A FRAMEWORK FOR DESIGN OF SCHEDULED OUTPUT FEEDBACK MODEL PREDICTIVE CONTROL

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Abstract: We present a stabilizing scheduled output feedback Model Predictive Control (MPC) algorithm for constrained nonlinear systems with large operating regions. We design a set of local output feedback predictive controllers with their estimated regions of stability covering the desired operating region, and implement them as a single scheduled output feedback MPC which on-line switches between the set of local controllers and achieves nonlinear transitions with guaranteed stability. This algorithm provides a general framework for scheduled output feedback MPC design.

Keywords: Model predictive control, constrained nonlinear systems, output feedback, scheduling, linear matrix inequalities

1. INTRODUCTION

Most practical control systems with large operating regions must deal with nonlinearity and constraints under output feedback control. Nonlinear Model Predictive Control (NMPC) is a powerful design technique that can stabilize processes in the presence of nonlinearities and constraints. Comprehensive reviews of state feedback NMPC algorithms can be found in (De Nicolao *et al.*, 2000). An output feedback NMPC algorithm can be formulated by combining the state feedback NMPC algorithm with a moving horizon observer (MHE)(Findeisen *et al.*, 2000*b*) or an extended Kalman filter (Kothare and Morari, 2000). For an output feedback NMPC algorithm, nonlinear programming (NLP) problems are solved at each sampling time.

Besides developing efficient techniques such as multiple shooting for solving NLP (Findeisen *et al.*, 2000*b*) and parallel programming for control of nonlinear PDE systems (Ma *et al.*, 2002), researchers have proposed various methods to simplify NMPC on-line computation. In (Scokaert *et al.*, 1999), it was proposed that instead of the global optimal solution, an improved feasible solution obtained at each sampling time is enough to ensure stability. In (Magni *et al.*, 2001), a stabilizing NMPC algorithm was developed with a few control moves and an auxiliary controller implemented over the finite control horizon. In (Jadbabaie *et al.*, 2001), stability is guaranteed through the use of an *a priori* control Lyapunov function (CLF) as a terminal cost without imposing terminal state constraints. In (Angeli *et al.*, 2000), nonlinear systems were approximated by linear time varying (LTV) models, and the optimal control problem was formulated as a min-max convex optimization. In (Lu and Arkun, 2002), nonlinear systems were approximated as linear parameter varying (LPV) models, and a scheduling quasi-min-max MPC was developed with the current linear model known exactly and updated at each sampling time.

For a control system with a large operating region, it is desired for the controller to achieve satisfactory performance of the closed-loop system around all setpoints while allowing smooth transfer between them. Pseudolinearization was used in the quasi-infinite horizon NMPC formulation to obtain a closed expression for the controller parameters as a function of the setpoint (Findeisen *et al.*, 2000*a*). A novel gain scheduling approach was introduced in (McConley *et al.*, 2000), in which a set of off-line local controllers are designed with their estimated regions of stability

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overlapping each other. In (El-Farra *et al.*, 2002), a bounded controller was used to define an explicit region of stability for a predictive controller.

In this paper, we extend the scheduled state feedback MPC design in (Wan and Kothare, 2003) to the output feedback case. For an individual local output feedback predictive controller, we characterize explicitly an estimate of its region of stability under the condition that the initial state estimation error is norm bounded, and we also develop a moving horizon scheme to on-line estimate the norm bound of the state estimation error. Then we design a set of local output feedback predictive controllers with their estimated regions of stability overlapping each other. On-line, we implement the resulting family of local controllers as a single scheduled output feedback MPC whose parameters are changed if certain switching criteria on the estimated state and the norm bound of the state estimation error are satisfied. This supervisory scheduling of the local controllers moves the state through the intersections of the estimated regions of stability of different controllers to the desired operating point with guaranteed stability. This algorithm provides a general framework for the scheduled output feedback MPC design, which can incorporate any local output feedback MPC design scheme once its explicit region of stability is characterized. Furthermore, by locally representing the nonlinear system as a set of linear time varying (LTV) models, the original nonlinear non-convex optimization problem can be reduced to a convex optimization problem involving Linear Matrix Inequalities (Wan and Kothare, 2003).

2. LOCAL OUTPUT FEEDBACK MPC FOR CONSTRAINED NONLINEAR SYSTEMS

2.1 State Feedback And No Disturbances

Consider a discrete-time nonlinear dynamical system described by

$$x(k+1) = f(x(k), u(k)) \qquad (1)$$

where $x(k) \in \mathcal{X} \subseteq \Re^n$, $u(k) \in \mathcal{U} \subseteq \Re^m$ are the system state and control input, respectively, $\mathcal{X}$ and $\mathcal{U}$ are compact sets. Assume $f(x,u) = \left[\ f_1(x,u)\ \cdots\ f_n(x,u)\ \right]^T$ are continuous differentiable in x and u.

Definition 1. Given a set $\mathcal{U}$, a point $x_0 \in \mathcal{X}$ is an equilibrium point of the system (1) if a control $u_0 \in int(\mathcal{U})$ exists such that $x_0 = f(x_0, u_0)$. We call a connected set of equilibrium points an equilibrium surface.

Suppose (x^{eq}, u^{eq}) is a point on the equilibrium surface. Within a neighborhood around (x^{eq}, u^{eq}), i.e., $\Pi_x = \{x \in \Re^n |\ |x_r - x_r^{eq}| \leq \delta x_r,\ r = 1,...,n\} \subseteq \mathcal{X}$, and $\Pi_u = \{u \in \Re^m |\ |u_r - u_r^{eq}| \leq \delta u_r,\ r = 1,...,m\} \subseteq \mathcal{U}$, let $\bar{x} = x - x^{eq}$ and $\bar{u} = u - u^{eq}$. The objective is to minimize the infinite horizon quadratic objective function

$$\min_{\bar{u}(k+i|k) = F(k)\bar{x}(k+i|k)} J_\infty(k)$$

subject to

$$|\bar{u}_r(k+i|k)| \leq \delta u_{r,\max},\ \ i \geq 0,\ r = 1,2,...,m \qquad (2)$$

$$|\bar{x}_r(k+i|k)| \leq \delta x_{r,\max},\ \ i \geq 0,\ r = 1,2,...,n \qquad (3)$$

where $J_\infty(k) = \sum_{i=0}^{\infty}[\bar{x}(k+i|k)^T \mathcal{Q} \bar{x}(k+i|k) + \bar{u}(k+i|k)^T \mathcal{R} \bar{u}(k+i|k)]$ with $\mathcal{Q} > 0$, $\mathcal{R} > 0$.We assume that at each sampling time k, a state feedback law $\bar{u}(k+i|k) = F(k)\bar{x}(k+i|k)$ is used to minimize $J_\infty(k)$. At sampling time k, to derive an upper bound on $J_\infty(k)$, define a quadratic function $V(\bar{x}) = \bar{x}^T Q(k)^{-1} \bar{x}$, $Q(k) > 0$. Suppose $V(x)$ satisfies the following exponential stability constraint

$$V(\bar{x}(k+i+1|k) \leq \alpha^2 V(\bar{x}(k+i|k)),\ V(\bar{x}(k|k)) \leq 1 \qquad (4)$$

with $0 < \alpha < 1$. There exists a $\gamma(k) > 0$ such that

$$V(\bar{x}(k+i+1|k)) - V(\bar{x}(k+i|k)) \leq -\frac{1}{\gamma(k)} \times \left[\bar{x}(k+i|k)^T \mathcal{Q} \bar{x}(k+i|k) + \bar{u}(k+i|k)^T \mathcal{R} \bar{u}(k+i|k)\right] \qquad (5)$$

Summing (5) from $i = 0$ to $i = \infty$ and requiring $\bar{x}(\infty|k) = 0$ or $V(\bar{x}(\infty|k)) = 0$, it follows that $J_\infty(k) \leq \gamma(k) V(\bar{x}(k|k)) \leq \gamma(k)$. Therefore, the optimization is formulated as

$$\min_{\gamma(k), F(k)} \gamma(k) \qquad (6)$$

subject to (2)-(5). An ellipsoidal feasible region of the optimization (6) is given by $\mathcal{S} = \left\{\bar{x} \in \Re^n \ \middle|\ \bar{x}^T R^{-1} \bar{x} \leq 1\right\}$, where R is the optimal solution Q of the following maximization

$$\max_{F(k)} det(Q(k)) \qquad (7)$$

subject to (2)-(4). Replacing the state constraint (3) by $\bar{x}(k+i|k) \in \mathcal{S},\ i \geq 0$, or, equivalently

$$R - Q > 0 \qquad (8)$$

which confines the current state and all future predicted states inside $\mathcal{S}$, we develop an exponentially stable MPC algorithm with an estimated region of stability.

Theorem 1. (Exponentially stable MPC) Consider the nonlinear system (1) within the neighborhood (Π_x, Π_u) around (x^{eq}, u^{eq}). Given the controller design parameter $0 < \alpha < 1$. At sampling time k, apply $u(k) = F(k)(x(k) - x^{eq}) + u^{eq}$ where $F(k)$ is obtained from $\min_{\gamma(k), F(k)} \gamma(k)$ subject to (2), (4), (5) and (8). Here R is obtained from the maximization (7) subject to (2)-(4).

Suppose (x^{eq}, u^{eq}) is locally stabilizable, then there exist a neighborhood (Π_x, Π_u) around (x^{eq}, u^{eq}) and $0 < \alpha < 1$ such that the above local state feedback controller exponentially stabilizes the closed-loop system with an estimate of its region of stability $\mathcal{S} = \left\{x \in \Re^n \ \middle|\ (x - x^{eq})^T R^{-1} (x - x^{eq}) \leq 1\right\}$.

Proof. The proof can be found in the Appendix. ■

2.2 State Feedback And Asymptotically Decaying Disturbances

Consider the nonlinear system (1) subject to the unknown additive asymptotically decaying disturbance $d(k)$, $x^p(k+1) = f(x^p(k),u(k)) + d(k)$, where we have made a distinction between the state of the perturbed system, $x^p(k)$, and the state of the unperturbed system, $x(k)$. In order for $x^p(k+1)$ to remain in the region of stability $\mathcal{S}$, we develop a sufficient condition between the norm bound of $d(k)$ and the controller design parameter α. Let $\bar{x}^p(k) = x^p(k) - x^{\text{eq}}$, $\bar{x}(k+1) = f(x^p(k),u(k)) - x^{\text{eq}}$. Suppose $\bar{x}^p(k) \in \mathcal{S}$, (i.e., $\|\bar{x}^p(k)\|^2_{R^{-1}} \le 1$), $\|\bar{x}^p(k+1)\|^2_{R^{-1}} = \|\bar{x}(k+1) + d(k)\|^2_{R^{-1}} = \|\bar{x}(k+1)\|^2_{R^{-1}} + 2\bar{x}(k+1)^T R^{-1} d(k) + \|d(k)\|^2_{R^{-1}}$, where $u(k)$ is computed by Theorem 1. From (8) and (4), we know that $\|\bar{x}(k+1)\|^2_{R^{-1}} \le \|\bar{x}(k+1)\|^2_{Q(k)^{-1}} \le \alpha^2 \|\bar{x}^p(k)\|^2_{Q(k)^{-1}} \le \alpha^2$. Therefore, invariance is guaranteed if $\|\bar{x}^p(k+1)\|^2_{R^{-1}} \le \alpha^2 + 2\alpha\|d(k)\|_{R^{-1}} + \|d(k)\|^2_{R^{-1}} = (\alpha + \|d(k)\|_{R^{-1}})^2 \le 1$. A sufficient condition is $\|d(k)\|_{R^{-1}} \le 1-\alpha$, which means that the disturbance should be bounded in a region $\mathcal{S}^d \triangleq \{d \in \Re^n \,|\, d^T R^{-1} d \le (1-\alpha)^2\}$. Furthermore, since $d(k)$ is asymptotically decaying, the closed-loop trajectory asymptotically converges to the equilibrium $(x^{\text{eq}}, u^{\text{eq}})$.

2.3 Output Feedback

Consider the nonlinear system (1) with a nonlinear output map

$$y(k) = h(x(k)) \in \Re^q \tag{9}$$

where $h(x,u) = \begin{bmatrix} h_1(x) & \cdots & h_q(x) \end{bmatrix}^T$ are continuous differentiable. For all $x, \hat{x} \in \Pi_x$ and $u \in \Pi_u$, consider a full order nonlinear observer with a constant observer gain L_p,

$$\hat{x}(k+1) = f(\hat{x}(k),u(k)) + L_p(h(x(k)) - h(\hat{x}(k))) \tag{10}$$

The error dynamic system is $e(k+1) = f(x(k),u(k)) - f(\hat{x}(k),u(k)) - L_p(h(x(k)) - h(\hat{x}(k)))$. Define a quadratic function $V_e(e) = e^T P e$, $P > 0$. Suppose for all time $k \ge 0$, $x(k), \hat{x}(k) \in \Pi_x$ and $u(k) \in \Pi_u$, and $V_e(e)$ satisfies the following exponential convergent constraint

$$V(e(k+i+1)) \le \rho^2 V(e(k+i)),\ 0 < \rho < 1 \tag{11}$$

In order to facilitate the establishment of the relation between $\|d\|_{R^{-1}}$ and $\|e\|_P$ in §2.4, we want to find a P as close to R^{-1} as possible. Therefore, we minimize γ such that

$$\gamma R^{-1} \ge P \ge R^{-1} \tag{12}$$

Theorem 2. Consider the nonlinear system (1) and (9) within the neighborhood (Π_x, Π_u) around $(x^{\text{eq}}, u^{\text{eq}})$. Given the observer design parameter $0 < \rho < 1$, the constant observer gain L_p of the full order observer (10) is obtained by solving

$$\min_{\gamma, P, L_p} \gamma \tag{13}$$

subject to (11) and (12).

Suppose $(x^{\text{eq}}, u^{\text{eq}})$ is locally observable, then there exist a neighborhood (Π_x, Π_u) around $(x^{\text{eq}}, u^{\text{eq}})$ and an observer design parameter $0 < \rho < 1$ such that the minimization (13) is feasible. Furthermore, if for all time $k \ge 0$, $x(k), \hat{x}(k) \in \Pi_x$ and $u(k) \in \Pi_u$, then the above observer is exponentially convergent.

Proof. The proof can be found in the Appendix. ■

Now we combine the state feedback MPC in Theorem 1 with the observer in Theorem 2 to form a local output feedback MPC for the constrained nonlinear system.

Algorithm 1. (Local output feedback MPC for constrained nonlinear systems) Consider the nonlinear system (1) and the output map (9) within the neighborhood (Π_x, Π_u) around $(x^{\text{eq}}, u^{\text{eq}})$. Given the controller and observer design parameters $0 < \alpha < 1$ and $0 < \rho < 1$. At sampling time $k > 0$, apply $u(k) = F(k)\,(\hat{x}(k) - x^{\text{eq}}) + u^{\text{eq}}$, where $\hat{x}(k)$ is solved by the observer in Theorem 2 with the output measurement $y(k-1)$ and $F(k)$ is solved by the state feedback MPC in Theorem 1 based on $\bar{x}(k) = \hat{x}(k) - x^{\text{eq}}$.

2.4 Stability Analysis of Output Feedback MPC

For the output feedback MPC in Algorithm 1 to be feasible and asymptotically stable, it is required that for all time $k \ge 0$, $x(k), \hat{x}(k) \in \Pi_x$. In this subsection, we study conditions on $x(0)$ and $\hat{x}(0)$ such that $x(k), \hat{x}(k) \in \mathcal{S}$ is satisfied for all times $k \ge 0$.

Consider the closed-loop system based on the output feedback MPC in Algorithm 1,

$$x(k+1) = f(\hat{x}(k),u(k)) + d_1(k)$$
$$\hat{x}(k+1) = f(\hat{x}(k),u(k)) + d_2(k)$$

with $d_1(k) = f(x(k),u(k)) - f(\hat{x}(k),u(k))$ and $d_2(k) = L_p(h(x(k)) - h(\hat{x}(k)))$. At time k, $u(k)$ is obtained by using the state feedback MPC in Theorem 1 based on $\hat{x}(k) - x^{\text{eq}}$.

Suppose $x(k), \hat{x}(k) \in \mathcal{S}$. Since f is continuous differentiable, there exist $\beta_1, \beta_2 > 0$ such that $\|d_1(k)\|_{R^{-1}} \le \beta_1 \|e(k)\|_P$ and $\|d_2(k)\|_{R^{-1}} \le \beta_2 \|e(k)\|_P$. Suppose initially $x(0), \hat{x}(0) \in \mathcal{S}$ and $\|e(0)\|_P \le \eta := \frac{1-\alpha}{\max\{\beta_1,\beta_2\}}$, then $\|d_1(0)\|_{R^{-1}} \le 1-\alpha$ and $\|d_2(0)\|_{R^{-1}} \le 1-\alpha$, which in turn lead to $x(1), \hat{x}(1) \in \mathcal{S}$ (see §2.2) and $\|e(1)\|_P \le \eta$ (see §2.3), and so on. Since for all time $k \ge 0$, $x(k), \hat{x}(k) \in \mathcal{S}$, the state feedback MPC in Theorem 1 is exponentially stable, the observer in Theorem 2 is exponentially convergent, and the combination of both asymptotically stabilizes the closed-loop system.

Theorem 3. Consider the nonlinear system (1) and (9). Suppose $(x^{\text{eq}}, u^{\text{eq}})$ is locally stabilizable and observable, then there exist a neighborhood (Π_x, Π_u) around $(x^{\text{eq}}, u^{\text{eq}})$ and controller and observer design parameters $0 < \alpha < 1$ and $0 < \rho < 1$ such that the output feedback MPC in Algorithm 1 asymptotically stabilizes the closed-loop system for any $x(0), \hat{x}(0) \in \mathcal{S} = \left\{ x \in \Re^n \,\middle|\, (x - x^{\text{eq}})^T R^{-1} (x - x^{\text{eq}}) \leq 1 \right\}$ satisfying $\|x(0) - \hat{x}(0)\|_P \leq \eta$.

Proof. Refer to the derivation in §2.4. ■

2.5 Observability Analysis of Output Feedback MPC

For the stabilizing output feedback MPC in Theorem 3, the state is not measured, but from the output of the system and the estimated state, we can observe the exponential decay of the norm bound of the state estimation error, and thus observe the real state incrementally.

Consider the stabilizing output feedback MPC in Theorem 3. Let the current time be $k \geq \mathcal{T} > 0$. From $k - \mathcal{T}$ to k, an input sequence $\{u(k-\mathcal{T}), ..., u(k-1)\} \subset \Pi_u$ is obtained by the controller based on $\{\hat{x}(k-\mathcal{T}), ..., \hat{x}(k-1)\} \subset \mathcal{S}$. The state evolution starting from $x(k-\mathcal{T})$ driven by $\{u(k-\mathcal{T}), ..., u(k-1)\}$ is inside $\mathcal{S} \subset \Pi_x$. Consider an auxiliary system $\tilde{x}(k+1) = f(\tilde{x}(k), u(k))$ starting from $\tilde{x}(k-\mathcal{T}) = \hat{x}(k-\mathcal{T})$ driven by $\{u(k-\mathcal{T}), ..., u(k-1)\}$. Suppose the state evolution of the auxiliary system from $k - \mathcal{T}$ to k is also inside Π_x, then we can get

$$
\begin{aligned}
x(k+1) - \tilde{x}(k+1) &= f(x(k), u(k)) - f(\tilde{x}(k), u(k)) \\
y(k) - \tilde{y}(k) &= h(x(k)) - h(\tilde{x}(k))
\end{aligned}
$$

Let $V_{\mathcal{T}} := \sum_{j=k-\mathcal{T}}^{k-1} \|y(j) - \tilde{y}(j)\|^2$. Suppose $(x^{\text{eq}}, u^{\text{eq}})$ is locally observable, then there exist a neighborhood (Π_x, Π_u) around $(x^{\text{eq}}, u^{\text{eq}})$ and $\mathcal{T}, \mu > 0$ such that $V_{\mathcal{T}} \geq \mu \|x(k-\mathcal{T}) - \hat{x}(k-\mathcal{T})\|_P^2$, or equivalently, $\|x(k) - \hat{x}(k)\|_P^2 \leq \frac{\rho^{\mathcal{T}} V_{\mathcal{T}}}{\mu}$.

Theorem 4. Consider the nonlinear system (1), (9) and the stabilizing output feedback MPC in Theorem 3. On-line, at time $k \geq \mathcal{T}$, if the state evolution starting from $\hat{x}(k-\mathcal{T})$ driven by the input sequence $\{u(k-\mathcal{T}), ..., u(k-1)\}$ from the controller is inside Π_x, then $\|x(k) - \hat{x}(k)\|_P^2 \leq \frac{\rho^{\mathcal{T}} V_{\mathcal{T}}}{\mu}$.

Proof. The proof can be found in the Appendix. ■

3. SCHEDULED OUTPUT FEEDBACK MPC FOR CONSTRAINED NONLINEAR SYSTEMS

Algorithm 2. (Design of scheduled output feedback MPC) For the nonlinear system (1) and the output map (9), given an equilibrium surface and a desired equilibrium point $(x^{(0)}, u^{(0)})$. Let $i := 0$.

(1) Specify a neighborhood $\left(\Pi_x^{(i)}, \Pi_u^{(i)}\right)$ around $(x^{(i)}, u^{(i)})$ satisfying $\Pi_x^{(i)} \subseteq \mathcal{X}$ and $\Pi_u^{(i)} \subseteq \mathcal{U}$.

(2) Given the controller and observer design parameter $0 < \alpha^{(i)} < 1$ and $0 < \rho^{(i)} < 1$, design Controller #i (Algorithm 1) with its estimated region of stability

$$
\mathcal{S}^{(i)} = \left\{ x \in \Re^n \,\middle|\, \left(x - x^{(i)}\right)^T \left(R^{(i)}\right)^{-1} \times \left(x - x^{(i)}\right) \leq 1 \right\}
$$

Store $\left(x^{(i)}, u^{(i)}, \left(R^{(i)}\right)^{-1}, P^{(i)}, \eta^{(i)}, \mathcal{T}^{(i)}, \mu^{(i)}\right)$ in a lookup table;

(3) Select an equilibrium point $(x^{(i+1)}, u^{(i+1)})$ satisfying $x^{(i+1)} \in int\left(\mathcal{S}_\theta^{(i)}\right)$ with

$$
\mathcal{S}_\theta^{(i)} = \left\{ x \in \Re^n \,\middle|\, \left(x - x^{(i)}\right)^T \left(R^{(i)}\right)^{-1} \times \left(x - x^{(i)}\right) \leq \theta^{(i)} < 1 \right\}
$$

Let $i := i + 1$ and go to step 1, until the region $\cup_{i=0}^{M} \mathcal{S}^{(i)}$ with $M = \max i$ covers a desired portion of the equilibrium surface.

On-line, we implement the resulting family of local output feedback predictive controllers as a single controller whose parameters are changed if certain switching criteria are satisfied. We call such a controller scheme *a scheduled output feedback MPC.*

For the case that $x(0), \hat{x}(0) \in \mathcal{S}^{(i)}$, $i \neq 0$ satisfying $\|x(0) - \hat{x}(0)\|_{P^{(i)}} \leq \eta^{(i)}$, Controller #$i$ asymptotically converges the closed-loop system to the equilibrium $\left(x^{(i)}, u^{(i)}\right)$. Because $x^{(i)} \in int\left(\mathcal{S}_\theta^{(i-1)}\right)$, both $x(k)$ and $\hat{x}(k)$ will enter $\mathcal{S}_\theta^{(i-1)}$ in finite time. At time k, in order to switch from Controller #i to #$(i-1)$, we need to make sure that the initial conditions for stability of Controller #$(i-1)$ are satisfied, i.e., $x(k), \hat{x}(k) \in \mathcal{S}^{(i-1)}$ and $\|x(k) - \hat{x}(k)\|_{P^{(i-1)}} \leq \eta^{(i-1)}$.

Suppose $\hat{x}(k) \in \mathcal{S}_\theta^{(i-1)}$. We know that

$$
\begin{aligned}
&\left\| x(k) - x^{(i-1)} \right\|_{\left(R^{(i-1)}\right)^{-1}} \\
&\leq \|x(k) - \hat{x}(k)\|_{\left(R^{(i-1)}\right)^{-1}} + \left\| \hat{x}(k) - x^{(i-1)} \right\|_{\left(R^{(i-1)}\right)^{-1}} \\
&\leq \|x(k) - \hat{x}(k)\|_{P^{(i-1)}} + \left\| \hat{x}(k) - x^{(i-1)} \right\|_{\left(R^{(i-1)}\right)^{-1}}
\end{aligned}
$$

and $\|x(k) - \hat{x}(k)\|_{P^{(i-1)}}^2 \leq \zeta^{i \to (i-1)} \|x(k) - \hat{x}(k)\|_{P^{(i)}}^2$ with $\zeta^{i \to (i-1)}$ obtained by solving $\min_{\zeta^{i \to (i-1)}} \zeta^{i \to (i-1)}$ subject to $\zeta^{i \to (i-1)} > 0$ and $\zeta^{i \to (i-1)} P^{(i)} - P^{(i-1)} \geq 0$. Hence $x(k) \in \mathcal{S}^{(i-1)}$ and $\|x(k) - \hat{x}(k)\|_{P^{(i-1)}} \leq \eta^{(i-1)}$ are satisfied, if

$$
\begin{aligned}
&\|x(k) - \hat{x}(k)\|_{P^{(i)}}^2 \\
&\leq \frac{1}{\zeta^{i \to (i-1)}} \min\left(1 - \theta^{(i-1)}, \left(\eta^{(i-1)}\right)^2\right). \quad (14)
\end{aligned}
$$

From Theorem 4, we know that if Controller # i has been implemented for at least $\mathcal{T}^{(i)}$ time steps, and if the state evolution starting from $\hat{x}(k - \mathcal{T}^{(i)})$ driven by the input from Controller #i is inside $\Pi_x^{(i)}$, then $\|x(k)-\hat{x}(k)\|^2_{P^{(i)}} \leq \frac{\left(\rho^{(i)}\right)^{\mathcal{T}^{(i)}} V_{\mathcal{T}}^{(i)}}{\mu^{(i)}}$. By imposing an upper bound $\delta^{i\to(i-1)}$ on $V_{\mathcal{T}}^{(i)}$, we can upper bound the state estimation error at current time k. If

$$\delta^{i\to(i-1)} = \frac{\mu^{(i)}}{\zeta^{i\to(i-1)}\left(\rho^{(i)}\right)^{\mathcal{T}^{(i)}}} \times \min\left(1-\theta^{(i-1)}, \left(\eta^{(i-1)}\right)^2\right) \quad (15)$$

then the satisfaction of (14) is guaranteed. Furthermore, because the observer is exponentially converging, for any finite $\delta^{i\to(i-1)}$, there exists a finite time such that $V_{\mathcal{T}}^{(i)} \leq \delta^{i\to(i-1)}$ is satisfied.

Theorem 5. (Scheduled output feedback MPC) Off-line, construct $M+1$ local predictive controllers by Algorithm 2. On-line, given $x(0), \hat{x}(0) \in \mathcal{S}^{(i)}$ satisfying $\|x(0)-\hat{x}(0)\|_{P^{(i)}} \leq \eta^{(i)}$ for some i. Apply Controller #i. Let $T^{(i)}$ be the time period during which Controller #i is implemented. If for Controller #$i>0$, (1) $T^{(i)} \geq \mathcal{T}^{(i)}$, (2) $\hat{x}(k) \in \mathcal{S}_\theta^{(i-1)}$, (3) the state evolution starting from $\hat{x}(k-\mathcal{T}^{(i)})$ driven by the input from Controller #i is inside $\Pi_x^{(i)}$, and $V_{\mathcal{T}}^{(i)} \leq \delta^{i\to(i-1)}$, then, at the next sampling time, switch from Controller #i to Controller #$(i-1)$; Otherwise, continue to apply Controller #i. The above scheduled output feedback MPC asymptotically stabilizes the closed-loop system to the desired equilibrium $(x^{(0)}, u^{(0)})$.

Proof. Refer to the derivation in §3. ■

Remark 1. If for the $M+1$ local predictive controllers constructed by Algorithm 2, not only $x^{(i)} \in int\left(\mathcal{S}_\theta^{(i-1)}\right)$, $i=1,\ldots,M$, but also $x^{(i)} \in int\left(\mathcal{S}_\theta^{(i+1)}\right)$, $i=0,\ldots,M-1$, then on-line, the scheduled MPC in Theorem 5 can not only switch from Controller #i ($i=1,\ldots,M$) to Controller #$(i-1)$, but also switch from Controller #i ($i=0,\ldots,M-1$) to Controller #$(i+1)$ with $\delta^{i\to(i+1)}$ defined as (15) with $i-1$ replaced by $i+1$.

4. EXAMPLE

Consider the highly nonlinear model of a continuous stirred tank reactor (CSTR) (Magni *et al.*, 2001).

$$\dot{C}_A = \frac{q}{V}(C_{Af}-C_A) - k_0 \exp\left(-\frac{E}{RT}\right) C_A$$
$$\dot{T} = \frac{q}{V}(T_f - T) + \frac{(-\Delta H)}{\rho C_p} k_0 \exp\left(-\frac{E}{RT}\right) C_A + \frac{UA}{V\rho C_p}(T_c - T) \quad (16)$$

where C_A is the concentration of A in the reactor, T is the reactor temperature, and T_c is the temperature of the coolant stream. The parameters are $q = 100$ l/min, $V = 100$ l, $C_{Af} = 1$ mol/l, $T_f = 400$ K, $\rho = 10^3$g/l, $C_p = 1$J/(g K), $k_0 = 4.71 \times 10^8$ min^{-1}, $E/R = 8000$ K, $\Delta H_{rxn} = -2\times 10^5$J/mol, , $UA = 10^5$J/(min K). Sampling time is 0.03 min. We design seven local output feedback predictive controllers to cover the operating region $360K \leq T \leq 400$ and 0.3mol/l$\leq C_A \leq$ 1mol/l. Let $\alpha = 0.98$, $\theta = 0.9$, $\mathcal{Q}$ =diag($\frac{1}{0.5}, \frac{1}{400}$) and $\mathcal{R} = \frac{1}{300}$ for all the controller designs and $\rho^{(0)},\ldots,\rho^{(6)} = 0.91,\ldots,0.97$ for the observer designs. Let $\mathcal{T} = 10$, and we use the linearized models at the equilibrium points to estimate $\mu^{(i)}$. Let $\delta = 1$ for all switches. Figure 1 shows the seven local output feedback predictive controllers with their explicit regions of stability, and the nonlinear transitions by using the scheduled MPC. Figure 2 shows the time responses. Close-up views of the responses of the real and estimated temperatures are provided to show fast convergence of the observer.

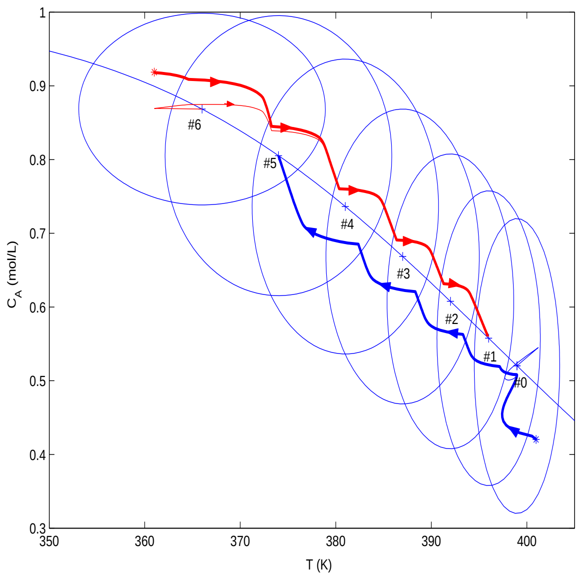

Fig. 1. Phase plots of the nonlinear transitions from $(0.92, 361)^T$ to $(C_A^{(1)}, T^{(1)})^T$ and from $(0.42, 401)^T$ to $(C_A^{(5)}, T^{(5)})^T$. Lines with width 2 for $(C_A, T)^T$ and lines with width 0.5 for $(\hat{C}_A, \hat{T})^T$.

5. CONCLUSIONS

In this paper, we have proposed a stabilizing scheduled output feedback MPC formulation for constrained nonlinear systems with large operating regions. Since we were able to characterize explicitly an estimated region of stability of the designed local output feedback predictive controller, we could expand it by designing multiple predictive controllers, and on-line switch between the local controllers and achieve nonlinear transitions with guaranteed stability. This algorithm provides a general framework for the scheduled output feedback MPC design. Furthermore, we have shown that this scheduled MPC is easily implementable by applying it to a highly nonlinear CSTR process.

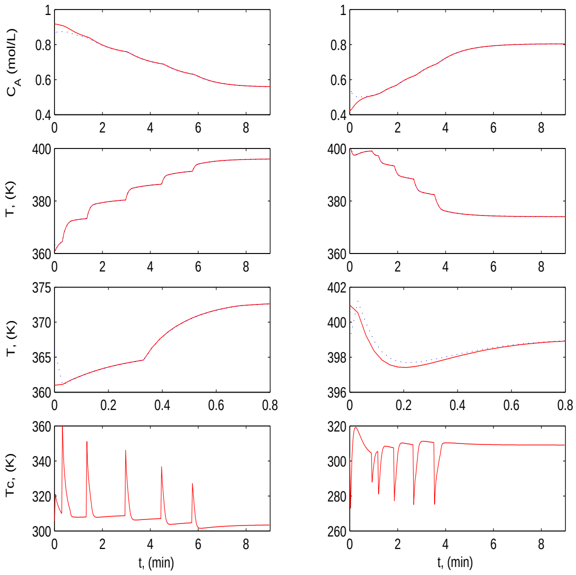

Fig. 2. Time responses for the nonlinear transitions from $(0.92, 361)^T$ to $(C_A^{(1)}, T^{(1)})^T$ and from $(0.42, 401)^T$ to $(C_A^{(5)}, T^{(5)})^T$ with solid lines for the real states and dotted lines for the estimated states.

APPENDIX

Proof of Theorem 1: Within a neighborhood (Π_x, Π_u) around $(x^{\text{eq}}, u^{\text{eq}})$, we locally represent the nonlinear system (1) by a LTV model $\bar{x}(k+1) = A(k)\bar{x}(k) + B(k)\bar{u}(k)$ with $\left[\, A(k) \;\; B(k) \,\right] \in \Omega$. For all $x \in \Pi_x$ and $u \in \Pi_u$, the Jacobian matrix $\left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}\right] \in \Omega$ with

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \text{ and } \frac{\partial f}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}.$$

It is straight forward to establish the closed-loop exponential stability within $\mathcal{S}$ based on the LTV model. Since the LTV model is a representation of a class of nonlinear systems including the given nonlinear system (1) within the neighborhood (Π_x, Π_u), the closed-loop nonlinear system is exponentially stable within $\mathcal{S}$. ■

Proof of Theorem 2 and 4: Following the same procedure as in the proof for Theorem 1, we locally represent the nonlinear error dynamics as a LTV model $e(k+1) = (A(k) - L_p C(k))\, e(k)$ with $\left[\, A(k)^T \;\; C(k)^T \,\right]^T \in \Psi$. For all $x, \hat{x} \in \Pi_x$ and $u \in \Pi_u$, the Jacobian matrix

$$\left[\left(\frac{\partial f}{\partial x}\right)^T \; \left(\frac{\partial h}{\partial x}\right)^T \right]^T \in \Psi \text{ with } \frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

and $\frac{\partial h}{\partial x} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_q}{\partial x_1} & \cdots & \frac{\partial h_q}{\partial x_n} \end{bmatrix}$. It is straight forward to establish the exponential convergency of the observer and the norm bound of the state estimation error within $\mathcal{S}$ based on the LTV model and the nonlinear model. ■

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