre-selected nominal operating point, i.e. for $u(t) << 1$, the process can be accurately modeled by the linear part of the state-affine model. The uncertainty of the system will be assumed to be the difference between the nonlinear model and the nominal linear model. It is also assumed that all of the uncertainty in the state-affine model is due to the time-varying nonlinearity of the state-affine model around this operating point. It is therefore possible to describe the model uncertainty perturbation $\delta_{i,t}$ in the following form:

$$\delta_{i,t} = u(t)^i \quad (2)$$

Some of the advantages of this model for the purpose of analysis are: 1)-it can be easily identified from data, 2)- by considering the high order of u to be the uncertainty elements, it is easy to split the model above into a linear part and a nonlinear part and to formulate analysis tests for robust stability and performance (Budman and Knapp, 2001) and, 3)- the uncertainty elements can be easily bounded based on the limits of the manipulated variable u. Also, step models for designing the gain scheduled controller mentioned above can be easily extracted from the model given by (1).

The closed-loop system composed of the state-affine model combined with a state-space formulation of the gain-scheduled MPC controller is studied for robust stability and performance through LMI's tests. Thus, the proposed gain-scheduled MPC controller is designed to ensure closed-loop system robust stability and suitable performance.

This paper is organized as follows. Section 2 develops the state-space formulation of the unconstrained MPC control law. The closed-loop system equations, composed of the state-affine model and the MPC controller, are formulated as an affine parameter-dependent system. In Section 3, the procedures for the design and optimization of the robust gain-scheduled MPC are detailed. In Section 4, the above proposed approach is applied to a 2x2 system, leading to results and conclusions summarized in Section 5.

2. UNCONSTRAINED MPC ALGORITHM IN STATE-SPACE FORM

In order to formulate robust stability and robust performance tests, a state-space formulation of the MPC controller (Zanovello and Budman, 1999) is desired. A standard unconstrained MPC formulation is used. Consider a multiple-input-multiple-output (MIMO) system with n_u inputs and n_y outputs, to be controlled by a MPC controller, with prediction horizon p and control horizon m.

The model update vector is defined as follows:

$$
\begin{aligned}
&\overline{\mathbf{Y}}(t)=\mathbf{M}_I\overline{\mathbf{Y}}(t-1)+\mathbf{s}^u\Delta\mathbf{u}(t-1)\\
&\mathbf{M}_I=\begin{bmatrix}
\mathbf{0}_{n_y\times n_y} & \mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_y} & \cdot & \mathbf{0}_{n_y\times n_y}\\
\mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_y} & \mathbf{I}_{n_y} & \cdot & \mathbf{0}_{n_y\times n_y}\\
\cdot & \cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot & \mathbf{I}_{n_y}\\
\mathbf{0}_{n_y\times n_y} & \mathbf{0}_{n_y\times n_y} & \cdot & \mathbf{0}_{n_y\times n_y} & \mathbf{I}_{n_y}
\end{bmatrix}_{nn_y\times nn_y}
\end{aligned}
\qquad (3)
$$

The step response coefficient $\mathrm{S}_\mathrm{i}^\mathrm{u}$ and impulse response coefficient h_i are defined as follows:

$$
\mathbf{S}_i^u=\begin{bmatrix}
S_{1,1,i}^u & S_{1,2,i}^u & \cdot & S_{1,n_u,i}^u\\
S_{2,1,i}^u & \cdot & \cdot & 0\\
\cdot & \cdot & \cdot & \cdot\\
S_{n_y,1,i}^u & S_{n_y,2,i}^u & \cdot & S_{n_y,n_u,i}^u
\end{bmatrix}_{n_y\times n_u}
\quad
\mathbf{h}_i=\begin{bmatrix}
h_{1,1,i} & h_{1,2,i} & \cdot & h_{1,n_u,i}\\
h_{2,1,i} & \cdot & \cdot & 0\\
\cdot & \cdot & \cdot & \cdot\\
h_{n_y,1,i} & h_{n_y,2,i} & \cdot & h_{n_y,n_u,i}
\end{bmatrix}_{n_y\times n_u}
\qquad (4)
$$

where $S_{l,k,i}^u$ and $\mathbf{h}_{l,k,i}$ are the i^{th} response coefficient describing the effect of k^{th} input on l^{th} output. The step response vector $\mathbf{s}^u$ and the step response matrix $\mathbf{S}^u$ are given as follows:

$$
\begin{aligned}
&\mathbf{s}^u=\begin{bmatrix}\mathbf{S}_1^u & \cdots & \mathbf{S}_n^u\end{bmatrix}^T\\
&\mathbf{S}^u=\begin{bmatrix}
\mathbf{S}_1^u & 0 & \cdot & 0\\
\mathbf{S}_2^u & \mathbf{S}_1^u & \cdot & 0\\
\cdot & \cdot & \cdot & \cdot\\
\mathbf{S}_n^u & \mathbf{S}_{n-1}^u & \cdot & \mathbf{S}_{n-m+1}^u
\end{bmatrix}_{nn_y\times mn_u}
\end{aligned}
\qquad (5)
$$

where m is referred to as the control horizon. The p-step-ahead prediction vector $\overline{\mathbf{Y}}(t+1/t)$ is defined as follows:

$$
\begin{aligned}
&\overline{\mathbf{Y}}(t+1/t) = \mathbf{M}_p\overline{\mathbf{Y}}(t) + \mathbf{W}(t+1/t) + \mathbf{S}_p^u\Delta\mathbf{U}(t) \\
&\mathbf{M}_p = \begin{bmatrix}\mathbf{I}_{pn_y\times pn_y} & 0\end{bmatrix}_{pn_y\times nn_y}\mathbf{M}_I
\end{aligned} \quad (6)
$$

$\mathbf{S}_p^u$ is the sub-matrix of the first p rows of $\mathbf{S}^u$ and p is the prediction horizon. The vector $\mathbf{W}(t+1/t)$ is defined to represent the unmeasured disturbance and model/plant mismatch. Making the common assumption that the disturbances are step-like, the disturbance vector $\mathbf{W}(t+1/t)$ is given as follows:

$$
\begin{aligned}
&\mathbf{W}(t+1/t) = \mathbf{N}_2[\mathbf{y}(t) - \overline{\mathbf{y}}(t)] \\
&\mathbf{N}_2 = \begin{bmatrix}\mathbf{I}_{n_y} & . & \mathbf{I}_{n_y}\end{bmatrix}^T_{n_y\times pn_y}
\end{aligned} \quad (7)
$$

where $\mathbf{y}(t)$ is the vector of the n_y measured values, and $\overline{\mathbf{y}}(t)$ is the vector composed of the first n_y elements of the vector $\overline{\mathbf{Y}}(t)$.

The objective function is given as follows:

$$
\min_{\Delta\mathbf{u}(t)} \frac{1}{2}\{\left\|\mathbf{\Gamma}[\overline{\mathbf{Y}}(t+1\,|\,t) - \mathbf{R}(t+1)]\right\|^2 + \left\|\mathbf{\Lambda}\Delta\mathbf{U}\right\|^2\} \quad (8)
$$

where $\mathbf{\Lambda}, \mathbf{\Gamma}$ are positive-definite weighting matrices for the manipulated and controlled variables respectively. Then, the least squares solution of the minimization problem with the cost function given by (8) is given as follows:

$$
\begin{aligned}
&\Delta\mathbf{u}(t) = \mathbf{K}_{MPC}\boldsymbol{\varepsilon}(t+1\,|\,t) \\
&\mathbf{K}_{MPC} = \mathbf{e}({\mathbf{S}_p^u}^T\mathbf{\Gamma}^T\mathbf{\Gamma}\mathbf{S}_p^u + \mathbf{\Lambda}^T\mathbf{\Lambda})^{-1}{\mathbf{S}_p^u}^T\mathbf{\Gamma}^T\mathbf{\Gamma} \\
&\mathbf{e} = \begin{bmatrix}\mathbf{I}_{n_u} & \mathbf{0}_{n_u} & \cdots & \mathbf{0}_{n_u}\end{bmatrix}_{n_u\times mn_u}
\end{aligned} \quad (9)
$$

$\boldsymbol{\varepsilon}(t+1\,|\,t)$ is the feedback error vector defined as follows:

$$
\boldsymbol{\varepsilon}(t+1\,|\,t) = \mathbf{R}(t+1) - \mathbf{M}_p\overline{\mathbf{Y}}(t) - \mathbf{W}(t+1\,|\,t) \quad (10)
$$

The controller state vector $\mathbf{U}(t-1)$ is defined as follows:

$$
\mathbf{U}^T(t-1) = \begin{bmatrix}\mathbf{u}(t-1) & \ldots & \mathbf{u}(t-n)\end{bmatrix}_{1\times nn_u} \quad (11)
$$

Assuming $\mathrm{R}=0$ without loss of generality, the following is obtained from (11), (9) and (10):

$$
\begin{aligned}
&\mathbf{U}(t) = \mathbf{T}_2\mathbf{U}(t-1) + \\
&\mathbf{T}_1\mathbf{K}_{MPC}[-\mathbf{M}_p\overline{\mathbf{Y}}(t) - \mathbf{W}(t+1\,|\,t)]
\end{aligned} \quad (12)
$$

$$
\mathbf{T}_1 = \begin{bmatrix}\mathbf{I}_{n_u} \\ \mathbf{0} \\ . \\ . \\ \mathbf{0}\end{bmatrix}
\mathbf{T}_2 = \begin{bmatrix}
\mathbf{I}_{n_u} & \mathbf{0} & \mathbf{0} & . & \mathbf{0} \\
\mathbf{I}_{n_u} & \mathbf{0} & \mathbf{0} & . & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{n_u} & \mathbf{0} & . & . \\
. & . & . & . & . \\
\mathbf{0} & . & . & \mathbf{I}_{n_u} & \mathbf{0}
\end{bmatrix}
$$

The controller output $\mathrm{u(t)}$ is defined as follows:

$$
\begin{aligned}
&\mathbf{u}(t) = \mathbf{e}_1\mathbf{U}(t-1) + \Delta\mathbf{u}(t) \\
&\mathbf{e}_1 = \begin{bmatrix}\mathbf{I}_{n_u} & \mathbf{0}_{n_u\times n_u} & . & \mathbf{0}_{n_u\times n_u}\end{bmatrix}_{n_u\times nn_u}
\end{aligned} \quad (13)
$$

Using the relation $\mathbf{S}_{l,k,j} = \sum_{i=1}^{j}\mathbf{h}_{l,k,i}$, $j = 1,2,\ldots,n$, the model update vector $\overline{\mathbf{Y}}(t)$ is also given by:

$$
\begin{aligned}
&\overline{\mathbf{Y}}(t) = \begin{bmatrix}\overline{\mathbf{y}}(t) & \cdots & \overline{\mathbf{y}}(t+n-1)\end{bmatrix}^T \\
&= \begin{bmatrix}
\mathbf{S}_1^u & \mathbf{h}_2 & . & . & \mathbf{h}_n \\
\mathbf{S}_2^u & \mathbf{h}_3 & . & \mathbf{h}_n & \mathbf{0} \\
. & . & . & . & . \\
. & . & . & . & . \\
\mathbf{S}_n^u & \mathbf{0} & . & . & \mathbf{0}
\end{bmatrix}
\begin{bmatrix}\mathbf{u}(t-1) \\ \mathbf{u}(t-2) \\ . \\ . \\ \mathbf{u}(t-n)\end{bmatrix} \\
&\overline{\mathbf{Y}}(t) = \mathbf{H}_{nn_y\times nn_u}\mathbf{U}(t-1)
\end{aligned} \quad (14)
$$

The predicted output is given as follows:

$$
\begin{aligned}
&\overline{\mathbf{y}}(t) = \begin{bmatrix}\mathbf{I}_{n_y} & \mathbf{0}_{n_y\times n_y} & . & \mathbf{0}_{n_y\times n_y}\end{bmatrix}_{n_y\times nn_y}\overline{\mathbf{Y}}(t) \\
&= \mathbf{e}_2\mathbf{H}\mathbf{U}(t-1)
\end{aligned} \quad (15)
$$

To analyze the robustness of the closed loop system the state affine model given in equation 1 is used to model the process. This model (Knapp and Budman, 2001; Gao and Budman, 2005) has been shown to correctly describe the nonlinear behavior of the process.

From (14), (7) and (15), equation (12) can be rewritten as:

$$
\begin{aligned}
&\mathbf{U}(t) = \mathbf{E}_2\mathbf{U}(t-1) - \mathbf{T}_1\mathbf{K}_{MPC}\mathbf{N}_2(\mathbf{H}_0\mathbf{x}(t) + \mathbf{W}_\mathbf{f}d(t)) \\
&\mathbf{E}_2 = \mathbf{T}_2 - \mathbf{T}_1\mathbf{K}_{MPC}(\mathbf{M}_p\mathbf{H} - \mathbf{N}_2\mathbf{e}_2\mathbf{H})
\end{aligned} \quad (16)
$$

The control action can be calculated from the following expression:

$$
\begin{aligned}
&\mathbf{u}(t) = \mathbf{C}_{u1}\mathbf{U}(t-1) - \mathbf{K}_{MPC}\mathbf{N}_2(\mathbf{H}_0\mathbf{x}(t) + \mathbf{W}_\mathbf{f}d(t)) \\
&\mathbf{C}_{u1} = \mathbf{e}_1 - \mathbf{K}_{MPC}(\mathbf{M}_p\mathbf{H} - \mathbf{N}_2\mathbf{e}_2\mathbf{H})
\end{aligned} \quad (17)
$$

Then, a state-space representation of the MPC controller can be obtained by combining (16) and (17) as follows:

$$\begin{bmatrix}\mathbf{U}(t)\\ \mathbf{u}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{E}_2 & -\mathbf{T}_1\mathbf{K}_{mpc}\mathbf{N}_2\\ \mathbf{C}_{u1} & -\mathbf{K}_{mpc}\mathbf{N}_2\end{bmatrix}\begin{bmatrix}\mathbf{U}(t-1)\\ \mathbf{y}(t)\end{bmatrix} \quad (18)$$

From (1) and (18), the closed-loop system is obtained by combining the state-affine model and the MPC controller equations into the following equation:

$$\begin{bmatrix}\mathbf{x}(t+1)\\ \mathbf{U}(t)\\ d(t+1)\\ \hline \mathbf{y}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{A}(\boldsymbol{\delta}_t) & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \mathbf{U}(t-1)\\ d(t)\\ \hline \nu(t)\end{bmatrix}$$

$$\mathbf{A}(\boldsymbol{\delta}_t)=\begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{13}\\ -\mathbf{T}_1\mathbf{K}_{MPC}\mathbf{N}_2\mathbf{H}_0 & \mathbf{E}_2 & -\mathbf{T}_1\mathbf{K}_{MPC}\mathbf{N}_2\mathbf{W_f}\\ \mathbf{0} & \mathbf{0} & BW\end{bmatrix} \quad (19)$$

$$\mathbf{A}_{11}=\mathbf{F}(\delta)-\mathbf{G}(\delta)\mathbf{K}_{MPC}\mathbf{N}_2\mathbf{H}_0,\mathbf{A}_{12}=\mathbf{G}(\delta)\mathbf{C}_{u1}$$

$$\mathbf{A}_{13}=-\mathbf{G}(\delta)\mathbf{K}_{MPC}\mathbf{N}_2\mathbf{W_f}$$

$$\mathbf{B}=\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\\ 1-BW\end{bmatrix},\mathbf{C}=\begin{bmatrix}\mathbf{H}_0^T\\ \mathbf{0}^T\\ \mathbf{W_f^T}\end{bmatrix}^T,\mathbf{D}=[\mathbf{0}]$$

The above state-space system representation is used for robust stability and performance analysis as described in the next section.

3. ROBUST GAIN-SCHEDULED MPC DESIGN

For open-loop stable plants, the stability and performance of the closed-loop system depends on the MPC design parameters, m, p, Λ and Γ, and step response coefficients. In this work, for simplicity, Λ is the only parameter considered for tuning whereas the other parameters are assumed constant.

For the design of a gain-scheduled MPC controller for a MIMO system with n_u inputs, the overall range of change of each input variable $u_i(t), i=1,2,\dots,n_u$ is discretized into $k_i, i=1,2,\cdots,n_u$ regions. The impulse model H(u) is identified in each of these sub-ranges from equation (1). An optimal value of the input weight matrix $\mathbf{\Lambda}$ is selected for each one of these sub-ranges resulting in a gain-scheduled MPC algorithm.

The closed-loop system given by (19) has affine-parameter dependence with respect to the uncertain parameters $\delta_{i,t}$'s, and this allows the formulation of the robust stability and performance conditions developed by Gao and Budman (2005) to the design of MPC controllers given by (18). The robust stability condition (Gao and Budman, 2005) is:

$$\mathbf{A}(\omega)^T\mathbf{PA}(\omega)-\mathbf{P}<0 \quad \textit{for all } \omega\in\Re \quad (20)$$

where $\Re$ denotes the vertices or corners of the parameter box. For robust performance analysis, the system performance index is defined from the ratio between the error to disturbances: $\|\mathbf{e}\|_{\ell_2}<\gamma\|\mathbf{v}\|_{\ell_2}$. The objective of the controller optimization problem is to minimize the parameter γ according to a GEVP (Generalized Eigenvalue Problem) that can be formulated and solved using MATLAB as follows:

$$\min_{\mathbf{P}}\gamma^2$$

$$s.t.\quad \mathbf{K}(\omega,\mathbf{P})<\gamma^2\mathbf{L}, \textit{for} \quad \textit{all} \quad \omega\in\Re$$

$$\mathbf{K}=\begin{bmatrix}\mathbf{A}(\omega)^T\mathbf{PA}(\omega)-\mathbf{P} & \mathbf{A}(\omega)^T\mathbf{PB} & \mathbf{C}^T\\ \mathbf{B}^T\mathbf{PA}(\omega) & \mathbf{B}^T\mathbf{PB} & \mathbf{D}^T\\ \mathbf{C} & \mathbf{D} & -\mathbf{I}\end{bmatrix} \quad (21)$$

$$\mathbf{L}=\begin{bmatrix}0&0&0\\0&\mathbf{I}&0\\0&0&0\end{bmatrix}$$

4. DESIGN CASE STUDY

To illustrate the design technique, a simple 2-input-2-output example is used. The state-affine model has the form of with $\delta_{1,t}=u_1(t),\delta_{2,t}=u_2(t)$, and the model coefficient matrices are as follows:

$$\mathbf{F}_0=\begin{bmatrix}0.1188 & -0.0346\\ -2.3416 & 0.0937\end{bmatrix},\mathbf{F}_1=\begin{bmatrix}0.1076 & 0\\ 1.2289 & 0\end{bmatrix}$$

$$\mathbf{G}_1=\begin{bmatrix}1&0\\0&1\end{bmatrix},\mathbf{G}_2=\begin{bmatrix}0&0.1\\0.1&0\end{bmatrix}, \quad (22)$$

$$\mathbf{G}_3=\begin{bmatrix}-0.01 & -0.0159\\ -0.0508 & -0.0928\end{bmatrix}$$

$$\mathbf{H}_0=\begin{bmatrix}0.1755 & -0.0382\\ 0 & 0.1\end{bmatrix}, \mathbf{W_f}=\begin{bmatrix}1 & 0\end{bmatrix}^T, BW=0.8$$

Assuming even discretization of each one of the manipulated variables u into two regions, the following controller, referred to as GSMPC, is proposed for a total of four operating regions:

$$\begin{array}{lll} I & \begin{bmatrix}-0.3\le u_1\le 0\\ -0.3\le u_2\le 0\end{bmatrix} & MPC_{11}(\mathbf{K}_{MPC11},\mathbf{\Lambda}_{11})\\ II & \begin{bmatrix}-0.3\le u_1\le 0\\ 0<u_2\le 0.3\end{bmatrix} & MPC_{12}(\mathbf{K}_{MPC12},\mathbf{\Lambda}_{12})\\ III & \begin{bmatrix}0<u_1\le 0.3\\ -0.3\le u_2\le 0\end{bmatrix} & MPC_{21}(\mathbf{K}_{MPC21},\mathbf{\Lambda}_{21})\\ IV & \begin{bmatrix}0<u_1\le 0.3\\ 0<u_2\le 0.3\end{bmatrix} & MPC_{22}(\mathbf{K}_{MPC22},\mathbf{\Lambda}_{22})\end{array} \quad (23)$$

where $\mathrm{MPC_{ij}}(\mathrm{K_{MPCij}},\Lambda_{\mathrm{ij}})$ refers to the ij^{th} MPC controller, where u_1 evolves within the i^{th} region, and u_2 evolves within the j^{th} region. Each Λ_{ij} is of the following form:

$$\mathbf{\Lambda}_{ij}=\begin{bmatrix}\lambda_{1i} & \\ & \lambda_{2j}\end{bmatrix} \quad \begin{matrix}i=1,2\\ j=1,2\end{matrix} \quad (24)$$

K_{MPCij} is calculated based on the step response conducted between the extreme values of u of each of the regions defined in (23) in terms of u. For example, K_{MPC12} will be calculated using a step response corresponding to

$$\left[-0.3 \le u_1 \le 0, \quad 0 < u_2 \le 0.3\right].$$

For comparison, a linear MPC controller, referred to as LMPC in the sequel, will be designed with the following form:

$$for \ \begin{bmatrix} -0.3 \le u_1 \le 0.3 \\ -0.3 \le u_2 \le 0.3 \end{bmatrix} \ MPC(\mathbf{K}_{MPC}, \mathbf{\Lambda}(\lambda_{11}, \lambda_{22})) \quad (25)$$

where, λ_{11} and λ_{22} are the weights of the manipulated variables u_1 and u_2 respectively. For this linear controller, K_{MPC} is calculated based on step responses carried out between the limits -1 to 1 for each u.

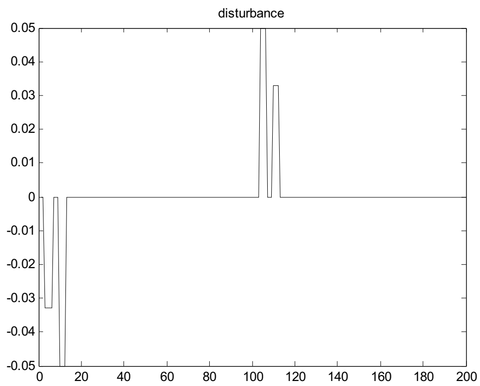

Fig. 1. Disturbance signal for the simulation

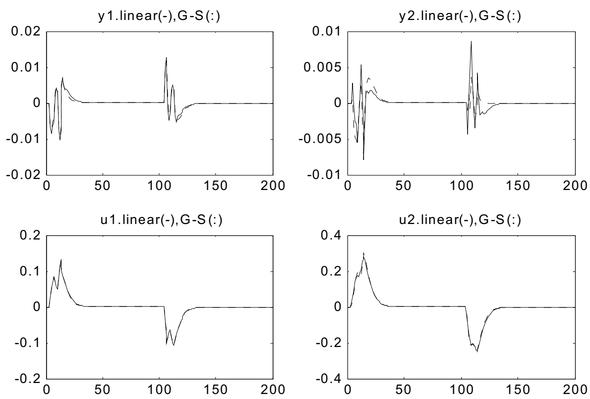

Fig. 2. Comparison of the LMPC (solid line) and GSMPC (dotted line) controllers designed based on the minimization of $\overline{\gamma}_{\mathrm{local}}$.

The input weights of the GSMPC and LMPC controllers are optimized to minimize the performance index γ calculated according to the GEVP problem given by (21). The calculation includes 16 LMI's according to all the vertices ω defined by (23) and the resulting optimal γ will be referred to as γ_{global}.

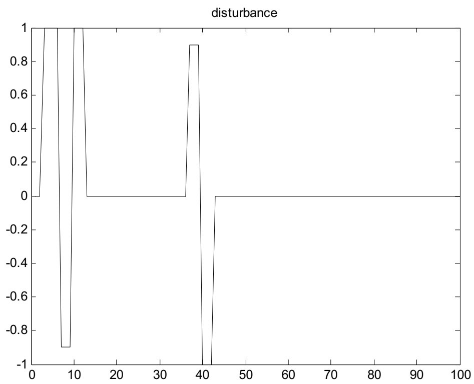

Fig. 3. Disturbance signal for the simulation

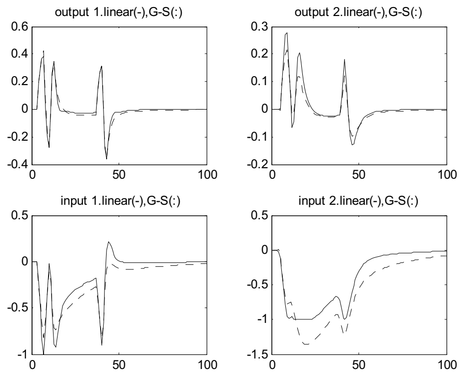

Fig. 4 Comparison of the LMPC (solid line) and GSMPC (dotted line) controllers designed based on the minimization of γ_{global}.

The robust performance analysis and simulation results of the gain scheduled and linear MPC controllers for the MIMO process are summarized in Table 1. It can be seen from Table 1 that the performance index γ_{global} for GSMPC controllers is 0.8494, larger than the value of 0.7472 for γ_{global} obtained for LMPC. This seems to indicate that a better performance can be achieved with LMPC. This result is not completely surprising since the gain scheduling controller is not necessarily better when large changes in u values occurred during dynamic transitions between the different regions defined in (23). One way to improve the GSMPC is by optimizing further the step response models used to calculate the controllers. However, this is a difficult optimization problem and it is beyond the scope of this study.

On the other hand, a scenario where GSMPC is expected to perform better than the LMPC is when the process is operated within each of the regions described by (23) for long periods of time. This is due to the fact that the GSMPC is based on "local" step response models identified in each of these regions. This scenario can be assessed by computing a value of γ for each of the regions given in (23) by using the corresponding set of 4 LMI's for each one of the combinations of u values in each of these 4 regions. An overall performance index, to be referred as $\bar{\gamma}_{local}$, can be obtained by calculating the average of the four resulting γ 's. Then, LMPC and GSMPC controllers that minimize $\bar{\gamma}_{local}$ can be designed. The results, shown in Table 1, confirm that the GSMPC is better than the LMPC when the controllers are designed based on this $\bar{\gamma}_{local}$ index, instead of the global performance index γ_{global} calculated for the whole range of operation.

Table 1 MPC controller optimization

based on γ_{global}	LMPC	GSMPC
$[\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}]$	[0.5009,-,-,0.4983]	[0.5164,0.5029, 0.4980,0.5034]
γ_{global} at optimum	0.7472	0.8694
$\gamma_{simulation}$	0.3396	0.3238
based on $\bar{\gamma}_{local}$	LMPC	GSMPC
$[\lambda_{11},\lambda_{12},\lambda_{21},\lambda_{22}]$	[0.1054,-,-,0.0958]	[0.0988,0.1006, 0.1044, 0.1006]
$\bar{\gamma}_{local}$ at optimum	0.4765	0.4021
$\gamma_{simulation}$	0.2524	0.2490

To assess the correctness of the analysis, a large number of disturbances were simulated for the controllers described above. Ideally, the disturbances that lead to the worst γ values are sought. However, this is a very difficult optimization problem and therefore the γ values calculated from simulations and shown in Table 1 as $\gamma_{simulation}$, are the largest found during the simulations but not necessarily the largest possible. The worst disturbances found in the simulations are given in Fig. 1 and Fig.3, for the controllers based on $\bar{\gamma}_{local}$ and γ_{global} respectively.

Both GSMPC, based on γ_{global} and $\bar{\gamma}_{local}$, showed improvement in terms of $\gamma_{simulation}$ compared to the LMPC. Fig. 2 and Fig.4 show the results for the simulations carried out to compute $\gamma_{simulation}$ for controllers that are designed based on the minimization of $\bar{\gamma}_{local}$ and γ_{global} respectively. It should be noticed in Fig.2 that corresponds to the design based on $\bar{\gamma}_{local}$ that during 0<t<100 the *u's* are evolving within region I described in (23) and during 100<t<200 the *u's* are within region IV in (23). The analytical γ_{global} and $\bar{\gamma}_{local}$ are clearly larger than the $\gamma_{simulation}$ indicating that the analytical bounds may be conservative but they can still be useful as an indicator of the relative performance of the controllers described above.

5. CONCLUSIONS

An approach is proposed to design gain-scheduled MPC controllers for nonlinear processes using process data. It is based on empirical state-affine models of the process. Gain-scheduled MPC controllers are obtained using a GEVP based optimization algorithm. The analysis show that the gain scheduled MPC performs better than the linear MPC when the local performance in small regions of operation is considered.

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