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FEASIBLE MODEL PREDICTIVE CONTROL WITH BOUNDED DISTURBANCES

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Abstract: This paper shows how to calculate feasible regions, parameterized in terms for the present state x_k, for MPC controllers for constrained linear systems. The dependence of the feasible region on the prediction horizon is also made clear. It is also shown how the procedure may be modified to find guaranteed feasible regions in the presence of unknown, bounded disturbances. These 'robust' feasible regions are used to propose a very simple MPC controller which achieves robust feasibility. *Copyright© 2005 Author*

Keywords: Model predictive control, feasible region, prediction horizon, disturbances.

1. INTRODUCTION

Model Predictive Control (MPC) has been a remarkable industrial success.A distinguishing feature of MPC controllers is the relative ease with which constraints in both states/outputs and inputs are handled. This paper starts by addressing the calculation of feasible regions and corresponding required prediction horizon for MPC. It is then shown how to calculate feasible regions that are robust to unkonwn, bounded disturbances. Subsequently, the parametrization of the robustly feasible regions are used to propose a simplified MPC controller. We start from a fairly typical MPC formulation:

$$\min_{u_0,u_1,\cdots,u_{N-1}} \sum_{k=0}^{N-1} \left(u_k^T R u_k + x_k^T Q x_k\right) + x_N^T Q_f x_N \tag{1}$$

with constraints

$$Gx_k + Hu_k \leq b, k \in [0,\ldots,N,\ldots,N+j] \tag{2}$$

$$x_{k+1} = Ax_k + Bu_k, \;\; x_0 = given \tag{3}$$

$$Q \succeq 0, \; Q_f \succeq 0 \; R \succ 0 \tag{4}$$

Input constraints are normally present in in real-life problems, these are usually the only constraints in (2) that are enforced at $k = 0$ (the corresponding rows in G are zero). For (2) to be meaningful, i.e., for the constraints to be fulfilled for a time horizon beyond the prediction horizon N, the control action in the interval $N \leq k \leq N+j$ needs to be defined. Here the common assumption is made that the infinite horizon LQ-optimal controller for the weighting matrices R and Q is used, and that Q_f is the solution of the corresponding algebraic Riccati equation. Additional assumptions are

A1 The system described by (3) is stabilizable.
A2 $(Q^{1/2}, A)$ is observable.
A3 The constraints defined by (2) constitute a closed and bounded polyhedron in the space

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spanned by $\{x, u\}$, and $\{0, 0\}$ is in the interior of this polyhedron.

A1 and A2 are necessary to guarantee closed loop stability, A1 is implied by the assumption that a (stabilizing) LQ-optimal controller can be found. A3 is not very restrictive. Manipulated variables in practice have a limited range of manipulation. In cases where the physical constraints do not impose constraints on all states, artificial constraints can be added so far from the origin that operation in the region of the added constraints is unlikely and/or represent unacceptably poor performance.

The purpose of extending the constraint horizon in (2) from N to $N + j$ is to ensure that the (unconstrained) LQ-optimal controller K will not violate any constraints. When these constraints are fulfilled, applying at each time step u_0 from the solution of (1) and applying the LQ-optimal controller K will result in the same closed loop trajectories for states and manipulated variables after time $k = N$. Closed loop stability of the MPC controller then follows from the stability of the LQ-optimal controller (since the constraints never become active for time steps $k \geq N$), see (Scokaert and Rawlings, 1998) for a proof. The matrices Q and R play much the same role in MPC as in conventional LQ-optimal control, and Q_f and K follow from Q and R. Issues that are particular to MPC (or, constrained LQ regulation) are

I1 The determination of j in (2).
I2 Efficient representation of the constraints for the period $N \leq k \leq N + j$, since many of the constraints in (2) may be redundant.
I3 Determination of the prediction horizon N , and the relationship between N and the feasible region.

The focus of this paper is on issue 3 above, and will be addressed in section 3. However, tackling this issue will require that issues 1 and 2 have been addressed first. For completeness, section 2 will therefore address issues 1 and 2, by specializing the results of (Gilbert and Tan, 1991) to constrained linear systems. The calculations involved in addressing issues 1-3 can be performed offline, at the design stage when there is presumably ample time for computations. The focus in this paper will therefore be on the clarity of presentation rather than finding the computationally most efficient algorithms.

In (Grieder *et al.*, 2004), it is stated that no method for calculating N is known, and a method of calculating the prediction horizon and associated feasible region was proposed, based on an explicit MPC formultion. The method described next only aims at describing the relationship between the prediction horizon N and the feasible region, without engaging the machinery of explicit MPC. In this author's opinion, explicit MPC is one of the most exciting developments in advanced control in recent years, but nevertheless the majority of applications still rely on on-line solution of optimization problems. The authors of (Grieder *et al.*, 2004) are prominent in the MPC community. Although well known mathematical tools are used in this paper, it is therefore fair to assume that the application of these tools in the present context is not generally understood by the MPC community.

2. MAXIMAL OUTPUT ADMISSIBLE SETS FOR CONSTRAINED LINEAR SYSTEMS

In (Rawlings and Muske, 1993) a conservative criterion for estimating a value for j in (3) was proposed. This criterion depends on the predicted value of x_N, and is thus impractical for on-line use. While the criterion in (Rawlings and Muske, 1993) makes it simple to check on-line whether a sufficiently large parameter j is in use, it is hard to know at the design stage what will be a sufficiently large value for j. Introducing assumption A3 above allows us to determine a non-conservative value for j at the design stage, and also to simplify the quadratic programming problem by removing redundant constraints, i.e., constraints that are always fulfilled whenever other constraints are fulfilled.

Applying the state feedback controller $u_k = Kx_k$ for $k \geq N$, the resulting closed loop system can be considered as an unforced linear system provided constraints are not active for $k \geq N$. The largest set of initial conditions for which an unforced linear system satisfies all constraints for all future times is called the Maximal Output Admissible Set, often denoted $\mathcal{O}_\infty$. Correspondingly, the set of initial conditions for which all constraints are fulfilled up until time t is denoted $\mathcal{O}_t$. Obviously, $\mathcal{O}_\infty \subseteq \mathcal{O}_{t+1} \subseteq \mathcal{O}_t$. The determination of output admissible sets was addressed in (Gilbert and Tan, 1991). We will apply their results to linear systems subject to linear inequality constraints. Assumption 3 above, together with the fact that the system is stable in closed loop, allow us to use some of the results of Gilbert and Tan (Gilbert and Tan, 1991) in the following:

R1 $\mathcal{O}_\infty$ is closed and bounded (and is convex due to the linearity of the constraints).
R2 $\mathcal{O}_\infty$ is *finitely determined* if $\mathcal{O}_\infty = \mathcal{O}_t$ for finite t. For the cases studied here, $\mathcal{O}_\infty$ is finitely determined by construction.
R3 If $\mathcal{O}_t = \mathcal{O}_{t+1}$ then $\mathcal{O}_\infty = \mathcal{O}_t$.

In our case, we are interested in the set to which x_N must belong in order for (2) to hold for all

$k \geq N$, assuming that the inputs are determined by $u_k = Kx_k$. We will in the following denote the set of states x_N for which all inequality constraints (2) are fulfilled for $N \leq k \leq N+t$ by $\mathcal{O}_t$. A straight forward way of determining the maximal output admissible set is therefore given by Algorithm 1.

Algorithm 1. Maximal Output Admissible Set.

(1) Set $t = 0$, and let $\mathcal{O}_0$ be parameterized by (2) for $k = N$.
(2) Increment the time index t, and express the constraints at time t in terms of x_N, using the system model (3) and the equation for the state feedback controller.
(3) Remove any redundant constraints for time t. If all constraints for time index t are redundant, $\mathcal{O}_{t-1} = \mathcal{O}_t$, and hence $\mathcal{O}_\infty = \mathcal{O}_{t-1}$. Stop. Otherwise, augment the set of constraints describing $\mathcal{O}_{t-1}$ by the non-redundant constraints for time t to define $\mathcal{O}_t$. Go to Step 2.

Due to R2 above, this algorithm will terminate in finite time for the problems considered here. Checking for redundancy of constraints is also straight forward for linear systems subject to linear inequality constraints.

Checking redundancy at step 3 in Algorithm 1 above does not necessarily guarantee that the final set of constraints is minimal (i.e., does not contain any redundant constraints). Redundant constraints may still be present in the description of $\mathcal{O}_\infty$ due to the possible presence of redundant constraints in the description of the original polyhedron, or because a constraint that was not redundant at time i was made redundant by adding constraints at later times. Clearly, it is simple to identify and remove any such redundant constraints in the description of $\mathcal{O}_\infty$, if necessary.

3. FEASIBLE REGIONS FOR MPC CONTROLLERS

With the description of the 'terminal set' $\mathcal{O}_\infty$ within which the predicted state at time N must lie, we are ready to address the issue of determining the feasible region for an MPC controller, and how this feasible region depends on the value of N. To this end, we will use what is known as Fourier-Motzkin elimination. This is a procedure for eliminating variables from sets of inequalities, originally discovered by Fourier in the first half of the 1800's. The use of Fourier-Motzkin elimination has been proposed in the control literature previously (e.g., (Keerthi and Gilbert, 1987), (Kerrigan and Maciejowski, 2000)), and it should be well known to people working with invariant sets (also in the MPC context). Nevertheless, as argued above, recent literature show that its application in the present context is not widely known.

3.1 Application to MPC controllers

When applying Fourier-Motzkin elimination in the design and analysis of MPC controllers, we assume that we start from a description of the maximal output admissible set $\mathcal{O}_\infty$, as well as a predefined feasible region (2) and the model equations (3). A typical problem may then be to find the required prediction horizon N such that a feasible solution to the MPC QP problem exists for all x_0 such that

$$\mathcal{A}_r x_0 \leq b_r \tag{5}$$

Naturally, it is assumed that the required feasible region is consistent with the constraints in (2). The most straight forward approach would then be to guess at a value for N, use (3) to eliminate x_k from the constraints (in (2) as well as $\mathcal{O}_\infty$) for all $k > 0$, use Fourier-Motzkin elimination to eliminate $u_0, \cdots, u_{N-1}$, and finally check whether the resulting feasible region is sufficiently large. Instead, we will work 'backwards' from the prediction horizon, as this will allow a stage-wise removal of redundant constraints, and allows terminating the analysis once a sufficiently large feasible region has been found. The stage-wise removal of redundant constraints is important in this context, as it reduces the complexity of the description of the feasible region. The resulting algorithm is as follows:

Algorithm 2. Calculating the required prediction horizon N.

(1) Start with $N = 0$ (ordinary LQ-optimal control) and corresponding feasible region $\mathcal{O}_\infty$.
(2) Check whether the feasible region is sufficiently large. If yes, terminate.
(3) Set $N \leftarrow N+1$ and correspondingly $x_0 \leftarrow x_1$.
(4) Use (3) to express the 'new' x_1 in the constraints in terms of x_0 and u_0.
(5) Use Fourier-Motzkin elimination to remove u_0 from the constraints.
(6) Remove any redundant constraints. Go to Step 2.

For Algorithm 2 to terminate in a finite number of steps is clearly critically dependent on the assumption that the desired feasible region is consistent with the constraints in (2). For unstable systems, this assumption also implies that the desired feasible region is within the region that can be stabilized by constrained inputs.

Applying Algorithm 2 now only requires a method for checking whether the feasible region is sufficiently large. A simple way of checking this is to check that all the constraints defining the feasible region are 'redundant' relative to the desired feasible region described by (5).

Example 1.
We will here consider the example in (Grieder *et al.*, 2004). The system is given by

$$x_{k+1} = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} x_k + \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix} u_k$$

The system should be regulated to the origin while adhering to the constraints $|u_k| \leq 2 \ \forall k \geq 0$ The desired feasible region is given by $|x_{k,i}| \leq 1000$ and we want to determine the parameters N in (1) and j in (2) such that this feasible region is achieved. To fulfill assumption A3 above we add the artificial constraints $|x_{k,i}| \leq 2000$. The weight matrices are set to $Q = I$ and $R = 0.01$.

Applying Algorithm 1, we find that the parameter j in (2) should be set to $j = 3$, and that the maximal output admissible set $\mathcal{O}_\infty$ is defined by eight inequalities. These inequalities all arise from the constraints in the manipulated variable, and are hence not influenced by the artificial constraints. Performing the Fourier Motzkin elimination, starting from the previously calculated $\mathcal{O}_\infty$, we get that the prediction horizon N needs to be set to $N = 68$ to achieve the desired feasible region. Note that $N + j = 71$, which is the same as the required horizon found in (Grieder *et al.*, 2004).

4. ROBUSTNESS TO DISTURBANCES

The methods presented in sections 2 and 3 do not consider disturbances, and are hence quite idealized and optimistic. In this section, we will describe how the methods can be modified to account for disturbances. In a similar fashion as done above, the maximal output admissible set (in the face of disturbances) will be considered first, since $\mathcal{O}_\infty$ is the starting point for calculating prediction horizons N and corresponding feasible regions. Maximal output admissible sets with disturbance inputs has been studied previously in (Kolmanovsky and Gilbert, 1995), whereas accounting for disturbances when calculating prediction horizons and feasible regions represents an extension of the results in (Grieder *et al.*, 2004). The same MPC formulation as in Section 1 will be used, with the modifications that (3), is replaced by

$$x_{k+1} = Ax_k + Bu_k + Ed_k, \quad x_0 = given \tag{6}$$

Clearly, bounded control can guarantee neither feasibility nor performance with unbounded disturbances. We therefore assume that the disturbances are bounded, and are confined to a polytope defined by the linear inequalities

$$\mathcal{A}_d d_k \leq b_d \tag{7}$$

The zero disturbance, $d_k = 0$, is assumed to lie in the interior of this polytope.

4.1 Robust output admissible sets

Application of the state feedback controller $u_k = Kx_k$ to the system described by (6) yields

$$x_{k+l} = (A + BK)^l x_k + \sum_{i=0}^{l-1} (A + BK)^i E d_{k+l-i-1} \tag{8}$$

Substituting (8) into (2) gives for timestep $k + l$

$$\begin{aligned} (G + HK)(A + BK)^l x_k \leq b \\ - (G + HK) \sum_{i=0}^{l-1} (A + BK)^i E d_{k+l-i-1} \end{aligned} \tag{9}$$

Clearly, the right hand side of (9) cannot be evaluated at time k without advance knowledge of the disturbances. However, to ensure feasibility of the constraints, we need only consider worst case disturbances (Kolmanovsky and Gilbert, 1995), i.e., the sequence of disturbances that minimizes the RHS of (9). Note that the worst case disturbances may be different for different constraints in (9), and we thus need to solve one LP for each of these constraints. Thus, we introduce the vector h, where $h = [h_1 \cdots h_m \cdots]^T$ and

$$h_m = \max_{[d_k, \cdots, d_{k+l-1}]} \sum_{i=0}^{l-1} \left[(A + BK)^i E \right]_m d_{k+l-i-1} \tag{10}$$

subject to the inequality (7) being fulfilled. Here the subscript m on the matrices $(A + BK)^i E$ indicates *row* number m of these matrices. Having modified the constraints accordingly, we can still use Algorithm 1 to determine the (robust) maximal output admissible set $\mathcal{O}_\infty$, and **R3** above still serves as a test for identifying $\mathcal{O}_\infty$ (Kolmanovsky and Gilbert, 1995). However, it is no longer obvious *a priori* that $\mathcal{O}_\infty$ is non-empty. We therefore assumed that $b - h > 0$, otherwise the state feedback controller cannot guarantee feasible operation even if $x_k = 0$. If this assumption is violated, it will be necessary either to change weights Q and R to retune the controller, or to take other measures (other than feedback control) to reduce the effects of the disturbances.

4.2 Robust feasible regions

The Fourier-Motzkin elimination can easily be used to calculate feasible regions that are robust to disturbances. The necessary modifications are:

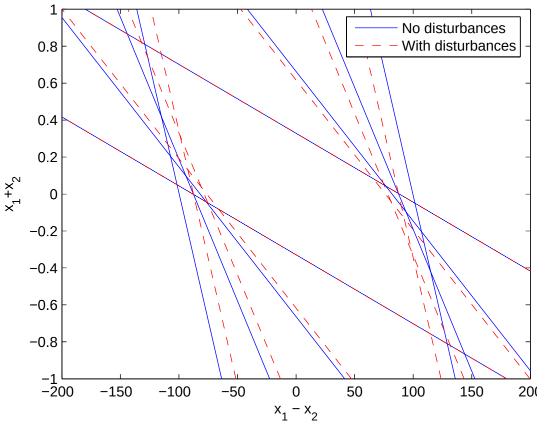

Fig. 1. Maximal output admissible sets $\mathcal{O}_\infty$ for Example 2 with and without disturbances.

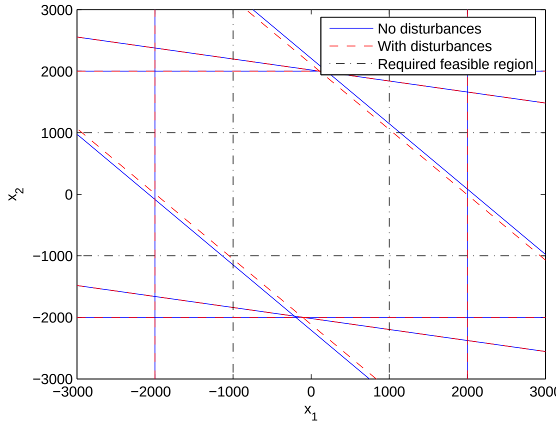

Fig. 2. Feasible regions for Example 2 with and without disturbances.

- Begin the calculations from the *robust* output admissible set $\mathcal{O}_\infty$, calculated as described above.
- The set of constraints used at each stage of Fourier-Motzkin elimination should be the intersection of the constraint sets over the vertices of the disturbance set. Since (8) is linear, the set of states for which there exists a feasible solution for all vertices of the constraint set will also admit a feasible solution for disturbances in the interior or along the edges of the disturbance set.

To illustrate the effects of considering disturbances in the calculation of output admissible sets and feasible regions, we modify Example 1 above by introducing disturbances, with $E = \mathrm{diag}(0.02, 0.003)$ in (6), and $|d_{k,i}| \leq 1$. Calculating j in (2), we still get $j = 3$, just as for the disturbance free case, but the maximal output admissible set is somewhat smaller. This can be seen from Fig. 1, where the two maximal output admissible sets are compared. Requiring the same feasible region as in the disturbance free case, we find that with disturbances we must increase the prediction horizon n from 68 to 69 to account for the disturbances. The resulting feasible regions are compared in Fig. 2.

5. A SIMPLE, ROBUSTLY FEASIBLE MPC

The MPC criterion in 1, when used with the plant model (6) cannot be optimized without *a priori* knowledge of the disturbances. This is clearly not realistic in most cases. Possible modifications include optimizing the *worst case* value of the criterion (leading to a *min-max* formulation), or optimizing the *expected value* of the MPC criterion 1. Both these modifications lead to very complex controllers with high computational load for linear constrained systems. Next, a very simple MPC controller is proposed, which retains the robust feasibility of a more complex formulation, while the control performance may be reduced during transients. Assume the following are given:

(1) A stabilizing controller K for the unconstrained linear system.
(2) The corresponding robust output admissible set, represented by linear inequalities $\mathcal{A}_0 x_k \leq b_0$.
(3) Robust feasible regions for MPC controllers of horizon N, represented by $\mathcal{A}_N x_k \leq b_N; N \in \left[1, \ldots, \bar{N}\right]$, where $\bar{N}$ is large enough for the corresponding feasible region to cover the required operating region.

The robust feasible region for the horizon-N controller is known from the calculations described above, and is parametrized by

$$\mathcal{A}_N x_k \leq b_N \tag{11}$$

At each timestep k, it is then very simple to identify the smallest prediction horizon N_k such that x_k lies within the corresponding feasible region. The basic idea behind the simplified MPC is then to use a prediction horizon of 1, but to constrain the optimization such that x_{k+1} is known to lie within the feasible region for an MPC of prediction horizon $N_k - 1$. Note that it is not sufficient to add the constraint

$$\mathcal{A}_{N_k-1} x_{k+1} \leq b_{N_k-1},$$

rather one has to ensure that that this constraint is fulfilled for all possible disturbances - but without knowledge of the actual disturbance values. With x_k given, (6) then gives

$$\mathcal{A}_{N_k-1} B u_k \leq b_{N_k-1} - \mathcal{A}_{N_k-1} A x_k - \mathcal{A}_{N_k-1} E d_k \tag{12}$$

In the same way as for the calculation in (10), solving a set of LP is (in general) needed to maximize the last term in (12). Let the results of these LP's be collected in the vector h_{N_k-1}. Thus, we get

$$
\begin{aligned}
\min_{u_0} \quad & \left(u_0^T R u_0 + x_{k+1}^T Q_f x_{k+1}\right) && (13)\\
s.t. \quad & \mathcal{A}_{N_k-1} A x_k + \mathcal{A}_{N_k-1} B u_k \\
& \leq b_{N_k-1} - h_{N_k-1}, && (14)\\
H_0 u_k \leq b_0 & && (15)
\end{aligned}
$$

where (15) represents the input constraints for $k = 0$ in (2). MPC controllers with robust feasibility in the face of bounded disturbances have also been proposed in previous works (Chischi *et al.*, 2001), (Sakizlis *et al.*, 2004). The controllers in these works use a prediction horizon of a fixed length N (as is common in predictive control), and constraints that are formulated to account for the worst possible disturbance sequence. In contrast, the controller proposed here uses a prediction horizon of 1, while the constraints are formulated to ensure that a feasible solution will exist also at future times if a feasible solution is found initially. The main advantage with the robust MPC formulation proposed here is that it requires the solution of quite small QP problems (whether solved on-line or *a priori* using parametric programming). The *worst case* performance is likely to be better with the controllers in (Chischi *et al.*, 2001), (Sakizlis *et al.*, 2004), since performance is optimized over an extended horizon N.

The idea of using a 'one step' controller to guarantee constraint fulfillment for an infinite horizon was recently proposed in the context of piecewise affine systems in (Grieder *et al.*, 2005). In some ways, the 'one step' controller proposed here is a specialization of their controller to linear systems. However, in order to achieve stability (Grieder *et al.*, 2005) assumes the origin to be an equilibrium state, which precludes the possibility of persistent bounded disturbances. The robust output admissible set is control invariant, and hence stability is guaranteed also in the face of bounded persistent disturbances for the 'one step' controller in this work.

6. CONCLUSIONS

This paper uses well known mathematical tools for determining the required prediction horizon in constrained linear MPC to guarantee a specified feasible region. It is shown how the calculation of the prediction horizon and feasible region can be modified to account for bounded disturbances. A simple MPC formulation is then proposed, which uses the robustly feasible regions to guarantee robust feasibility while keeping the computational load very low. The simple MPC controller does not guarantee any form of optimality during transients (when constraints are active), but at steady state it will inherit the optimality properties of the state feedback controller.

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