A COMBINED APPROACH TO SYSTEM IDENTIFICATION OF A CLASS OF HYBRID SYSTEM

Egbunonu Patrick, **Guay Martin** [1]

Department of Chemical Engineering, Queen's University, Kingston, Ontario, K7L 3N6 , Canada

Abstract: Hybrid Systems consist of continuous time and/or discrete time processes interfaced with some logical or decision making process. In this paper, a class of hybrid systems - switched linear systems is considered. It is shown that for this class of hybrid systems, it is possible to combine subspace methods with mixed integer programming. While most approaches are based on an input-output framework, we a state space identification approach is advocated. The states of the system are extracted from input-output data using sub-space methods. Once these states are known, the switched system is re-written as a mixed logical dynamical (MLD) system and the model parameters are solved for via mixed integer programming. An example is reported at the end of this paper.

Keywords: Hybrid Systems, System Identification, Subspace Identification, Mixed Integer Programming

1. INTRODUCTION

Most existing methods for system identification make use of an input-output framework where the input signals, u(t) to the system, and the output signals, y(t), from the system are observed data. For linear systems, a set of already established algorithms, are available in literature, with the prediction error method (Ljung, 1999) being very common. In this paper we focus on the development of system identification techniques for hybrid systems. Hybrid systems consist of a family of continuous/discrete-time subsystems and a rule that orchestrates the switching between them. In the particular case where all the individual subsystems are linear, a switched linear system is obtained. Considerable work has been performed in the area of hybrid system iden-

[1] corresponding author. email: guaym@chee.queensu.ca

tification (Trecate *et al.*, 2003; Bemporad *et al.*, 2001; Roll, 2003; Del Vecchio *et al.*, 2003; Vidal *et al.*, 2003; Huang *et al.*, 2004).

While most existing techniques are based on an input-output framework, the approach proposed in this paper advocates a state- space construction. A recent work by Huang *et al.* (2004) also looks at solving this problem using a state space approach. A major difference is that we approach this problem by advocating the combination of two different algorithms. Using a subspace identification technique, the identification problem is formulated in matrix form and the states of the system are then extracted. A basic assumption in this work is that the order of the system has a known upper bound, and the number of discrete states is known a priori. We assume that the states of the system is not known a-priori and focus on showing that for a very simple class of hybrid systems, it is possible to combine subspace methods with mixed integer programming. This paper is structured as follows: In section 2, we consider the class of hybrid system we are dealing with. In section 3, we present the contributions of this work, together with an example. Finally, we draw our conclusions in Section 4.

2. THE CLASS OF HYBRID SYSTEMS

Hybrid Systems are a class of systems that exhibit discrete/logical and continuous dynamics. Hybrid systems are ubiquitous in nature with most research work focusing on modeling and control of such systems. Examples of hybrid systems include the dynamics of a car, elevator, etc. Many chemical processes, which are not inherently hybrid, make use of hybrid controllers. An example can be found in Lennartson *et al.* (1996),where a hybrid controller is designed for a chemical mixing process.

In this paper we consider a special class of hybrid system, switched linear system. These consists of linear (state space) sub-models with a rule that determines which of the sub-models is active at any time t. A general form for the deterministic, discrete time switched linear systems is as follows:

$$x_{t+1} = A(\Upsilon_t)x_t + B(\Upsilon_t)u_t \quad (1)$$
$$y_t = C(\Upsilon_t)x_t + D(\Upsilon_t)u_t \quad (2)$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^l$ are the states, inputs and measured outputs of the system respectively. Υ_t, which we refer to as the active mode at time t, assumes its values in the set $\{1, ..., n_s\}$ [2] , so that the system parameters $A(\Upsilon_t) \in \mathbb{R}^{n\times n}$, $B(\Upsilon_t) \in \mathbb{R}^{n\times m}$, $C(\Upsilon_t) \in \mathbb{R}^{l\times n}$ and $D(\Upsilon_t) \in \mathbb{R}^{l\times m}$ switch among n_s different discrete states. The evolution of the discrete state Υ_t is modeled as a polyhedral partition of the hybrid state space (Bemporad and Morari, 1999).

Prior to solving this problem, we make the following assumptions:

Assumption 1. The system to be reformulated can be represented or approximated by the following equations

$$x_{t+1} = A(\Upsilon_t)x_t + Bu_t \quad (3)$$
$$y_t = Cx_t + Du_t \quad (4)$$

Assumption 2. Each of the modes is persistently excited for a long time in the data available.

Assumption 3. Each of the sub-linear system is controllable ,observable, and of the same order n

Assumption 4. The order, n, and the number of discrete states, n_s,of the system is known a priori

Remark 5. Assumptions 1, 3 and 4 are non-trivial as the purpose of this paper is to demonstrate and motivate -using a simple class of hybrid systems- the use of subspace and integer methods. Assumption 2 makes sure that all the modes are excited, enabling the different possible states to be identified. With

[2] $n_s \in \mathbb{R}$ is the number of discrete states in the the hybrid system

Assumption 3, we have that the controllability and observability matrices are full rank.

Remark 6. Comparing these assumptions with other methods; we note that in most of the methods mentioned earlier (Vidal *et al.*, 2003), the assumption is that the number of discrete states is not known. A major difference is that we use a state space framework, and look for a way of combining two algorithms.

The hybrid system deterministic identification problem is therefore as follows:

Problem 7. Given s measurements of the input $u_k \in \mathbb{R}^m$ and the output $y_k \in \mathbb{R}^l$ generated by the unknown deterministic system (3)-(4)of order n, and n_s discrete states, determine:

- The states of the system
- The system matrices $A(\Upsilon) \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$,$C \in \mathbb{R}^{l\times n}$,and $D \in \mathbb{R}^{l\times m}$.

To solve this problem, we first extract the states of the system from input-output data using subspace methods(VanOverschee and Moor, 1996). Once these states are known, the switched system is re-written as a mixed logical dynamical (MLD) system (Bemporad *et al.*, 2001)and the model parameters computed.

3. MATRIX ANALYSIS

In this section, we re-write the system described by (3)-(4) in Matrix form, and show how we can extract the states and the parameters of the system. We conclude this section with a relevant example. Let us define the following parameters:

Definition 8.

$$\begin{aligned}\widehat{\mathbb{A}}_L^K &= A(\Upsilon_{t_L})A(\Upsilon_{t_{L-1}})A(\Upsilon_{t_{L-2}}) \\ &\quad \dots A(\Upsilon_{t_{K+1}})A(\Upsilon_{t_K}) \qquad (5)\\ \widehat{\mathbb{A}}_K^K &= A(\Upsilon_{t_K}) \\ &= \widehat{\mathbb{A}}^K \qquad (6)\end{aligned}$$

where $\widehat{\mathbb{A}}_L^K \in \mathbb{R}^{n\times n}$.

Using results analogous to VanOverschee and Moor (1996), we define the Hankel Matrix $\widehat{Y_0^{2i}} \in \mathbb{R}^{2il\times j}$ containing the outputs $y_{t_i} \in \mathbb{R}^l$, as,

$$\widehat{Y_0^{2i}} := \begin{bmatrix} y_{t_0} & y_{t_1} & y_{t_2} & \cdots & y_{t_{j-1}} \\ y_{t_1} & y_{t_2} & y_{t_3} & \cdots & y_{t_j} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{t_{i-1}} & y_{t_i} & y_{t_{i+1}} & \cdots & y_{t_{i+j-2}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{t_i} & y_{t_{i+1}} & y_{t_{i+2}} & \cdots & y_{t_{i+j-1}} \\ y_{t_{i+1}} & y_{t_{i+2}} & y_{t_{i+3}} & \cdots & y_{t_{i+j}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y_{t_{2i-1}} & y_{t_{2i}} & y_{t_{2i+1}} & \cdots & y_{t_{2i+j-2}} \end{bmatrix} \qquad (7)$$

$$\widehat{Y_0^{2i}} := \begin{bmatrix} Y_p \\ \cdots \\ Y_f \end{bmatrix} \qquad (8)$$

$Y_p \in \mathbb{R}^{il\times j}$, and $Y_f \in \mathbb{R}^{il\times j}$ are Hankel matrices containing past and future output data respectively. i, l, and k are parameters defined by the user. We also define the *time-dependent 'hybrid observability matrix'*, $\widehat{\mathcal{O}}_w^r \in \mathbb{R}^{l(w-r+1)\times n}$, between times t_w and t_r,(irrespective of whether switching occurs or not) as

$$\widehat{\mathcal{O}}_w^r := \begin{bmatrix} C \\ C\widehat{\mathbb{A}}^r \\ C\widehat{\mathbb{A}}_{r+1}^r \\ \vdots \\ C\widehat{\mathbb{A}}_{w-2}^r \\ C\widehat{\mathbb{A}}_{w-1}^r \end{bmatrix} \qquad (9)$$

w and r are user defined indexes.

Let

$$
\mathbf{U}_0 = \begin{bmatrix}
u_{t_0} & 0 & 0 & \dots & 0 \\
0 & u_{t_1} & 0 & \dots & 0 \\
0 & 0 & u_{t_2} & \dots & 0 \\
\vdots & \vdots & \dots & \ddots & \vdots \\
0 & 0 & 0 & \dots & u_{t_{j-1}} \\
u_{t_1} & 0 & 0 & \dots & 0 \\
0 & u_{t_2} & 0 & \dots & 0 \\
0 & 0 & u_{t_3} & \dots & 0 \\
\vdots & \vdots & \dots & \ddots & \vdots \\
0 & 0 & 0 & \dots & u_{t_j} \\
\dots & \dots & \dots & \dots & \dots \\
u_{t_{i-1}} & 0 & 0 & \dots & 0 \\
0 & u_{t_i} & 0 & \dots & 0 \\
0 & 0 & u_{t_{i+1}} & \dots & 0 \\
\vdots & \vdots & \dots & \ddots & \vdots \\
0 & 0 & 0 & \dots & u_{t_{i+j-2}}
\end{bmatrix} \tag{10}
$$

and

$$
\mathbf{U}_i = \begin{bmatrix}
u_{t_i} & 0 & 0 & \dots & 0 \\
0 & u_{t_{i+1}} & 0 & \dots & 0 \\
0 & 0 & u_{t_{i+2}} & \dots & 0 \\
\vdots & \vdots & \dots & \ddots & \vdots \\
0 & 0 & 0 & \dots & u_{t_{i+j-1}} \\
u_{t_{i+1}} & 0 & 0 & \dots & 0 \\
0 & u_{t_{i+2}} & 0 & \dots & 0 \\
0 & 0 & u_{t_{i+3}} & \dots & 0 \\
\vdots & \vdots & \dots & \ddots & \vdots \\
0 & 0 & 0 & \dots & u_{t_{i+j}} \\
\dots & \dots & \dots & \ddots & \dots \\
u_{t_{2i-1}} & 0 & 0 & \dots & 0 \\
0 & u_{t_{2i}} & 0 & \dots & 0 \\
0 & 0 & u_{t_{2i+1}} & \dots & 0 \\
\vdots & \vdots & \dots & \ddots & \vdots \\
0 & 0 & 0 & \dots & u_{t_{2i+j-2}}
\end{bmatrix} \tag{11}
$$

and define the matrix

$$
\hat{\mathcal{X}}_i = \begin{bmatrix}
x_{t_i} & 0 & 0 & \dots & 0 \\
0 & x_{t_{i+1}} & 0 & \dots & 0 \\
0 & 0 & x_{t_{i+2}} & \dots & 0 \\
\vdots & \vdots & \ddots & \dots & \vdots \\
0 & 0 & 0 & 0 & x_{t_{i+j-1}}
\end{bmatrix} \in \mathbb{R}^{nj \times j} \tag{12}
$$

where $x_{t_i}, x_{t_i+1}, .., x_{t_{i+j-1}}$ are the states and t_i is the first element of the state sequence. Similar to past outputs, we define $\mathcal{X}_p \in \mathbb{R}^{nj \times j}$ (contains the past states $x_{t_0}, x_{t_1}, .., x_{t_{j-1}}$), and $\mathcal{X}_f \in \mathbb{R}^{nj \times j}$ (containing the future state sequence $x_{t_i}, x_{t_i+1}, .., x_{t_{i+j-1}}$) as:$\mathcal{X}_p = \hat{\mathcal{X}}_0, \mathcal{X}_f = \hat{\mathcal{X}}_i$. also $\mathcal{U}_p \in \mathbb{R}^{mj(i-1) \times j}$ and $\mathcal{U}_f \in \mathbb{R}^{mj(i-1) \times j}$ as the matrix of past inputs and future inputs respectively,and

$$
\mathcal{U}_p = \mathbf{U}_0 \tag{13}
$$

$$
\mathcal{U}_f = \mathbf{U}_i \tag{14}
$$

The results in this section of the paper can be summarized by the following theorem which shows how the switched linear state space system of (3)-(4) can be reformulated in matrix form.

Proposition 9. If the switched linear system of (3)-(4) satisfies assumptions (1)-(6), the system can be represented by the following Matrix input-output equations(Egbunonu, 2005).

$$
Y_p = \Delta_0 \mathcal{X}_p + \mathbf{\Gamma}_0 \mathcal{U}_p \tag{15}
$$

$$
Y_f = \Delta_i \mathcal{X}_f + \mathbf{\Gamma}_i \mathcal{U}_f \tag{16}
$$

$$
\mathbb{I}_i \mathcal{X}_f = \mathcal{A} \mathcal{X}_p + \mathfrak{F} \mathcal{U}_p \tag{17}
$$

where

$$
\Delta_i = \left[\widehat{\mathcal{O}}^{i}_{i+k-1} \; \widehat{\mathcal{O}}^{i+1}_{i+k} \; \widehat{\mathcal{O}}^{i+2}_{i+k+1} \; \dots \; \widehat{\mathcal{O}}^{i+j-1}_{i+k+j} \right] \in \mathbb{R}^{lk \times nj}
$$

and

$$
\mathbf{\Gamma}_i = \begin{bmatrix}
D & \dots & 0 \\
CB & \dots & 0 \\
C\widehat{\mathbb{A}}^{i+1}B & \dots & 0 \\
\dots & \dots & \dots \\
C\widehat{\mathbb{A}}^{i+1}_{i+k-2}B & \dots & D
\end{bmatrix} \tag{18}
$$

$$
\mathfrak{F} := \left[\widehat{\mathbb{A}}^{1}_{i-1}B \; \widehat{\mathbb{A}}^{i}B \; \widehat{\mathbb{A}}^{i+1}B \; \dots \; \widehat{\mathbb{A}}^{i+j-2}B \; \dots \; B \right]
$$

$$\mathbb{I}_i := \begin{bmatrix} I \ I \ I \ \ldots \ I \end{bmatrix} \tag{19}$$

where $\mathfrak{F} \in \mathbb{R}^{n \times mjk}, \mathbf{\Gamma}_i \in \mathbb{R}^{lk \times mjk}, I \in \mathbb{R}^{n \times n}$ is the identity matrix and $\mathbb{I}_i \in \mathbb{R}^{n \times nj}$

3.1 Extraction of The States of the Hybrid System

Once our system is reformulated into matrix form, using the Theorems from the preceding section, we extract the states of the hybrid system by making use of the following simple technique (VanOverschee and Moor, 1996; Egbunonu, 2005) summarized in the following theorem

Theorem 10. Given (15) to (17) and assuming that the row space of the future inputs and the row space of the past states is empty, we have that(Egbunonu, 2005)

(1) The matrix Δ_i is equal to

$$\Delta_i = L.S_1 \tag{20}$$

(2) The matrix $\mathcal{X}_f$ is equal to

$$\mathcal{X}_f = \Delta_i^{\dagger} W_i \tag{21}$$

(3) The state sequence $\mathbb{X}_f$ is equal to

$$\mathbb{I}_i \mathcal{X}_f \tag{22}$$

$W_i \in \mathbb{R}^{li \times j}$, $L \in \mathbb{R}^{li \times li}$,and $S \in \mathbb{R}^{li \times j}$

3.2 Mixed Logical Dynamical Systems

After extracting the states, we now re-formulate our system of (3)-(4) as a Mixed Logical Dynamical (MLD) systems (Bemporad and Morari, 1999) i.e. we can re-write equation (3)-(4) in MLD form as

$$x(t+1) = \sum_{i=1}^{n_s} z_i(t) \tag{23}$$

where

$$\begin{aligned} & z_i(t) \leq M\delta_i(t) \\ & z_i(t) \geq m\delta_i(t) \\ & z_i(t) \leq A_i x(t) + B_i u(t) - m(1 - \delta_i(t)) \\ & z_i(t) \geq A_i x(t) + B_i u(t) - M(1 - \delta_i(t)) \end{aligned} \tag{24}$$

δ_i(t) are $0 - 1$ variables and the vectors $M = [M_1, \ldots, M_n]'$, $m = [m_1, ..., m_n]'$ are defined as

$$M_j \triangleq \max_{i=1,\ldots,n_s} \left\{ \max_{[x \quad u]' \in \Im} A_i^j x + B_i^j u \right\} \tag{25}$$

$$m_j \triangleq \min_{i=1,\ldots,n_s} \left\{ \max_{[x \quad u]' \in \Im} A_i^j x + B_i^j u \right\} \tag{26}$$

With this formulation, we can now solve for the parameters of the system, using least squares method. The result is a Mixed Integer Programming Problem. One must mention here that this method has a drawback which is its computational complexity. The branch and bound method utilized in solving the problem, increases the computational time and makes the solution algorithm potentially more complex. To aid the algorithm, several bounds and assumptions are introduced.

3.3 Example

In this section we look at an example of a linear hybrid system and how we can use the Matrix Form of the system to extract the states of the system.

Example 11. Consider the discrete switched linear systems shown below:

$$x(t+1) = A_{n_a} x + \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} u \tag{27}$$

$$y = \begin{bmatrix} -2 \ 1 \end{bmatrix} x + 5u \tag{28}$$

where $x \in \mathbb{R}^2$, $u \in \mathbb{R}$, $y \in \mathbb{R}$, $A \in \mathbb{R}^{2\times2}$ $n_a = \{1, 2\}$, and

$$A_1 = \begin{bmatrix} -0.3 & -0.2 \\ -0.1 & 0.5 \end{bmatrix} \text{if } K_1 x + J_1 u \leq T_1 \tag{29}$$

$$A_2 = \begin{bmatrix} -0.5 & -0.4 \\ -0.1 & 0.3 \end{bmatrix} \text{if } K_2 x + J_2 u \leq T_2 \tag{30}$$

$K_1 = \begin{bmatrix} 0.2 & 0.3 \end{bmatrix}$; $K_2 = \begin{bmatrix} -0.2 & -0.3 \end{bmatrix}$;$J_1 = 5$;$J_2 = -5$, $T_1 = 0.8$, $T_2 = -0.8$

Using the methods described in the previous sections, coupled with the aforementioned assumptions, we are able to extract the states of the system, and generate a set of parameters describing the system. (Results were generated by using the TOMLAB/Xpress software to solve the mixed integer problem). Figure 1 shows a validation graph of the outputs generated by the actual and predicted system, with mean error value of -0.006.

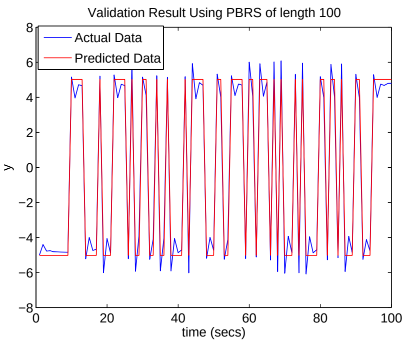

Fig. 1. Validation of Predicted Switched Linear System (2)

4. CONCLUSIONS

In this paper, we have proposed a combined subspace and mixed integer methods for identification of a set of switched linear systems. We made use of a state space modeling framework, and assumed that the order of our system and the number of discrete states where known. We also assumed a deterministic model. Open issues includes the performance of the algorithm in the presence of noise and/or disturbances. Also, a possible extension of the algorithm would be to cases where the order of the system and perhaps the number of discrete states are unknown. A positive note is that since we are dealing with a state space model, the availability of tested algorithm for analysis and control of linear systems makes this option very applicable and use-able.

REFERENCES

Bemporad, A. and M. Morari (1999). Control of systems integrating logic, dynamics, and constraints. **35**(3), 407–427.

Bemporad, Alberto, Jacob Roll and L. Ljung (2001). Identification of hybrid systems via mixed-integer programming. Proceedings of the IEEE Conference on Decision and Control, Orlando, Florida, USA, pp. 786-792.

Del Vecchio, Domitila., Richard M. Murray and P. Perona (2003). Decomposition of human motion into dynamcs based primitives with application to drawing tasks. **39**(12), 2085–2098.

Egbunonu, P. (2005). Identification of hybrid systems using subspace and integer methods. MSc. Thesis, Queen's University.

Huang, K., A. Wagner and Y. Ma (2004). Identification of hybrid linear time-invariant systems via subspace embedding and segmentation. *In Proc. 43st IEEE Conf. on Decision and Control* **3**, 3227–3234.

Lennartson, B., M. Tittus, B. Egardt and S. Pettersson (1996). Hybrid systems in process control. Control Systems Magazine, IEEE, Volume: 16 Issue: 5. 45-56.

Ljung, Lennart (1999). *System Identification-Theory For The User.* Prentice Hall. Upper Saddle River, New Jersey.

Roll, Jacob (2003). Local and piecewise affine approaches to system identification. Ph.D. Thesis, Linköping University, Sweden.

Trecate, G.F., M. Muselli, D. Liberati and M. Morari (2003). Identification of piecewise affine and hybrid systems. **39**(2), 205–217.

VanOverschee, P. and B. De Moor (1996). *Subspace Identification for Linear Systems Theory, Implementation, Applications.* Kluwer Academic Publishers. London.

Vidal, Rene., Y. Soatto, Y. Ma and S. Sastry (2003). An algebraic gemetric approach to the identification of a class of hybrid systems. Proceedings of the IEEE Conference on Decision and Control.