# ADAPTIVE EXTREMUM SEEKING OUTPUT FEEDBACK CONTROL FOR A CONTINUOUS STIRRED TANK BIOREACTOR

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Abstract: An adaptive extremum seeking controller is presented for the optimization of the production rate of a continuous stirred tank bioreactor. This controller is saturated outside a domain of interest and a reduced-order high-gain observer is designed to estimate the substrate concentration of the bioreactor. Semiglobal asymptotic stability is proved and recovery of the performance achieved under state feedback is shown when the speed of the high gain observer is sufficiently high. Simulation experiment is given to illustrate the proposed approach.

Keywords: Adaptive extremum seeking, parameter estimation, persistence of excitation, output feedback, separation principle.

## 1. INTRODUCTION

Adaptive extremum seeking control of nonlinear systems has received the attention of many researchers. The potential benefits of extremum seeking techniques in the maximization of the production rate in a continuous stirred tank bioreactor has been demonstrated by (Wang *et al.*, 1999) and (Zhang *et al.*, 2001). Practical implementation of the controller scheme designed in (Zhang *et al.*, 2001) requires the measurement of substrate concentration and production rate. However, knowledge of the substrate concentration is not always possible. The extension of these results to the output feedback requires the construction of an observer to estimate the unmeasured state of the system from its output.

Owing to nonlinearity (Lee and Khalil, 1997), a separation principle cannot be applied in the design of output feedback control as in linear control theory, but a certain degree of separation can be achieved by designing high-gain observers. High-gain observers, however, exhibit peaking in their transient behavior (Esfandiari and Khalil, 1992). Fortunately, this peaking phenomenon in certain classes of systems.

In this work, an adaptive extremum-seeking output feedback controller is designed by the application of a similar separation principle. The design is achieved in two steps. First, we saturate the controller scheme and the right hand side of the adaptation rules designed in (Zhang *et al.*, 2001) for the continuous stirred tank bioreactor. Second, we use an high-gain observer to estimate the substrate concentration, based on the measurement of the production rate. Using Lyapunov theory, we prove that the output feedback controller recovers the performance achieved under state feedback when the gain of the observer is large enough. The rest of the paper is organized as follows. Section 2 presents some notation and the problem formulation for the state feedback case. In Section 3, the reduced order high gain observer is designed. The performance recovery is shown in Section 4,

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followed by simulation results in Section 5 and a brief conclusion in Section 6.

## 2. STATE FEEDBACK CONTROL

We consider the following microbial growth models for a continuous stirred tank bioreactor (Zhang *et al.*, 2001)

$$\dot{y} = -uy + \frac{\theta_\mu s^2 y - \theta_k y^2 + (s_0 - s)uy}{s(1+\theta_s s)} \quad (1)$$

$$\dot{s} = -\theta_k y + u(s_0 - s) \quad (2)$$

where the states $s > 0$ and $y > 0$ denote the substrate concentration, and the production rate of the reaction product, respectively. The input of the system is the dilution rate $u \geq 0$, and $s_0$ denotes the concentration of the substrate in the feed.

The constant parameter $\theta_k$ is known, while the constant parameters $\theta_s$, $\theta_\mu$ are unknown. However, the vector $\theta = [\theta_s \quad \theta_\mu]^T$ belongs to $\Omega$, a known compact convex subset of $R^2$. Let $\hat{\Omega}$ be a convex subset of $R^2$ which contains $\Omega$ in its interior.

The adaptive extremum seeking controller and the adaptation rules for the parameters of the system are designed in (Zhang *et al.*, 2001) for the state feedback case. The state feedback controller is

$$u = \frac{1}{(s_0 - s)}(\theta_k y - a(t) + d - k_z z_s) \quad (3)$$

where $a(t)$ and $z_s$ corresponds to the dither signal (to be designed later) and the error in the set-point $s^*$, respectively

$$z_s = s - s^* + d, \quad s^* = \frac{1}{\hat{\theta}_s}\left(\sqrt{1 + s_0\hat{\theta}_s} - 1\right) \quad (4)$$

Let $\hat{\theta}$ denote the estimate of the true parameter $\theta$ and let $\hat{y}$ be the prediction of the state $y$ by using the estimated parameters $\hat{\theta}_s$ and $\hat{\theta}_\mu$. The predicted state $\hat{y}$ and $d$ are generated by

$$\dot{\hat{y}} = -uy + \frac{\hat{\theta}_\mu s^2 y - \theta_k y^2 + (s_0 - s)uy}{s(1+\hat{\theta}_s s)} + k_y e_y \quad (5)$$

$$\dot{d} = -\dot{\hat{\theta}}_s \beta(\hat{\theta}_s) + a(t) - d \quad (6)$$

where $e_y = y - \hat{y}$.

We suppose $\Omega_s$ and $\Omega_\mu$ are convex hypercubes, (see (Khalil, 1996)) $Omega_i = \{\theta \mid a_i \leq \theta_i \leq b_i\}$ for $i = \ s, \ \mu$. Let

$$\Omega_{\delta-i} = \{\theta \mid a_i - \delta_i \leq \theta_i \leq b_i + \delta_i\} \qquad for\ i = s\ \mu$$

where $\delta_s > 0$ and $\delta_\mu > 0$ are chosen such that $\Omega_{\delta-s} \subset \hat{\Omega}_s$ and $\Omega_{\delta-\mu} \subset \hat{\Omega}_\mu$.

The parameter adaptation rule for $\hat{\theta}_i$ with $i = s, \ \mu$, is taken as

$$\dot{\hat{\theta}}_i = \begin{cases} \Gamma_i & \text{if } a_i \leq \hat{\theta}_i \leq b_i \text{ or} \\ & \text{if } \hat{\theta}_i > b_i \text{ and } \Gamma_i \leq 0 \text{ or} \\ & \text{if } \hat{\theta}_i < a_i \text{ and } \Gamma_i \geq 0 \\ (1 - c_i(\hat{\theta}_i))\Gamma_i & \text{if } \hat{\theta}_i > b_i \text{ and } \Gamma_i > 0 \text{ or} \\ & \text{if } \hat{\theta}_i < a_i \text{ and } \Gamma_i < 0 \end{cases} \quad (7)$$

for $\hat{\theta}_i > b_i$ and $\Gamma_i > 0$

$$c_i(\hat{\theta}_i) = \left(\frac{\hat{\theta}_i - b_i}{\delta_i}\right) sign(\Gamma_i) \quad (8)$$

and for $\hat{\theta}_i < a_i$ and $\Gamma_i < 0$

$$c_i(\hat{\theta}_i) = \left(\frac{\hat{\theta}_i - a_i}{\delta_i}\right) sign(\Gamma_i) \quad (9)$$

Equation (7) is a smooth projection algorithm (Pomet and Praly, 1992).

The nominal value for $\dot{\hat{\theta}}_i$ is $\Gamma_i$ where

$$\Gamma_s = \frac{\gamma_s \phi_s y e_y}{(1+\hat{\theta}_s s)}, \ \Gamma_\mu = \frac{\gamma_\mu \phi_\mu y e_y}{(1+\hat{\theta}_s s)} \quad (10)$$

with $\phi_s = -u(s_0 - s) - \hat{\theta}_\mu s^2 + \theta_k y$ and $\phi_\mu = (1 + \hat{\theta}_s s)s$. It can be seen from equations (8) and (9) that $0 \leq c_i(\hat{\theta}_i) \leq 1$ and $c_i(\hat{\theta}_i) = 0$ for $\dot{\hat{\theta}}_i = \Gamma_i$.

Equations (1)-(10) represent the system under state feedback. Let the vector $\psi = [s\ y\ d\ \hat{y}\ \hat{\theta}_s\ \hat{\theta}_\mu]^T$ represent the trajectories of the closed loop system. Then considering $\chi = [z_s \quad \tilde{\theta}_s \quad \tilde{\theta}_\mu \quad e_y]^T$, we have

$$\dot{\chi} = \begin{bmatrix} \dot{z}_s \\ \dot{\tilde{\theta}}_s \\ \dot{\tilde{\theta}}_\mu \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} \dot{s} - \dot{s}^* + \dot{d} \\ -\dot{\hat{\theta}}_s \\ -\dot{\hat{\theta}}_\mu \\ \dot{y} - \dot{\hat{y}} \end{bmatrix} = \begin{bmatrix} f_1(\psi) \\ f_2(\psi) \\ f_3(\psi) \\ f_4(\psi) \end{bmatrix} \quad (11)$$

For simplicity, we can define

$$f_r(\psi) = [f_1(\psi)\ f_2(\psi)\ f_3(\psi)\ f_4(\psi)]^T$$

and express equation (11) as

$$\dot{\chi} = f_r(\psi) \quad (12)$$

For the system (12) we consider the following Lyapunov function

$$V(\chi, t) = \frac{1}{2}\left[z_s^2 + \frac{\tilde{\theta}_s^2}{\gamma_s} + \frac{\tilde{\theta}_\mu^2}{\gamma_\mu} + (1+\theta_s s)e_y^2\right] \quad (13)$$

The rate of change of the Lyapunov function (13) is

$$\dot{V} = \frac{\partial V}{\partial \chi} f_r(\psi) + \frac{\partial V}{\partial s}\dot{s} \leq -U_3(\chi) \quad (14)$$

where $U_3(\chi) = k_z z_s^2 + k_{y0} e_y^2$.

*Remark 1.* The functions $f_1(\psi), f_2(\psi), f_3(\psi)$, and $f_4(\psi)$ are locally Lipschitz in their arguments over the domain of interest.

*Remark 2.* Assuming that the persistency of excitation condition developed in (Zhang *et al.*, 2001) is met, the origin ($z = 0, \tilde{\theta}_s = 0, \tilde{\theta}_\mu = 0, e_y = 0$) is an equilibrium point of the closed loop system. The asymptotic stability of the origin for the state feedback system (12) was proved in (Zhang *et al.*, 2001).

## 3. OUTPUT FEEDBACK CONTROL

We consider the case where only $y$ is measurable, the substrate concentration $s$ is not available for feedback control. By the locally observability condition (Marino and Tomei, 1995), the system is observable for $y > 0$. To implement the state feedback adaptive controller (3), we need to estimate the unmeasured state $s$. The estimation of the states $y$ and $s$ are given by $\hat{y}_{obs}$ and $\hat{s}_{obs}$. We use the reduced-order high-gain observer $\hat{x} = [\hat{y}_{obs} \quad \hat{s}_{obs}]^T$

$$\dot{\hat{y}}_{obs} = -uy + \frac{\hat{\theta}_\mu \hat{s}_{obs}^2 y - \theta_k y^2 + (s_0 - \hat{s}_{obs})uy}{\hat{s}_{obs}(1 + \hat{\theta}_s \hat{s}_{obs})} + \frac{\alpha_y}{\epsilon}\tilde{y} \quad (15)$$

$$\dot{\hat{s}}_{obs} = -\theta_k y + u(s_0 - \hat{s}_{obs}) + \frac{\alpha_s}{\epsilon^2}\tilde{y} \quad (16)$$

where $\tilde{y}$, $\tilde{s}$ are defined as $\tilde{y} = y - \hat{y}_{obs}$ and $\tilde{s} = s - \hat{s}_{obs}$ and $\alpha_s, \alpha_y, \epsilon$ are positive constants.
For the output feedback, the dynamics for the production rate is represented by (1) and the dynamics of the substrate concentration is represented by (2). The controller for the output feedback system is

$$u = \frac{1}{(s_0 - \hat{s}_{obs})}(\theta_k y - a(t) + d - k_z z_s) \quad (17)$$

In order to avoid the singularity that may happen in the controller when the estimation of the substrate concentration increases, we bound the state $\hat{s}_{obs}$ below and above by the positive bounds $\hat{s}_{obs-min}$ and $0.99 s_0$ respectively.
To overcome the peaking phenomenon associated with the high gain observer, we saturate the controller and the rate of change of $\hat{y}$, $d$, $\hat{\theta}_s$, and $\hat{\theta}_\mu$ outside the domain of interest. The rate of change of $\hat{y}$ and $d$ are

$$\dot{d} = -\dot{\hat{\theta}}_s \beta(\hat{\theta}_s) + a(t) - d \quad (18)$$

$$\dot{\hat{y}} = -uy + \frac{\hat{\theta}_\mu \hat{s}_{obs}^2 y - \theta_k y^2 + (s_0 - \hat{s}_{obs})uy}{\hat{s}_{obs}(1 + \hat{\theta}_s \hat{s}_{obs})} + k_y e_y \quad (19)$$

The parameter adaptation rule for the output feedback case is the same as that for the state feedback case. However, the nominal updating laws for $\dot{\hat{\theta}}_s$ and $\dot{\hat{\theta}}_\mu$ are

$$\Gamma_s = \frac{\gamma_s \phi_s y e_y}{(1 + \hat{\theta}_s \hat{s}_{obs})}, \Gamma_\mu = \frac{\gamma_\mu \phi_\mu y e_y}{(1 + \hat{\theta}_\mu \hat{s}_{obs})} \quad (20)$$

with

$$\phi_s = -u(s_0 - \hat{s}_{obs}) - \hat{\theta}_\mu \hat{s}_{obs}^2 + \theta_k y \quad (21)$$

$$\phi_\mu = (1 + \hat{\theta}_s \hat{s}_{obs})\hat{s}_{obs} \quad (22)$$

The error dynamics for the observer are

$$\dot{\tilde{e}} = \begin{bmatrix} \dot{\tilde{y}} \\ \dot{\tilde{s}} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha_y}{\epsilon} & F_1 \\ -\frac{\alpha_s}{\epsilon^2} & -u \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{s} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} G \quad (23)$$

where $F_1 = \frac{y}{(1+\theta_s s)(1+\hat{\theta}_s \hat{s}_{obs})}\theta_\mu$ and $G$ is defined as $G = \frac{y}{(1+\theta_s s)(1+\hat{\theta}_s \hat{s}_{obs})}[\theta_\mu \hat{\theta}_s \hat{s}_{obs} s + \tilde{\theta}_\mu \hat{s}_{obs} - \hat{\theta}_\mu \theta_s \hat{s}_{obs} s] + \frac{-\theta_k y^2 + (s_0 - s)uy}{s(1+\theta_s s)} - \frac{-\theta_k y^2 + (s_0 - \hat{s}_{obs})uy}{\hat{s}_{obs}(1+\hat{\theta}_s \hat{s}_{obs})}$.
We scale the observer dynamics as $\tilde{y} = \xi_1$ and $\tilde{s} = \frac{\xi_2}{\epsilon}$. Replacing equation (23) by its scaled equivalent, we get

$$\epsilon\dot{\xi} = A(t)\xi + \epsilon BG \quad (24)$$

where $\xi = [\xi_1 \quad \xi_2]^T$, $A(t) = \begin{bmatrix} -\alpha_y & F_1 \\ -\alpha_s & -u\epsilon \end{bmatrix}$ and $B = [1 \quad 0]^T$.

## 4. PERFORMANCE RECOVERY

In this section, we follow the procedure used in (Atassi and Khalil, 1999) and (Khalil, 1996) to show semi-global asymptotic stability of the origin.

**1. BOUNDEDNESS**

Considering the equations (1), (2), (18), (19) and the parameter updating laws (7) with nominal updating laws (20), the rate of change of the vector $\chi$ for the output feedback becomes

$$\dot{\chi} = \begin{bmatrix} \dot{z}_s \\ \dot{\tilde{\theta}}_s \\ \dot{\tilde{\theta}}_\mu \\ \dot{e}_y \end{bmatrix} = \begin{bmatrix} \dot{s} - \dot{s}^* + \dot{d} \\ -\dot{\hat{\theta}}_s \\ -\dot{\hat{\theta}}_\mu \\ \dot{y} - \dot{\hat{y}} \end{bmatrix} = f_r(\psi, D(\epsilon)) \quad (25)$$

and also

$$\dot{s} = h_r(\psi, D(\epsilon)) \quad (26)$$

The initial conditions for equation (25) are $\chi(0) = (z_s(0), \tilde{\theta}_s(0), \tilde{\theta}_\mu(0), e_y(0)) = (z_{s0}, \tilde{\theta}_{s0}, \tilde{\theta}_{\mu 0}, e_{y0}) \in \mathcal{U}$. Related to the set $\mathcal{U}$ there is $\mathcal{U}'$ which is the set of initial conditions for the states $\psi$. In other

words, $\psi(0) = (s(0), y(0), d(0), \hat{y}(0), \hat{\theta}_s(0), \hat{\theta}_\mu(0)) \in \mathcal{U}'$. The initial states for the estimated parameters are $\hat{x}(0) = (\hat{y}_{obs}(0), \hat{s}_{obs}(0)) = \hat{x}_0 \in \mathcal{Q}$.
The system (24), (25) and (26) is a standard singularly perturbed one. It can be noticed that $\xi = 0$ is the unique solution of (24) when $\epsilon = 0$. If we substitute $\epsilon = 0$ in (25) we get the closed-loop system under state feedback, equation (12). Then, the reduced system is given by

$$\dot{\chi} = f_r(\psi, 0) \tag{27}$$

The boundary-layer system obtained by applying to (24) the change of time variable $\tau = t/\epsilon$ then setting $\epsilon = 0$, is given by

$$\frac{d\xi}{d\tau} = A(t)\xi \tag{28}$$

We denote $(\chi(t,\epsilon), \xi(t,\epsilon))$ the trajectory of system (24) and (25) starting from $(\chi(0), \xi(0))$. The recovery of the boundedness of trajectories is summarized in the following theorem.

*Theorem 3.* Let *Remark 1* and *Remark 2* hold, then there exists $\epsilon_1^* > 0$ such that, for every $0 < \epsilon \leq \epsilon_1^*$, the trajectories $(\chi, \xi)$ of system (25) and (24), starting in $\mathcal{U} \times \mathcal{Q}$ are bounded for all $t \geq 0$.

**PROOF.** The origin of (12) is asymptotically stable with a region of attraction $\mathcal{R}$. Based on equations (13), and (14) there are three positive functions $U_1(\chi)$, $U_2(\chi)$ and $U_3(\chi)$, all defined and continuous on $\mathcal{R}$ such that

$$U_1(\chi) \leq V(\chi, t) \leq U_2(\chi) \tag{29}$$

$$\lim_{\chi \to \partial\mathcal{R}} U_1(\chi) = \infty \tag{30}$$

$$\dot{V} = \frac{\partial V}{\partial \chi} f_r(\psi) + \frac{\partial V}{\partial s}\dot{s} \leq -U_3(\chi) \tag{31}$$

where $U_3(\chi)$ is defined above. The functions $U_1(\chi)$ and $U_2(\chi)$ are

$$U_1(\chi) = k_{u1}\left[z_s^2 + \frac{\tilde{\theta}_s^2}{\gamma_s} + \frac{\tilde{\theta}_\mu^2}{\gamma_\mu} + e_y^2\right]$$

$$U_2(\chi) = k_{u2}\left[z_s^2 + \frac{\tilde{\theta}_s^2}{\gamma_s} + \frac{\tilde{\theta}_\mu^2}{\gamma_\mu} + (1 + \theta_s s_0)e_y^2\right]$$

with $0 < k_{u1} < 1/2$ and $1/2 < k_{u2}$. Equations (29), (30) and (31) are satisfied for all $\chi \in \mathcal{R}$. The properness of $V(\chi, t)$ in $\mathcal{R}$ guarantees that with any finite $c > max_{\chi \in \mathcal{U},\ s \in \mathcal{U}'} V(\chi, t)$, the set $\Sigma = \{\chi \in \mathcal{R} : V(\chi, t) \leq c\}$ is a compact subset of $\mathcal{R}$ and $\mathcal{U}$ is in the interior of $\Sigma$. Similarly, we can prove that there exists a compact set $\Sigma'$ which is a subset of $\mathcal{R}$ and $\mathcal{U}'$ is in the interior of $\Sigma'$.

For the boundary layer system we define the Lyapunov function

$$W(\xi) = \xi^T P_0 \xi \tag{32}$$

where $P_0 = P_0^T$ is the positive definite solution of the Lyapunov equation $P_0 A(t) + A(t)^T P_0 = -Q(t)$. The matrix $Q(t)$ is symmetric and positive definite. This function satisfies

$$\lambda_{min}(P_0)\|\xi\|^2 \leq W(\xi) \leq \lambda_{max}(P_0)\|\xi\|^2 \tag{33}$$

$$\frac{\partial W}{\partial \tau} = -\xi Q(t)\xi \leq -\lambda_{min}(Q(t))\|\xi\|^2 \tag{34}$$

Let $\Lambda = \Sigma \times \{W(\xi) \leq \rho\epsilon^2\}$. Due to Remarks 1-2 we have, for all $\chi \in \Sigma$, all $\psi \in \Sigma'$ and all $\xi \in \mathcal{R}^2$

$$\|f_r(\psi, D(\epsilon)\xi)\| \leq k_1 \tag{35}$$

$$\|G(\psi, D(\epsilon)\xi)\| \leq k_2 \tag{36}$$

$$\|h_r(\psi, D(\epsilon)\xi)\| \leq k_3 \tag{37}$$

where $k_1, k_2$ and $k_3$ are positive constants independent of $\epsilon$. Moreover, for any $0 < \tilde{\epsilon} < 1$, there is $L_1$, independent of $\epsilon$, such that, for all $(\chi, \xi) \in \Lambda$ and every $0 < \epsilon \leq \tilde{\epsilon}$, we have

$$\|f_r(\psi, D(\epsilon)\xi) - f_r(\psi, 0)\| \leq L_1\|\xi\| \tag{38}$$

$$\|h_r(\psi, D(\epsilon)\xi) - h_r(\psi, 0)\| \leq L_2\|\xi\| \tag{39}$$

Proceeding as in (Atassi and Khalil, 1999), we show that there exists $0 \leq \epsilon \leq \epsilon_1^*$ such that the trajectory $(\chi(t,\epsilon), \xi(t,\epsilon))$ enters $\Lambda$ during the interval $[0, T(\epsilon)]$ and remains there for all $t \geq T(\epsilon)$ where

$$T(\epsilon) = \frac{\epsilon}{\sigma_1} ln\Big(\frac{\sigma_2}{\rho\epsilon^4}\Big) \leq T_0. \tag{40}$$

Thus the trajectory is bounded for all $t \geq T(\epsilon)$. On the other hand, for $t \in [0, T(\epsilon)]$, the trajectory $(\chi(t,\epsilon), \xi(t,\epsilon))$ is bounded.

## 2. ULTIMATE BOUNDEDNESS

Next, we show that the trajectories of system (25) and (24), starting in $\mathcal{U} \times \mathcal{Q}$, come arbitrarily close to the origin as time progresses. This is summarized in the following theorem.

*Theorem 4.* Under the conditions of *Theorem 1*, given any $\eta > 0$, there exists $\epsilon_2^* = \epsilon_2^*(\eta) > 0$ and $T_1 = T_1(\eta)$ such that, for every $0 < \epsilon \leq \epsilon_2^*$, we have

$$\|\chi(t,\epsilon)\| + \|\xi(t,\epsilon)\| \leq \eta, \qquad \forall \ \ t \geq T_1. \tag{41}$$

**PROOF.** Due to space restrictions we omit the proof which proceeds as in (Atassi and Khalil, 1999).

## 3. TRAJECTORY CONVERGENCE

Let $\chi_r(t)$ be the solution of (27) starting from $\chi(0)$. In this section we follow the procedure used in (Atassi and Khalil, 1999) to prove that $\chi(t,\epsilon)$ converges to $\chi_r(t)$ as $\epsilon \to 0$ uniformly in $t$, for all $t \geq 0$. As in (Atassi and Khalil, 1999), we divide the interval $[0,\infty]$ into three intervals $[0,T(\epsilon)]$, $[T(\epsilon),T_2]$ and $[T_2,\infty]$, and based on *Theorem 1* and *Theorem 2*, we show $\|\chi(t,\epsilon)-\chi_r(t)\| \leq \eta$ for each interval.

## 4. ASYMPTOTIC STABILITY

We define $F_2^T = \left[\frac{\Phi^T(\hat{s}_{obs},y,\hat{\theta})y}{(1+\theta_s s)(1+\hat{\theta}_s\hat{s}_{obs})}\right]$ where where $\Phi^T = [\phi_s \quad \phi_\mu]$, $\tilde{\theta} = [\tilde{\theta}_s \quad \tilde{\theta}_\mu]$ and $F_3$ is a function $\psi$. From equations (1) and (19),

$$\dot{e}_y = -k_y e_y + F_{2e}^T\tilde{\theta} + (F_2 - F_{2e})^T\tilde{\theta} + F_3\tilde{s} \quad (42)$$

The subscript $e$ indicates that the function is evaluated at steady state. From the projection algorithms (7) with the nominal updating laws (20) we define new state variables

$$\frac{\partial(F_{2e}^T\tilde{\theta})}{\partial t} = -F_{2e}^T R_e N_e e_y - F_{2e}^T(RN - R_eN_e)e_y + \left(\frac{\partial F_{2e}}{\partial t}\right)^T\tilde{\theta} \quad (43)$$

with $R = \frac{y}{(1+\hat{\theta}_s\hat{s}_{obs})}$ and $N = \begin{bmatrix}\gamma_s\phi_s(1-c_s(\hat{\theta}))\\ \gamma_\mu\phi_\mu(1-c_\mu(\hat{\theta}))\end{bmatrix}$
Re-arranging equations (42) and (43) in a matrix form, we get

$$\dot{w} = C(t)w + E(t) + F_6\tilde{s} \quad (44)$$

where $w = [e_y \quad F_{2e}^T\tilde{\theta}]^T$, $C(t) = \begin{bmatrix}-k_y & 1\\ -F_{2e}^TR_eN_e & 0\end{bmatrix}$, $E(t) = \begin{bmatrix}(F_2-F_{2e})^T\tilde{\theta}\\ -F_{2e}^T(RN-R_eN_e)e_y + \left(\frac{\partial F_{2e}}{\partial t}\right)^T\tilde{\theta}\end{bmatrix}$ and $F_6 = [1 \quad 0]^T F_3$ Equation (44) is a linear time variant system. It can be noticed that when time$\to\infty$, $E(t)\to 0$. Matrix $C(t)$ is Hurwitz if and only if $F_{2e}^TR_eN_e > 0$.
In equation (24), the function $G$ can be written as $G = F_4\tilde{\theta} + F_5\tilde{s}$, where $F_4$ and $F_5$ are functions of $\psi$. Then equation (24) becomes

$$\dot{\xi} = \frac{1}{\epsilon}A(t)\xi + BF_4\tilde{\theta} + BF_5\tilde{s} \quad (45)$$

For the system (44) and (45), we define a new Lyapunov function

$$V_T = V_w^{\frac{1}{2}} + W^{\frac{1}{2}} \quad (46)$$

where $V_w = w^TM_0w$, and $W$ corresponds to the Lyapunov function for the boundary layer system,equation (32). The constant matrix $M_0$ is positive definite and symmetric. We select the matrix $L(t)$, a positive definite and symmetric matrix such that $C(t)^TM_0 + M_0C(t) = -L(t)$. It can be verified that the rate of change of the Lyapunov function $V_T$ is

$$\dot{V} \leq -K_3\|w\| - K_4\|\xi\| + K_2\|2E(t)^TM_0\| \quad (47)$$

where $K_3$ and $K_4$ are positive bounds for the states over the domain of interest. Let $K_5 = min(K_3,K_4)$, and let

$$K_6 = K_5(max(\sqrt{\lambda_{max}(M_0)}, \sqrt{\lambda_{max}(P_0)}))$$

then equation (47) becomes

$$\dot{V}_T \leq -K_6V_T + K_2\|2E(t)^TM_0\| \quad (48)$$

Integration of equation (48), yields

$$V_T(t) \leq V_T(t_0)e^{-K_6(t-t_0)} + \int_{t_0}^{t} e^{-K_6(t-\tau)}K_2\|2E(\tau)^TM_0\|d\tau \quad (49)$$

When time$\to\infty$, $E(\tau)\to 0$. Then inequality (49) vanishes as time$\to\infty$. This means that $V_T = V^{\frac{1}{2}} + W^{\frac{1}{2}} \to 0$ or

$$\|w\| = \|[e_y \quad F_{2e}^T\tilde{\theta}]^T\| \to 0 \quad (50)$$

$$\|\xi\| = D(\epsilon)^{-1}\|[\tilde{y} \quad \tilde{s}]^T\| \to 0 \quad (51)$$

where $D(\epsilon)$ is a two dimensional diagonal matrix with the first element $D(\epsilon)_{11} = 1$ and the second element $D(\epsilon)_{22} = 1/\epsilon$.

*Remark 5.* Equation (50) implies that $e_y \to 0$ and $F_{2e}^T\tilde{\theta} \to 0$ when time$\to\infty$. Under a Persistence of Excitation condition for the output feedback case, $\tilde{\theta} = 0$ when time $\to\infty$.

*Remark 6.* It can be easily proved from equations (50), and (51), that $z_s$ under output feedback approaches $z_s$ under state feedback as time $\to\infty$. From the asymptotic stability of the origin under state feedback, $z_s \to 0$ as time $\to\infty$. As a result, $z_s$ under output feedback converges to zero as time $\to\infty$.

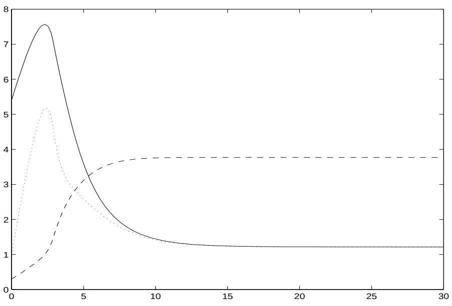
From equations (50), (51) and *Remarks 7* and *8*, the origin of $(\chi(t,\epsilon),\xi(t,\epsilon))$ is asymptotically stable.

## 5. SIMULATION RESULTS

A simulation study is performed using the experimental conditions provided in (Wang *et al.*, 1999). The following parameters and initial states are used in the simulation experiment.
$\epsilon = 0.01$, $\alpha_s = 1$, $\alpha_y = 50$, $K_s = 0.2$, $\mu_m = 1$, $k_1 = 2$, $k_2 = 1$, $s_0 = 10$, $s(0) = 1$, $y(0) = 0.3$, $\hat{s}_{obs}(0) = 5$, $\hat{y}_{obs}(0) = 0.1$, $\hat{y}(0) = 1.5$, $\hat{\theta}_\mu(0) = 3$,

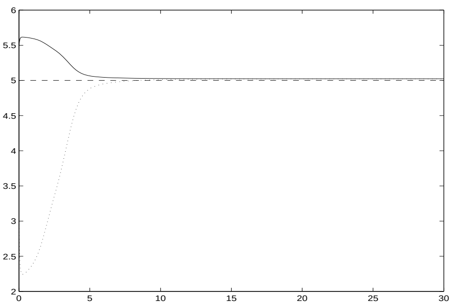
$\hat{\theta}_s(0) = 5.5$.

The dither signal is chosen as $a(t) = 0.01(sin(0.01t) + sin(0.05t))$. Figure 1 represents the simulation result of the substrate concentration ($s$), the estimation of the substrate concentration ($\hat{s}_{obs}$), the production rate ($y$) and the estimation of the production rate ($\hat{y}_{obs}$). Figure 2 shows that both $\hat{\theta}_s$ and $\hat{\theta}_\mu$ converge to their true value $\theta_s = \theta_\mu = 5$. From Figure 3, the trajectories under output feedback recover the trajectories under state feedback for the high gain observer with sufficiently large gain ($\epsilon = 0.01$). Furthermore, the maximum value for the production rate $y = 3.77$ is achieved under output and state feedback which confirms the effectiveness of the adaptive extremum seeking scheme.



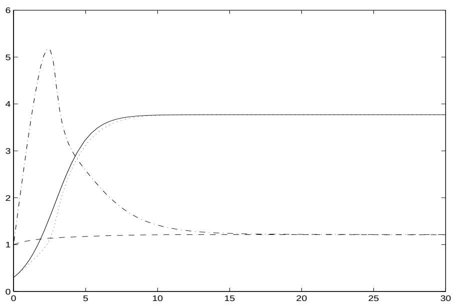
t(hr)

Fig. 1. Substrate concentration $s$("..") and its estimate $\hat{s}_{obs}$("–"), production rate $y$ and its estimate $\hat{y}_{obs}$ ("- -")



t(hr)

Fig. 2. Parameter $\theta_\mu$("- -") and its estimate $\hat{\theta}_\mu$("–"), parameter $\theta_s$("- -") and its estimate $\hat{\theta}_s$("..")



t(hr)

Fig. 3. $y$ under state feedback ("–") and $y$ under output feedback ("- -"), $s$ under output feedback("- .") and $s$ under state feedback ("- -")

## 6. CONCLUSION

An adaptive output-feedback extremum-seeking control was developed for a class of stirred tank bioreactors governed by Monod growth kinetics. The controller allows the stabilization of the system to its unknown optimal production rate.

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