DESIGN OF SUB-OPTIMAL ROBUST GAIN-SCHEDULED PI CONTROLLERS

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Abstract**:** A methodology is proposed for the analysis and design of a robust gain-scheduled PI controller for nonlinear chemical processes. The stability and performance tests can be formulated as a finite set of linear matrix inequalities (LMI) and hence, the resulting problem is numerically tractable. Input saturation and model error are explicitly incorporated into the analysis. A simulation study of a nonlinear CSTR (continuous stirred tank reactor) process indicates that this approach can provide useful sub-optimal robust controllers.

Keywords: robust control, nonlinear systems.

1. INTRODUCTION

This paper derives LMI-based tests, to test the closed-loop stability and performance of gain-scheduled Proportional-Integral (PI) controllers, when applied to nonlinear processes.

The design of gain-scheduled controllers for Linear Parameter Varying (LPV) systems has been reported in a number of publications (e.g. Shamma and Athans, 1992) and software is available, e.g. Matlab, to design these controllers using LMI. Two main problems in the application of these techniques to chemical engineering processes are: i- models of chemical systems are often not available in LPV form ready for the LMI's tests, ii- The LMI-based methodology results in controller structures that are significantly more complex than the PI or PID control forms, which are widely accepted by the chemical industry.

Following these, Knapp and Budman (2001) have proposed to model nonlinear processes with a special class of state-affine nonlinear discrete model. These state-affine models are in LPV form where the manipulated variable fulfills the role of the time-varying parameter. They showed that by using these models in combination with a discrete PI controller, the analysis of the closed loop system can be reduced to the solution of a set of LMI. These models are nonlinear with respect to the manipulated variables and then, this input nonlinearity is treated as model uncertainty with respect to a linear nominal model. Then, the robust stability and performance of the closed loop system can be analyzed with respect to this model uncertainty.

Using these state-affine models in combination with the proposed gain-scheduled PI controller, the closed-loop system can be represented by a class of discrete-time systems state-space equations with a state vector η.

For time-varying real uncertainty, a quadratic stability test seeks a fixed quadratic Lyapunov function $V(t) = \eta(t)^T P \eta(t)$ that proves stability for all admissible uncertainties. It is shown that finding an adequate P, amounts to solving a

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convex problem involving a system of LMI. This system of LMI can be extended to test robust performance as well.

In the current paper we have expanded the work of Knapp (2001) by considering a special class of scheduled PI controllers, defined in section 2, where the tuning coefficients of the controller are linear functions of the manipulated variable. These linear functions are defined in terms of 4 parameters. Then, this work also addresses the optimization of these parameters. The parameterization of the controller in terms of a small number of parameters greatly facilitates the optimization step.

The paper is organized as follows. Section 2 presents the state-affine model realization and the gain-scheduled PI controller structure. Section 3 derives LMI-based stability condition. Section 4 develops the performance condition and addresses the performance optimization problem. Section 5 integrates input saturation and modeling error into the analysis. Section 6 illustrates the validity of the design approach by a case study example. Section 7 summarizes the conclusions and future work.

2. STATE-AFFINE MODEL AND GAIN-SCHEDULED PI CONTROLLER

Based on Knapp and Budman's (2000, 2001) work, a state-affine model for a nonlinear process is obtained as follows

$$
\begin{aligned}
x(t+1) &= \{F_0 + \sum_{i=1}^{n} F_i u(t)^i\} x(t) + \{G_1 + \sum_{i=1}^{n} G_{i+1} u(t)^i\} u(t) \\
y(t) &= H_0 x(t) + d(t)
\end{aligned}
\tag{1}
$$

where F, G, H are polynomial matrices. Disturbances of infinite frequencies can not be effectively rejected unless an infinite closed-loop bandwidth is used, because of robust stability limitations. Therefore, the actual disturbance $v(t)$ is filtered through a low-pass filter as follows:

$$d(t+1) = BWd(t) + (1 - BW)v(t) \tag{2}$$

Where $0 \le BW \le 1$, which is a bandwidth related weight.

A gain-scheduled PI controller of the form given by (3) is used. When $W_c = W_d = 0$, the control law $\hat{u}$ reduces to a conventional discrete PI controller with proportional gain K_c and reset time τ_I. Thus the coefficients C_c and D_c of the PI controller are augmented in equation (3) by a linear dependency with respect to the manipulated variable u to allow for scheduling as a function of u. $\hat{u}(t)$ stands for the control action calculated without saturation whereas $u(t)$ is computed with saturation limits.

$$
\begin{aligned}
&\xi(t+1) = A_c \xi(t) + B_c e(t) \\
&\hat{u}(t) = (C_c + W_c u(t))\xi(t) + (D_c + W_d u(t))e(t) \\
&e(t) = y_d(t) - y(t), y_d(t) = 0 \\
&A_c = 1, B_c = 1, C_c = \frac{K_c}{\tau_I}, D_c = K_c + \frac{K_c}{\tau_I}
\end{aligned}
\tag{3}
$$

For a process represented by the state-affine model (1), at the nominal operating point, it is valid to assume that the process can be accurately modeled by the linear part of the state-affine model given by (4). It is also assumed that most of the model uncertainty is due to the time-varying nonlinearity of the state-affine model around this operating point. It is therefore possible to describe the model uncertainty δ_i in the form of (5).

$$
\begin{aligned}
x(t+1) &= F_0 x(t) + G_1 u(t) \\
y(t) &= H_0 x(t)
\end{aligned}
\tag{4}
$$

$$\delta_i = u(t)^i, i = 1,2,\ldots,n \tag{5}$$

(5) represents the key advantage of the methodology used here. In general it is very difficult to quantify the uncertainty, δ_i, from mechanistic first-principle models (Doyle, 1990). In our case, since δ_i is equal to the powers of the input, it can be easily quantified. Each input in a process is known to lie between a lower and an upper limit known during the design stage due to, for example, actuator constraints or economic considerations. According to (5):

$$
\begin{aligned}
&u(t) \in \left[\underline{u} \quad \overline{u}\right] \rightarrow \delta_i \in \left[\underline{\delta_i} \quad \overline{\delta_i}\right] \\
&S := \{(\omega_1, \omega_2, \cdots, \omega_n) : \omega_i \in \{\underline{\delta_i}, \overline{\delta_i}\}\}
\end{aligned}
\tag{6}
$$

Rewriting (1) using (5) gives:

$$
\begin{aligned}
x(t+1) &= \{F_0 + \sum_{i=1}^{n} F_i \delta_i\} x(t) + \{G_1 + \sum_{i=1}^{n} G_{i+1} \delta_i\} u(t) \\
y(t) &= H_0 x(t) + d(t)
\end{aligned}
\tag{7}
$$

The closed-loop system of (7), (2) and (3) is then put into a form given by (8) suitable for analysis. The state matrix $A(\delta)$ depends on the uncertainties defined by (6).

$$A_{11} = F_0 + \sum_i F_i \delta_i - (G_1 + \sum_i G_{i+1}\delta_i)(D_c + W_d \delta_1) H_0 \psi$$
$$A_{12} = (G_1 + \sum_i G_{i+1}\delta_i)(C_c + W_c \delta_1)\psi$$
$$A_{13} = -(G_1 + \sum_i G_{i+1}\delta_i)(D_c + W_d \delta_1)\psi$$
$$A(\delta) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ -B_c H_0 \psi & A_c \psi & -B_c \psi \\ 0 & 0 & BW \end{bmatrix}$$
$$\begin{bmatrix} \eta(t+1) \\ e(t) \end{bmatrix} = \begin{bmatrix} A(\delta) & B \\ C & D \end{bmatrix} \begin{bmatrix} \eta(t) \\ \nu(t) \end{bmatrix} \Leftrightarrow$$
$$\begin{bmatrix} x(t+1) \\ \xi(t+1) \\ d(t+1) \\ \hline e(t) \end{bmatrix} = \begin{bmatrix} A(\delta) & B \\ C & D \end{bmatrix} \begin{bmatrix} x(t) \\ \xi(t) \\ d(t) \\ \hline \nu(t) \end{bmatrix}$$
$$B = \begin{bmatrix} 0^T & 0^T & (1-BW)^T \end{bmatrix}^T \quad C = \begin{bmatrix} -H_0 & 0 & -1 \end{bmatrix}$$
$$D = [0], \eta(0) = \eta_0 \qquad (8)$$

3. QUADRATIC STABILITY

Consider the uncertain nonlinear system (8). This system is quadratic stable if there exists a positive-definite quadratic Lyapunov function

$$V(t) = \eta(t)^T P \eta(t), \; P > 0 \qquad (9)$$

such that $V(t) > 0$ and $V(t+1) - V(t) < 0$ for all admissible uncertainties and for all initial conditions η_0.

Definition 3.1(quadratic stability): The system

$$\eta(t+1) = A(\delta)\eta(t), \eta(0) = \eta_0 \qquad (10)$$

is quadratically stable if there exists a symmetric matrix P such that

$$P > 0 \qquad (11)$$
$$A(\delta)^T P A(\delta) - P < 0 \qquad (12)$$

hold for all admissible uncertainties.

When δ ranges in a polytope with vertices in S, it suffices to enforce (12) at the vertices, and so (12) is equivalent to the following convex LMI problem

$$A(\omega)^T P A(\omega) - P < 0, \; for \quad all \quad \omega \in S \qquad (13)$$

A complete summary of LMI theory is given by Boyd et al. (1994).

4. QUADRATIC H_∞ PERFORMANCE

Definition 4.1 (quadratic H_∞ performance): System (1) with zero initial state has quadratic H_∞ performance γ if there exists a symmetric matrix P such that

$$P > 0 \qquad (14)$$
$$\begin{bmatrix} A(\delta)^T P A(\delta) - P & A(\delta)^T P B & C^T \\ B^T P A(\delta) & B^T P B - \gamma^2 I & D^T \\ C & D & -I \end{bmatrix} < 0 \qquad (15)$$

is satisfied for all admissible uncertainties.

This condition establishes that the closed-loop system defined by (8) satisfies $\|e\|_{L_2} < \gamma \|v\|_{L_2}$ for all L_2-bounded input ν, that is (15) guarantees

$$V(t+1) - V(t) + e^T(t)e(t) - \gamma^2 v^T(t) v(t) < 0 \qquad (16)$$

(15) is equivalent to the finite LMI as follows

$$\begin{bmatrix} A(\omega)^T P A(\omega) - P & A(\omega)^T P B & C^T \\ B^T P A(\omega) & B^T P B - \gamma^2 I & D^T \\ C & D & -I \end{bmatrix} < 0$$
$$for \quad all \quad \omega \in S \qquad (17)$$

Equation (17) is solved as a generalized eigenvalue problem (GEVP), to optimize γ.

5. INPUT SATURATION AND MODELING ERROR

Input saturation would occur when the controller outputs $\hat{u}(t)$ exceeded the limits. The gain-scheduled PI controller can be reformulated using a variable gain $\tilde{K}_c$. Define:

$$\psi = \frac{1}{|\hat{u}|} = \frac{1}{K_c} \frac{1}{\left| \frac{1}{\tau_I}\xi + (1+\frac{1}{\tau_I})e \right|} \tag{18}$$

Then the gain of the controller is given by:

$$\begin{aligned} &\textit{if} \quad 0 \le \psi \le 1 \quad \tilde{K}_c = K_c\psi \\ &\textit{else} \quad \psi > 1 \quad \tilde{K}_c = K_c = \text{constant} \end{aligned}$$

These definitions ensure that $|u|$ never exceeds the saturation limit of 1 whereas $|\hat{u}|$ can exceed the limit.

A lumped error δ_t in the output is considered as the modeling error so that the H matrix can be rewritten as follows:

$$H = H_0 \xrightarrow{\delta_t} H = H_0 + W_t\delta_t \tag{19}$$

δ_t can be easily calculated from the difference between the model prediction and the actual data from the process (Budman and Knapp, 2000 and 2001). Limits of ψ and δ_t need to be taken into account in the stability and performance analysis.

6. DESIGN CASE STUDY: CSTR

The case study under investigation is a CSTR from Doyle et al. (1989). A state-affine mode is first obtained, see (Budman and Knapp, 2000 and 2001). Input saturation with $\psi \in [0.4 \quad 1]$ and modeling error with $\delta_t = [-1 \quad 1]$ and $W_t = 0.025$, will also be considered. In principle, the lower limit of ψ should have been assumed to be equal to zero for the case that the calculated control action is infinite. When a lower limit of zero was assumed, robustness could not be achieved. Fortunately, the output in a real process is always bounded due to sensor saturation or the physical limitation of the process, e.g. conversion cannot be larger than 1. Accordingly, following equation (3), a finite upper limit for the control action exists and consequently a lower bound of ψ larger than zero can be assumed.

Fig.1 shows the robust stability and robust performance ($\gamma = 1$) regions for linear PI controllers, i.e. with W_c and W_d equal to zero, defined in terms of the proportional gain and reset time.

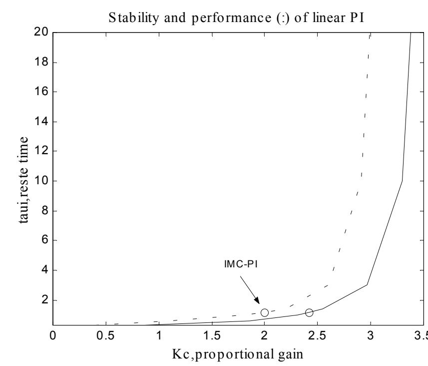

Fig.1. Stability and performance regions of linear PI controller parameters. Stability region is the area above the solid line including the solid line as limit. Performance region for a $\gamma = 1$ is the area above the dotted line including the dotted line as limit.

For the purpose of comparison with the gain-scheduled controller, a set of PI controller parameters was selected in the neighborhood of the robust performance boundary shown in Fig. 1. as follows: $K_c = 2$ and $\tau_I = 1.1545$. This point corresponds approximately to the Internal Model Control (IMC) tuning parameters around the nominal operating point based on the rules available in the literature (Morari and Zafiriou, 1989). Using these linear PI controller parameters in equation (3), gain-scheduled PI controller weights W_c, W_d can be calculated according to the stability and performance tests presented above. Accordingly, regions of robust stability and robust performance are computed in terms of different combinations of the weights and the results are shown in Fig.2 and 3. The circles shown in Fig.2 and 3 represent the linear PI controllers selected on the limit of robust stability and performance, respectively, i.e. $W_c = 0, W_d = 0$, also shown on the curves in Fig.1.

In order to improve upon the performance of the linear PI controller, a pair of gain scheduling weight values can be sought inside the robust performance region, corresponding to a point indicated by a star in Fig.3, that will provide a better performance. Since the performance of the controller is directly related to the parameter γ as shown by equation (17) the objective is to minimize this value.

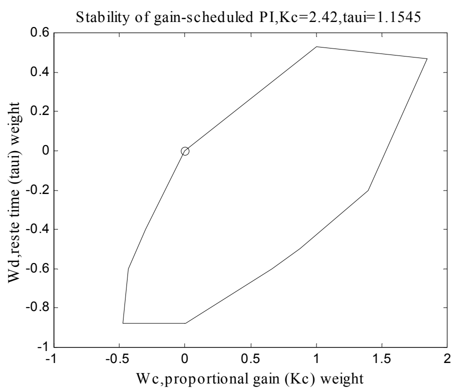

Fig.2. Stability region of gain-scheduled PI controller weights, that is, the area inside the solid box including the solid circle as limit.

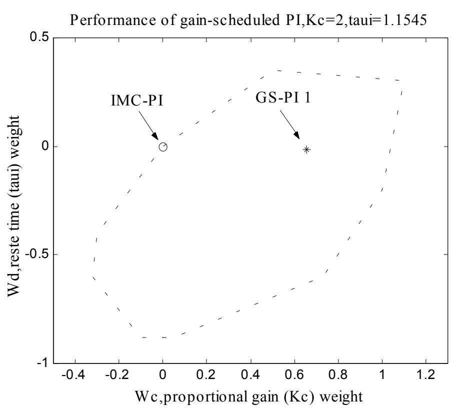

Fig.3. Performance region of gain-scheduled PI controller weights, that is, the area inside the dotted box including the dotted circle as limit.

The problem of searching for a $\gamma_{optimal}$ is not convex in terms of the controller parameters. The conditions result in a nonlinear matrix inequality for the controller parameters. Branch and bound methods have been proposed to solve LMI's systems of this type (Fukuda and Kojima, 2001; Braatz, et al., 1997). For simplicity, it was decided to limit the search to a sub-optimal design in the neighborhood of the selected linear PI controller using the FMIN optimization function in Matlab. This was done by using K_c and τ_I computed by the IMC rules and by optimizing the values of the weights W_c and W_d. The objective is to assess the improvement in performance over that obtained with this IMC-PI controller. Subsequently, an additional optimization was conducted where all the parameters, i.e. K_c, τ_I and the weights, were allowed to change to minimize γ.

The optimization of the controller weights using the GEVP procedure produces the best robust gain-scheduled PI controller in the neighborhood of the IMC design, shown as a star in Fig.3. For this design $\gamma^*_{optimal}$ =0.5890 and this is an improvement of 38.9% over $\gamma^o_{optimal}$ =0.9634 in robust performance obtained with the IMC-PI design. When all the parameters are optimized, an additional improvement in performance is obtained with $\gamma_{optimal}$ =0.3894. Table 1 summarizes the optimization results.

Table 1 Optimization design results

	IMC-PI	G-S PI 1	G-S PI 2
K_c	2	2	1.3723
τ_I	1.1545	1.1545	2.949
W_c	0	0.6547	-0.004
W_d	0	-0.015	0.001
$\gamma_{optimal}$	0.9634	0.5890	0.3894
$\gamma_{simulation}$	0.3787	0.3495	0.202

To assess the conservatism of the analysis a simulation study is conducted for the CSTR using the different controllers synthesized in this work. The performance is tested by investigating through a large number of simulations how the system rejects a bounded disturbance. $\gamma_{simulation}$ is used to refer to the performance limit obtained from the simulation.

$\gamma_{simulation}$ calculated based $\|e\|_{L_2} < \gamma \|v\|_{L_2}$ is always bounded by $\gamma_{optimal}$ in each case, indicating that the analysis tests produce a worst-performance bound as expected and it is not exceeded. The difference between $\gamma_{optimal}$ and $\gamma_{simulation}$ shows that the design procedure is conservative to some degree.

Simulations were conducted for a large number of different disturbances. A disturbance was sought that would result in the worst performance for each controller. Then for the worst case found from simulation, $\gamma_{simulation}$ was calculated. Simulation results for the IMC-PI controller and for the sub-optimal gain-scheduled PI controller are shown in Fig.4. These simulations correspond to a spike type disturbance also shown in Fig.4. Worse performance than the one shown in Fig.4 may be also possible but there is no systematic way to find the specific disturbance function that will lead to it.

Fig.4. Closed-loop simulations of state-affine model (lower two curves).
Linear PI controller (dotted line), $K_c = 2, \tau_I = 1.1545, \gamma_{simulation} = 0.3787$.
Gain-scheduled PI controller (solid line), $K_c = 1.3723, \tau_I = 2.949, W_c = -0.004$, $W_d = 0.001, \gamma_{simulation} = 0.202$.

Conservatism associated with the design approach comes from two main facts. First, a possible source of this conservatism is that simulation can only be done on a limited period of time, while the calculation of the performance condition requires an infinite simulation interval. Second, conservatism is obviously inherent to the robust control approach where several scenarios included in the analysis will not occur during actual closed-loop operation.

7. CONCLUSIONS

An approach is proposed to design gain-scheduled PI controllers for nonlinear processes using process data. It is based on empirical state-affine models of the process. Gain-scheduled PI controller with sub-optimal performance is obtained using a GEVP based optimization algorithm. Simulations show that the gain-scheduled controller provides better performance than a conventional PI controller found for robustness with IMC rules. A performance index γ, although conservative, has been found to be a good indicator of the relative performance of the different controllers.

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ADAPTIVE EXTREMUM SEEKING CONTROL OF CONTINUOUS STIRRED TANK BIOREACTORS [1]

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Abstract: In this paper, we present an adaptive extremum seeking control scheme for continuous stirred tank bioreactors. We assume limited knowledge of the growth kinetics. An adaptive learning technique is introduced to construct a seeking algorithm that drives the system states to the desired set-points that maximizes the value of an objective function. Lyapunov's stability theorem is used in the design of the extremum seeking controller structure and the development of the parameter learning laws. A simulation experiment is given to show the effectiveness of the proposed approach.

Keywords: Extremum seeking, Lyapunov function, adaptive learning, persistence of excitation

1. INTRODUCTION

The goal of extremum seeking is to find the operating set-points that maximize or minimize an objective function. Since the early research work on extremum control in the 1920's (Leblanc 1922), many successful applications of extremum control approaches have been reported (e.g., (Vasu 1957), (Astrom and Wittenmark 1995), (Sternby 1980) and (Drkunov *et al.* 1995)). Recently, Krstic et. al ((Krstic 2000), (Krstic and Deng 1998)) presented several extremum control schemes and stability analysis for extremum-seeking of linear unknown systems and a class of general nonlinear systems ((Krstic 2000) and (Krstic and Deng 1998)).

In this study, we investigate an alternative extremum seeking scheme for continuous stirred tank bioreactors. The proposed scheme utilizes an explicit structure information of the objective function that depends on system states and unknown plant parameters. However, it is assumed that the objective function is not available for measurement. Furthermore, no explicit knowledge of the microbial growth kinetics are assumed. A Lyapunov-based adaptive learning control technique is used to approximate the unknown kinetics and to steer the system to its unknown extremum. The technique ensures convergence of the system to an adjustable neighbourhood of its unknown optimum that depends on the approximation error. We also show that a certain level of persistence of excitation (PE) condition is necessary to guarantee the convergence of the extremum-seeking mechanism. The paper is organized as follows. Section 2 presents some notations and the problem formulation. In Section 3, an parameter estimation algorithm is developed. Sec-

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tion 4 presents the adaptive extremum seeking controller and the stability and convergence of the closed-loop extremum seeking system. Numerical simulation is shown in Section 5 followed by brief conclusions in Section 6.

2. PROBLEM

Consider the following microbial growth models

$$\dot{x} = \mu(s)x - ux \tag{1}$$

$$\dot{s} = -k_1\mu(s)x + u(s_0 - s) \tag{2}$$

$$y = k_2\mu(x, s)x \tag{3}$$

where states $x \in [0, +\infty)$ and $s \in [0, +\infty)$ denote biomass and substrate concentrations, respectively, $u \geq 0$ is the dilution rate, y is the production rate of the reaction product, s_0 denotes the concentration of the substrate in the feed, and $k_1, k_2 > 0$ are yield coefficients. We consider the case where only s and y are measurable, the biomass concentration x is not available for feedback control.

In this work, we consider the extremum seeking problem for plant (1)-(2) with an unknown substrate-dependent growth rate expression $\mu(s)$. One of the most common growth rate model is Monod's model given by

$$\mu(x, s) = \mu(s) = \frac{\mu_m s}{K_s + s} \quad \text{(Monod)} \tag{4}$$

where $\mu_m > 0$ is the maximum value of the specific growth rate, and positive constant K_s, K_c and K_0 to K_2 denote the coefficients for different growth rate models. While this simple model form is very useful in practice, a wide variety of growth patters and characteristics exist where the Monod expression is not applicable.

The extremum-seeking control of plants described by the Monod model have been investigated in (Zhang *et al.* 2002). In this study, we extend the result to a broad class of uncertain plants with unknown growth rate representations. As in (Zhang *et al.* 2002), the control objective is to design a controller, u, such that the production rate y achieves its maximum.

The strategy developed in this paper consists in approximating the growth rate expression using a neural network approximation technique. In this paper, radial basis function (RBF) neural networks presented in (Sanner and Slotine 1992) shall be used to approximate a continuous function $\phi(z) : R^p \to R$

$$\phi(z) = W^{*T}S(z) + \mu_l(t) \tag{5}$$

with NN approximation error $\mu_l(t)$, and basis function vector

$$S(z) = [s_1(z), s_2(z), \cdots, s_l(z)]^T$$

$$s_i(z) = \exp\left[\frac{-(z-\varphi_i)^T(z-\varphi_i)}{\sigma_i^2}\right], \qquad i = 1, 2, .. l \tag{6}$$

where φ_i is the center of the receptive field, and σ_i is the width of the Gaussian function. The ideal weight W^* in (5) is defined as

$$W^* := arg \min_{W\in\Omega_w} \left\{\sup \left|W^T S(z) - \phi(z)\right|\right\} \tag{7}$$

where $\Omega_w = \left\{W \;\middle|\; \|W\| \leq w_m\right\}$ with positive constant w_m to be chosen at the design stage. Universal approximation results stated in (Funahashi 1989) (Sanner and Slotine 1992) indicate that, if l is chosen sufficiently large, then $W^T S(z)$ can approximate any continuous function to any desired accuracy on a compact set.

We apply eq.(5) to develop an approximation of the growth rate expression given by

$$\mu(s(t)) = W^{*T}S(s(t)) + \mu_l(t) \tag{8}$$

where W^* and S are as defined in eqs.(6)-(7). Additionally, we make the following assumption about the approximation error $\mu_l(t)$.

Assumption 1: the NN approximation error satisfies $|\mu_l(t)| \leq \bar{\mu}_l$ with constant $\bar{\mu}_l > 0$ over a compact set in the state space.

We first calculate the system's equilibria corresponding to a constant dilution rate u_e. By setting the right-hand side of (1)-(2) to zero, we obtain two equilibria. The first is $x_e = 0$ and $s_e = s_0$ which is called the wash-out equilibrium. The second is

$$x_e = \frac{s_0 - s_e}{k_1}$$

where s_e is a positive solution of the equation

$$u_e = \mu(s_e).$$

At the steady-state, the production rate can be expressed by

$$y_e = \frac{k_2}{k_1}\mu(s_e)(s_0 - s_e) \tag{9}$$

Following eq.(8), the steady-state production rate is approximated by

$$y_e = \frac{k_2}{k_1}W^{*T}S(s_e)(s_0 - s_e) \tag{10}$$

From (2) and (4), we have

$$\frac{\partial y_e}{\partial s_e} = \frac{k_2}{k_1} W^{*T} \Big(dS(s_e)(s_0 - s_e) - S(s_e) \Big) \quad (11)$$

and

$$\frac{\partial^2 y_e}{\partial s_e^2} = \frac{k_2}{k_1} W^{*T} \Big(d^2S(s_e)(s_0 - s_e) - 2dS(s_e) \Big) (12)$$

where $dS = \frac{\partial S}{\partial s}$ and $d^2S = \frac{\partial^2 S}{\partial s^2}$. Assuming that the parameter vector W^* is such that $\frac{\partial^2 y_e}{\partial s_e^2} > 0, \forall s_e \geq 0$ then $y_e(s)$ has a maximum

$$y^* = y_e(s^*) = \frac{k_2}{k_1} W^{*T} S(s^*) x^* \quad (13)$$

with $x^* = \frac{s_0 - s^*}{k_1}$ at the system equilibrium.

The objective of this study is to develop a controller that maximizes the steady-state value of the production rate, y^*. However, since the exact values of the ideal weights, W^*, are not known *a priori* , they must be estimated. In the next section, we propose an adaptive extremum seeking algorithm is developed to search the unknown process set-point where the production rate, y, is optimized. The strategy attempts to estimate the gradient of the production rate with respect to the substrate concentration, s. A controller is then designed to bring the process to points where the gradient vanishes and where the second order derivatives of the production rate with respect to the substrate is negative. The resulting technique provides a real-time optimization techniques that can be used to a large class of bioreactors and chemical reactors.

3. CONTROLLER DESIGN

In this section, we design a control strategy that tracts the unknown optimum production rate. We first develop the parameter estimation algorithm for the unknown parameter vector W^*. Equations (1)-(2) can be re-expressed as

$$\dot{x} = (W^{*T} S(s) + \mu_l(t))x - ux \quad (14)$$

$$\dot{s} = -k_1(W^{*T} S(s) + \mu_l(t))x + u(s_0 - s) \quad (15)$$

We assume that the biomass and the substrate concentration are available for measurement.

Let $\hat{W}$ denote the estimate of the true parameter W^* and let $\hat{s}$ and $\hat{x}$ be the predictions of s and y. The predicted states $\hat{s}$ and $\hat{x}$ are generated by

$$\dot{\hat{x}} = \hat{W}^T Sx - ux + k_x e_x + c_1(t)^T \dot{\hat{W}} \quad (16)$$

$$\dot{\hat{s}} = -k_1 \hat{W}^T Sx + u(s_0 - s) + k_s e_s + c_2(t)^T \dot{\hat{W}} (17)$$

with gain functions $k_s, k_y > 0$, prediction errors $e_s = s - \hat{s}$ and $e_x = x - \hat{x}$ and $c_1(t)$, $c_2(t)$ time-varying functions to be assigned later. It follows from (14)-(17) that

$$\dot{e}_x = \tilde{W}^T Sx + \mu_l(t)x - k_x e_x - c_1(t)^T \dot{\hat{W}} \quad (18)$$

$$\dot{e}_s = -k_1 \tilde{W}^T Sx - k_1 \mu_l(t) - k_s e_s - c_2(t)^T \dot{\hat{W}} \quad (19)$$

where $\tilde{W} = W^* - \hat{W}$.

The objective of the extremum-seeking control is stabilize the closed-loop system around a point where the gradient of the production y with respect to s given in eq.(11) vanishes while attenuating the effect of the modelling uncertainty $\mu_l(t)$. Since the parameter vector W^* is unknown, we first design a controller to make the system states track points where the estimated gradient

$$z = \frac{k_2}{k_1} \hat{W}^T \Big(dS(s)(s_0 - s) - S(s) \Big) \quad (20)$$

vanishes. In order to ensure that the estimated gradient approaches the true gradient asymptotically, we have to ensure that the parameter estimates approach the optimal weight vector W^*. To achieve this objective, an excitation signal is designed and injected into the adaptive system to ensure convergence of the estimated parameters to their true value. The extremum seeking control objective is achieved when the system systems are stabilized at the optimal operating point $x^*, \ s^*$.

Define

$$z_s = \hat{W}^T \Big(dS(s)(s_0 - s) - S(s) \Big) - d(t) \quad (21)$$

where $\frac{k_2}{k_1} > 0$ has been removed for simplicity and $d(t) \in C^1$ is an excitation signal that will be assigned later. In the remainder, the dependence of the radial basis functions S on the substrate concentration s is implied and we write S, dS and d^2S.

Next we define the variables,

$$\begin{aligned} \eta_1 &= e_x - c_1(t)^T \tilde{W} \\ \eta_2 &= e_s - c_2(t)^T \tilde{W} \\ \eta_3 &= z_s - c_3(t)^T \tilde{W} \end{aligned} \quad (22)$$

where $c_3(t)$ is a vector of time-varying functions to be defined in the design procedure. We propose the Lyapunov function candidate

$$V = \frac{\eta_1^2}{2} + \frac{\eta_2^2}{2} + \frac{\eta_3^2}{2}. \quad (23)$$

We pose the following equation for the dither signal, $d(t)$,

$$\begin{aligned} \dot{d}(t) = & c_3(t)^T \dot{\hat{W}} + \Gamma_1^T \dot{\hat{W}} - (\hat{W}^T \Gamma_2)^2 d(t) \\ & + \hat{W}^T \Gamma_2 a(t) + k_z z_s \end{aligned} \quad (24)$$

where $a(t)$ is an external signal providing excitation to the process and $k_z > 0$ is a positive gain function to be assigned. We then assign $\dot{c}_1$, $\dot{c}_2$ and $\dot{c}_3$ as

$$\begin{aligned} \dot{c}_1^T &= -k_x c_1^T + xS^T \\ \dot{c}_1^T &= -k_s c_2^T - k_1 x S^T \\ \dot{c}_3^T &= -k_z c_3^T - k_1 x \hat{W}^T \Gamma_2 S^T \end{aligned} \tag{25}$$

and we let the control be given by

$$u = \frac{1}{(s_0 - s)} \Big(k_1 \hat{W}^T S x + a(t) - \hat{W}^T \Gamma_2 \big). \tag{26}$$

Taking the time derivative of V, we substitute eqs.(24)-(26) and we substitute e_x, e_s and z_s using eq.(22) to obtain

$$\begin{aligned} \dot{V} = \mu_l(t) x \eta_1 - k_x \eta_1^2 - k_1 x \mu_l(t) \eta_2 - k_s \eta_2^2 \\ -k_1 x \mu_l(t) \hat{W}^T \Gamma_2 \eta_3 - k_z \eta_3^2. \end{aligned} \tag{27}$$

where $\Gamma_1 = dS(s_0 - s) - S$ and $\Gamma_2 = d^2 S(s_0 - s) - 2dS$. Next, we complete the squares and assign the gain functions

$$\begin{aligned} k_x &= k_{x0} + \frac{k_4}{2} x^2 \\ k_s &= k_{s0} + \frac{k_3 k_5}{2} x^2 \\ k_z &= k_{z0} + \frac{k_3 k_6}{2} x^2 (\hat{W}^T \Gamma_2)^2 \end{aligned} \tag{28}$$

where $k_4 > 0$, $k_5 > 0$, $k_6 > 0$, $k_{x0} > 0$, $k_{s0} > 0$ and $k_{z0} > 0$ are positive constants. We finally obtain the inequality

$$\begin{aligned} \dot{V} \leq -k_{x0} \eta_1^2 - k_{s0} \eta_2^2 - k_{z0} \eta_3^2 \\ + \left(\frac{1}{2k_4} + \frac{1}{2k_5} + \frac{1}{2k_6} \right) \mu_l(t)^2 \end{aligned} \tag{29}$$

Eq.29) establishes that the state, η, converges to a small neighborhood of the origin. It remains to show that the original state variables, e_x, e_s and z_s and the parameter estimation errors $\tilde{W}$ converge to a small neighborhood of the origin. Note that it is not sufficient to check that e_x, e_s and z_s can be made small since the value of z_s depends on the parameter estimates, $\hat{W}$. To this end, we derive a persistency of excitation condition that guarantees the convergence of the parameter estimates to the ideal weights, W^*.

Consider the following matrix,

$$\Upsilon(t) = \begin{bmatrix} c_1(t)^T \\ c_2(t)^T \\ c_3(t)^T \end{bmatrix}$$

By construction, this matrix solves the matrix differential equation

$$\dot{\Upsilon}(t) = -K(t)\Upsilon(t) + B(t) \tag{30}$$

where

$$K(t) = \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_s & 0 \\ 0 & 0 & k_z \end{bmatrix}$$

and

$$B(t) = \begin{bmatrix} xS^T \\ -k_1 x S^T \\ -k_1 x \hat{W}^T \Gamma_2 S^T \end{bmatrix}.$$

A bound on the parameter estimates $\hat{W}$ can be ensured by choosing the following parameter update law.

$$\dot{\hat{W}} = \begin{cases} \gamma_w \Gamma & \text{if } \|\hat{W}\| \leq w_m \text{ or} \\ & \text{if} \|\hat{W}\| = w_m \text{ and } \hat{W}^T \Gamma \leq 0 \\ \gamma_w \left(I - \dfrac{\hat{W}\hat{W}^T}{\hat{W}^T \hat{W}} \right) \Gamma & \text{otherwise} \end{cases} \tag{31}$$

where $\Gamma = \Upsilon(t)^T e$. Eq.(31) is a projection algorithm which ensures that $\|\hat{W}\| \leq w_m$. The convergence of the parameter estimation scheme is considered in the sequel.

By the property of the projection algorithm and for the specific choice of basis function it is possible to show that the norm of $B(t)$ is bounded. Using the exponential stability of system eq.(30)and the bound on $B(t)$, an explicit bound for the solution of eq.(30) can be obtained as follows,

$$\|\Upsilon(t)\| \leq C_2 e^{-\lambda_2 (t - t_0)} + C_2 \frac{B_M}{\lambda_2}. \tag{32}$$

where $C_2 = \|\Upsilon(t_0)\| > 0$ and $\lambda_2 > 0$ is a positive constant. Next, we want to show that the parameter estimation error $\tilde{W}$ converges to a neighborhood of the origin.

Substituting for $e = \eta + \Upsilon(t)\tilde{W}$ we obtain the perturbed dynamics

$$\begin{aligned} \dot{\hat{W}} = -\gamma_w \Upsilon(t)^T \Upsilon(t) \tilde{W} - \gamma_w \Upsilon(t)^T \eta \\ + \begin{cases} 0 \quad \text{if } \|\hat{W}\| \leq w_m \text{ or} \\ \quad \text{if} \|\hat{W}\| = w_m \text{ and } \hat{W}^T \Upsilon(t)^T e \leq 0 \\ \gamma_w \dfrac{\hat{W}\hat{W}^T}{\hat{W}^T \hat{W}} \left(\Upsilon(t)^T \Upsilon(t) \tilde{W} + \Upsilon(t)^T \eta \right) \text{ otherwise} \end{cases} \end{aligned} \tag{33}$$

To establish the convergence of the parameter estimation, we make the following persistency of excitation assumption.

Assumption 3.1. The solution of eq.(30) is such that there exists positive constants $T > 0$ and $k_N > 0$ such that

$$\int_{t}^{t+T} \Upsilon(\tau)^T \Upsilon(\tau) d\tau \geq k_N I_N \quad (34)$$

where I_N is the N-dimensional identity matrix.

By a standard adaptive control argument, the persistency of excitation condition guarantees that the origin of the differential equation

$$\dot{\tilde{W}} = -\gamma_w \Upsilon(t)^T \Upsilon(t) \tilde{W} \quad (35)$$

is an exponentially stable equilibrium. Since $B(t)$ is a bounded function, it is shown that the parameter estimation error is guaranteed to decay exponentially as

$$\|\tilde{W}\| \leq \alpha_4 e^{-\lambda_4(t-t_0)} + \frac{|\bar{\mu}_l|}{\sqrt{2kmc_3}} \quad (36)$$

Hence the parameter estimation error and the redefined state variables, η, converge exponentially fast to an adjustable neighbourhood of the origin. By definition, convergence of η and $\tilde{W}$ to a neighbourhood of the origin implies that $\|e\| \leq \|\eta\| + \|\Upsilon(t)\|\|\tilde{W}\|$. Substituting for $\|\eta\|$, $\|\Upsilon(t)\|$ and $\tilde{W}$, we obtain

$$\|e\| \leq \alpha_5 e^{-\lambda_5(t-t_0)} + \beta_5 \quad (37)$$

where $\alpha_5 > 0$ and $\beta)5 > 0$ are computable positive constants.

The convergence of the error vector, e, implies that the convergence of the prediction errors, e_x and e_s and the exponential convergence of the closed-loop system to an adjustable neighbourhood of the unknown steady-state optimum. We summarize the result of the above analysis as follows.

Theorem 3.1. Consider the two-state bioreactor model eqs.(1)-(2) with production rate, eq.(3) in closed-loop with the state-observer eqs.(16)-(17), the controller eq.(26), the dither signal eq.(24) and the adaptive learning law eq.(31). Assume that the signal $a(t)$ is such that

$$\int_{t}^{t+T} \Upsilon(\tau)^T \Upsilon(\tau) d\tau \geq k_N I_N \quad (38)$$

for positive constants $T > 0$ and $k_N > 0$ where $\Upsilon(t)$ is the solution of eq.(30). Then

- the error dynamics eqs.(18)-(19) converge exponentially to a small neighbourhood of the origin
- the parameter estimation errors $\tilde{W}$ converge exponentially to a small neighbourhood of the origin
- the tracking error from the unknown steady-state, z_s, converges exponentially to a small neighbourhood of the origin.

4. SIMULATION RESULTS

To show the effectiveness of the proposed design, a simulation study is performed on three models.

In the first example, we consider a bioreactor with Haldane kinetics,

$$\mu(s) = \left(\frac{\mu_m s}{K_s + s + K_I s^2} \right)^{1.5}$$

The following parameters and initial states are used in the simulation experiment.

$$K_s = 0.2, \;\; \mu_m = 1.0, \;\; Y = 0.5, \;\; k_1 = 2.0,$$
$$k_2 = 1.0, \;\; K_I = 0.1, \;\; s_0 = 10.0, \;\; x(0) = 1.0$$
$$s(0) = 0.1, \;\; \hat{x}(0) = 0.5, \;\; \hat{s}(0) = 0.5$$

The design parameters in the adaptive controller (26) and the adaptive law eq.(31) are

$$\gamma_w = 100.0, \quad k_z 0 = k_{x0} = k_{s0} = k_4 = k_5 = k_6 = 2.0$$

The NN radial basis function approximation is of dimension 6 with parameters $\varphi_i = i$ and $\sigma_i = 1$ for $1 \leq i \leq 6$. The initial conditions for the adaptive learning weights are

$$\hat{W}_i(0) = 0, 1 \leq i \leq 6$$

The dither signal was set to

$$a(t) = exp(-0.1t) \sum_{i=1}^{6} (sin((0.5i)t) + cos((0.5i)t))$$

We let $d(0) = 0$ and $\Upsilon(0) = 0$.

Simulation results are shown in Figures 1-3. Figure 1 shows the value of the production rate y and its estimated value. The closed-loop system converges quickly to a small neighbourhood of the origin. Moreover, the estimated production rate is shown to converge to the a small neighbourhood of the true production rate. In this case, the true optimum, 3.036, was recovered by the adaptive learning scheme. The required control action of the extremum-seeking control is shown in Figure 2. The biomass concentration and the substrate concentration are shown in Figure 3.

5. CONCLUSION

We have solved a class of extremum seeking control problems for continuous stirred tank bioreactors represented by an unknown growth kinetic model. An adaptive learning technique is used to derive an extremum seeking controller that drives the production rate to an adjustable neighbourhood of the unknown optimal production. It has been shown that when the external dither signal is designed such that the persistent excitation condition is satisfied, the proposed adaptive extremum seeking controller guarantees the exponential convergence of the production rate of the bioreactor to an adjustable neighborhood of its maximum.

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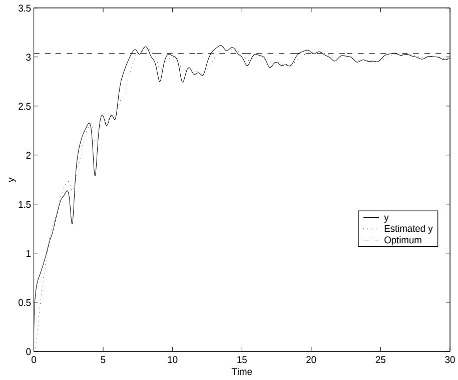

Fig. 1. Production rate y ("—") and its maximum y^* ("- -")

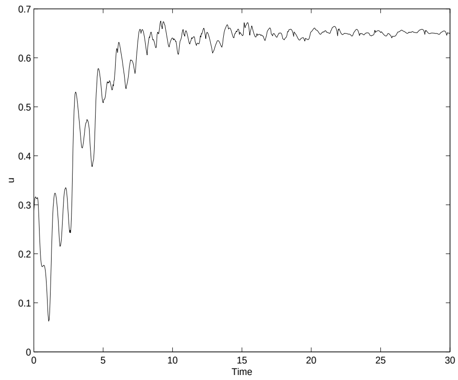

Fig. 2. Extremum-seeking dilution rate u

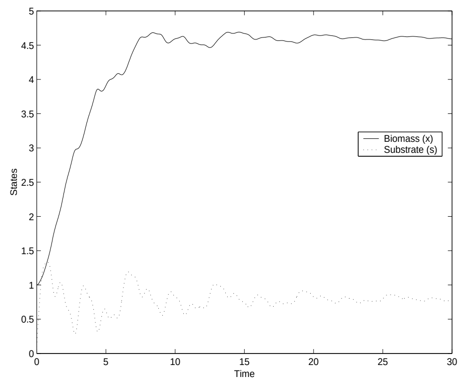

Fig. 3. Biomass x ("—") and substrate s ("- -") concentration

SET STABILIZATION OF A CLASS OF POSITIVE SYSTEMS

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Abstract: A specific class of positive systems is considered, where the system structure allows control of the distribution of "mass" in the system. Some robustness properties of the controller are pointed out, and the applicability of the model class is discussed. An example considering a CSTR modeled by mass- and energy balances illustrates the presented concepts.

Keywords: positive systems, set stabilization, van der Vusse reactor

1. INTRODUCTION

When modeling systems for control based on first principles, one often obtains nonlinear ordinary differential equations where the state variables (mass, pressure, level, energy, etc.) are *positive*. In addition, the control input will also often be positive (valve openings, amount of inflow, heat input, etc.). Hence, the class of positive systems (systems with nonnegative states and inputs) is a natural class of systems to consider in a control setting.

In this paper, we will consider a class of positive systems with strong structural constraints. Some of these constraints are natural from the perspective that the dynamics consist of "mass" flow between the different states, and other constraints are made to ensure controllability under constraints. We will argue that a diverse range of systems can be described by models in this class.

This is further funded by the fact that the intersection between the model class considered herein and the widely studied class of compartmental systems (Jacquez and Simon, 1993) is non-void. The interpretation of states as"masses" of compartments holds for both classes of systems, and similar assumptions (slightly stronger in the case of compartmental systems) concerning the flow of "mass" between the compartments are made. On the other hand, the controllability assumptions made herein, do not have their counterpart in the class of compartmental systems.

However, these controllability assumptions make it possible to specify a controller that controls the distribution of mass for the system class considered herein. The controller is related to the controllers in Bastin and Praly (1999) and De Leenheer and Aeyels (2002), but with distinct differences related to model class and controller specification.

The paper is outlined as follows: In Section 2 the class of systems we look at are specified, while the controller and the convergence result is recapitulated in Section 3. Some robustness properties are pointed out. The applicability of the model class is discussed in Section 4. The example in Section 5 illustrates the use of the controller for a system described by both mass and energy balances.

2. MODEL CLASS

We consider positive systems

$$\dot{x} = f(x, u), \tag{1a}$$

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that is, the state is positive ($x \in \mathbb{R}^n_+$), and the input is positive and upper bounded, $u \in U := \{u \in \mathbb{R}^m_+ \mid 0 \le u_j \le \bar{u}_j\}$. Each state can be interpreted as the "mass" (amount of material, or some measure of amount) in a "compartment". The controller we will propose exploits system structure, thus we assume the model equations to be on the following form:

$$f(x,u) = \Phi(x) + \Psi(x) + B(x)u. \tag{1b}$$

Loosely speaking, $\Phi(x)$ represents "interconnection structure" between compartments, $\Psi(x)$ represents *uncontrolled* external inflows to and outflows from compartments and $B(x)u$ represents *controlled* external inflows to and outflows from compartments.

Furthermore, we will assume that the state can be divided into m different parts, which will be denoted *phases*. Phase j will consist of r_j states, and have the control u_j associated with it, corresponding to *either* controlled inflow *or* outflow to compartments of that phase. The states in phase j will be denoted z^j, such that $x = [(z^1)^\top, (z^2)^\top, \ldots, (z^m)^\top]^\top$, and it follows that necessarily, $\sum_{j=1}^m r_j = n$. Corresponding to this structure, the vector functions $\Phi(x)$, $\Psi(x)$ and the matrix function $B(x)$ are on the form

$$\begin{aligned}
\Phi(x) &= \left[\phi^1(x)^\top, \phi^2(x)^\top, \ldots, \phi^m(x)^\top\right]^\top \\
\Psi(x) &= \left[\psi^1(x)^\top, \psi^2(x)^\top, \ldots, \psi^m(x)^\top\right]^\top \\
B(x) &= \text{blockdiag}\left(b^1(x), b^2(x), \ldots, b^m(x)\right).
\end{aligned}$$

Note that element j is (in general) a function of x, not (only) z^j. Also note that the partitioning into phases need not be unique.

We will state the assumptions on these functions on the set $D \subseteq \mathbb{R}^n_+$. In the case of global results, $D = \mathbb{R}^n_+$.

A1. **(Interconnection structure)** The function $\Phi : D \to \mathbb{R}^n$ is locally Lipschitz, $\phi_i^j(x) \ge 0$ for $z_i^j = 0$, and

$$\sum_{i=1}^{r_j} \phi_i^j(x) = 0, \; j = 1, \ldots, m.$$

A2. **(Controlled external flows)** The block diagonal matrix function $B(x) : D \to \mathbb{R}^{n\times m}$ is locally Lipschitz and satisfies:
  a. Phase j has controlled inflow:

$$\begin{aligned}
&b_i^j(x) \ge 0 \text{ for all } x \in D \\
&b_i^j(x) > 0 \text{ for all } x \in D \text{ for at least one } i
\end{aligned}$$

  b. Phase j has controlled outflow:

$$\begin{aligned}
&b_i^j(x) \le 0 \text{ for all } x \in D \\
&z_i^j = 0 \Rightarrow b_i^j(x) = 0 \\
&b_i^j(x) < 0 \text{ for all } x \in D \text{ with } z_i^j \ne 0
\end{aligned}$$

The uncontrolled external flows must satisfy some "controllability" assumption in relation to the controlled flows. Before we define this, it is convenient to define the "mass" of each phase, being the sum of the compartment masses of that phase:

$$M_j(x) := \sum_{i=1}^{r_j} z_i^j.$$

Our control objective will be to control $M_j(x)$ to some prespecified desired mass of phase j, denoted M_j^*, from initial conditions in D. For the control problem to be meaningful, the intersection of the set where $M_j(x) = M_j^*$ and D should be nonempty.

A3. **(Uncontrolled external flows)** For given $M^* = [M_1^*, M_2^*, \ldots, M_m^*]^\top$, $\Psi(x) : D \to \mathbb{R}^n$ is locally Lipschitz and satisfies that $\psi_i^j(x) \ge 0$ for $z_i^j = 0$, and in addition, if:
  a. Phase j has controlled inflow:
    1. For $x \in \{x \in D \mid M_j(x) > M_j^*\}$, $\sum_{i=1}^{r_j} \psi_i^j(x) \le 0$ and the set $\{x \in D \mid \sum_{i=1}^{r_j} \psi_i^j(x) = 0 \text{ and } M_j(x) > M_j^*\}$ does not contain an invariant set.
    2. For $x \in \{x \in D \mid M_j(x) < M_j^*\}$, $-\sum_{i=1}^{r_j} \psi_i^j(x) < \sum_{i=1}^{r_j} b_i^j(x)\bar{u}_j$.
  b. Phase j has controlled outflow:
    1. For $x \in \{x \in D \mid M_j(x) < M_j^*\}$, $\sum_{i=1}^{r_j} \psi_i^j(x) \ge 0$ and the set $\{x \in D \mid \sum_{i=1}^{r_j} \psi_i^j(x) = 0 \text{ and } M_j(x) < M_j^*\}$ does not contain an invariant set.
    2. For $x \in \{x \in D \mid M_j(x) > M_j^*\}$, $\sum_{i=1}^{r_j} \psi_i^j(x) < -\sum_{i=1}^{r_j} b_i^j(x)\bar{u}_j$.

It is straightforward to confirm that under the above assumptions, $x_i = 0$ implies $\dot{x}_i \ge 0$, that is, the system is positive.

3. STABILIZING STATE FEEDBACK CONTROLLER

In this section, the state feedback controller is defined, and a general convergence result is given for a general invariant set D that (is a subset of the set that) A1-A3 hold on. The set D could then be considered a region of attraction.

3.1 The controller and a convergence result

As mentioned in the previous section, our control objective is to control the total mass $M_j(x)$ of each phase to a prespecified value M_j^*.

To this end, the following constrained, positive state feedback control law is proposed:

$$u_j(x) = \begin{cases} 0 & \text{if } \tilde{u}_j(x) < 0 \\ \tilde{u}_j(x) & \text{if } 0 \le \tilde{u}_j(x) \le \bar{u}_j \\ \bar{u}_j & \text{if } \tilde{u}_j(x) > \bar{u}_j \end{cases} \tag{2}$$

where

$$\tilde{u}_j(x) = \frac{1}{\sum_{i=1}^{r_j} b_i^j(x)} \left(-\sum_{i=1}^{r_j} \psi_i^j(x) + \lambda_j (M_j^* - M_j(x)) \right) \tag{3}$$

and λ_j is a positive constant. Apparently, we can run into situations where the control is not defined if phase j is outflow controlled, since the term $\sum_{i=1}^{r_j} b_i^j(x)$ then might be zero. However, the continuity of the involved functions and the upper bound on the control ensures that the control in these cases unambiguously are defined by $u_j(x) = \bar{u}_j$.

Define the set

$$\Omega = \{x \in \mathbb{R}_+^n \mid M_1(x) = M_1^*, \ldots, M_m(x) = M_m^*\}.$$

Assumption 1. There exists a set D that is invariant for the dynamics (1) under the closed loop with control (2), and has a nonempty intersection with Ω.

Assumption 2. For $x \in \Omega \cap D$, $0 < \tilde{u}_j(x) < \bar{u}_j$.

Under the given assumptions, the convergence properties of the controller are summarized in the following Theorem, proved in Imsland (2002), see also Imsland and Foss (2002):

Theorem 1. Under the given assumptions, the state of the system (1), controlled with (2) and starting from some initial condition $x(0) \in D$, stays bounded and converges to the set $\Omega \cap D$ which is positively invariant.

To use this theorem, we need to find invariant sets D. In some cases, the assumptions hold globally and we can use $D = \mathbb{R}_+^n$. In other cases, it is possible to choose sets of the shape $D = D_1$ or $D = D_2$, where

$$D_1 := \{x \in \mathbb{R}_+^n \mid -\underline{c}_j \le M_j(x) - M_j^* \le \overline{c}_j,\ j = 1, \ldots, m\}$$

and

$$D_2 := \{x \in \mathbb{R}_+^n \mid \underline{z}_j^i \le z_j^i \le \overline{z}_j^i,\ i = 1, \ldots, r_j \text{ and } M_j^* - \underline{c}_j \le M_j(x) \le M_j^* + \overline{c}_j,\ j = 1, \ldots, m\}.$$

For further details and examples, we refer to Imsland (2002).

Note that the convergence result is convergence to the subset Ω, which often (somewhat inaccurate) is referred to as set stability. This does *not* imply convergence to an equilibrium. However, as pointed out in Imsland (2002) (see also De Leenheer and Aeyels (2002) and Chabour and Kalitine (2002) for similar issues), if the closed loop system has an equilibrium that is asymptotically stable with respect to Ω, this equilibrium will have D as an (estimate of) region of attraction.

3.2 Discussion of controller

The controller (2) can be seen as a generalization of the controller in Bastin and Praly (1999). The novelty is threefold:

a) The concept of phases allows to consider systems with multiple inputs. Furthermore, in Bastin and Praly (1999) the function $\Psi(x) = -Ax$ (A diagonal with nonnegative, at least one positive, diagonal elements) and $B(x) = b$ (a constant nonnegative vector with at least one positive element). Condition A3 (which in this case amounts to A3.a.1) is replaced by the system being *zero state detectable* through the output $[1\ 1 \ldots 1]Ax$, which has the same effect as A3.a.1. The results of Bastin and Praly (1999) are recently expanded in the direction of single-input compartmental systems in Bastin and Provost (2002).
b) Systems with controlled outflow can be considered.
c) Sufficient conditions are given to allow upper constraints on the input.

3.3 Robustness

The proposed feedback scheme is independent of the interconnection structure and hence robust [2] to model uncertainties in $\Phi(x)$ (as long as Assumption A1 holds). This is the most important robustness property. As mentioned in Bastin and Praly (1999), the interconnection terms are in practical examples often the terms that are hardest to model.

Assume in the rest of this section that the input saturations are not met, that is, $u_j(x) = \tilde{u}_j(x)$. This will always hold in a neighborhood of Ω. In this case, we can also show some robustness properties with respect to bounded uncertainties in $\Psi(x)$ and $B(x)$.

First note that in the nominal unconstrained case, the feedback (3) linearizes the dynamics of the mass of phase j,

$$\dot{M}_j(x) = \lambda_j \left(M_j^* - M_j(x)\right). \tag{4}$$

We assume further that the modeling errors in $\Psi(x)$ and $B(x)$ are bounded. Mark the "real" values of the terms involved in the controller (3) with a tilde, and assume that there exists norm-bounded $\Delta_j^\psi = \Delta_j^\psi(x,t)$ and $\Delta_j^b = \Delta_j^b(x,t)$ (the dependence on x and t is sometimes suppressed for notational simplicity in the following) such that the nominal values (used in the controller) are related to the real values as

$$\sum_{i=1}^{r_j} \tilde{\psi}_j^i(x) = \sum_{i=1}^{r_j} \psi_j^i(x) + \Delta_j^\psi(x,t),$$
$$\sum_{i=1}^{r_j} \tilde{b}_j^i(x) = (1 + \Delta_j^b(x,t)) \sum_{i=1}^{r_j} b_j^i(x).$$

The real dynamics of phase j can then be written

[2] Robust in the sense that convergence to Ω still holds. Note that changes in $\Phi(x)$ will typically move the equilibria on Ω.

$$\begin{aligned}\dot{M}_j(x) &= \sum_{i=1}^{r_j} \tilde{\psi}_j^i(x) + \sum_{i=1}^{r_j} \tilde{b}_j^i(x) u_j(x) \\ &= \lambda_j (M_j^* - M_j(x)) + \Delta_j^{\psi} \\ &\quad + \Delta_j^b \left(- \sum_{i=1}^{r_j} \psi_i^j(x) + \lambda_j (M_j^* - M_j(x)) \right)\end{aligned}$$

The last part is in general not bounded in terms of x. However, we assume that we can define

$$\delta_j(t) = \Delta_j^{\psi}(x(t), t) + \Delta_j^b(x(t), t) \Big(- \sum_{i=1}^{r_j} \psi_i^j(x(t)) + \lambda_j (M_j^* - M_j(x(t))) \Big)$$

such that $\delta_j(t)$ is norm-bounded, $\delta_j(t) \leq \bar{\delta}_j$. This requires either that $\Delta_j^b(x(t), t) \equiv 0$, or that we know that $x(t)$ is bounded (which is guaranteed by initial conditions in a bounded, invariant set).

The mass dynamics can under the above assumptions be written

$$\dot{M}_j(x(t)) = -\lambda_j (M_j(x(t)) - M_j^*) + \delta_j(t). \quad (5)$$

Since this is linear, it is easy to solve this to find

$$M_j(x(t)) = M_j^* + e^{-\lambda_j (t - t_0)} (M_j(x(t_0)) - M_j^*) + \int_{t_0}^{t} e^{-\lambda_j (t - \tau)} \delta_j(\tau) d\tau$$

where the last element is bounded,

$$\left| \int_{t_0}^{t} e^{-\lambda_j (t-\tau)} \delta_j(\tau) d\tau \right| \leq \frac{1 - e^{-\lambda_j (t - t_0)}}{\lambda_j} \bar{\delta}_j \leq \frac{\bar{\delta}_j}{\lambda_j}.$$

We see that $M_j(x(t))$ converges to the set $\{M_j \mid |M_j - M_j^*| \leq \frac{\bar{\delta}_j}{\lambda_j}\}$ which can be made arbitrarily close to M_j^* by choosing λ_j large. Of course, in choosing λ_j large, the system might become more vulnerable to the influence of measurement noise and unmodeled dynamics.

The above analysis is only valid as long as the input is not saturated. What happens when the input is saturated can be (conservatively) analyzed by examining if the "Lyapunov function" of Theorem 1 is still decreasing under the allowed perturbations. This can be done by checking if assumptions similar to Assumption A3 hold for the perturbed flows.

4. APPLICABILITY OF THE MODEL CLASS

The system class (1) and accompanying control design method has wide applicability. Referring to Imsland (2002) the class has been applied to a number of different examples. We will briefly summarize these results in the following.

- A system comprised of three tanks in series (three states) was investigated using either the inflow to the first tank, the outflow from the third tank, or both as control input(s). In the one control input cases the total mass in the three tanks was controlled. In the two control input case the masses in tank one, and tank two and three; or the masses in tank one and two, and tank three were controlled. Convergence from non-local regions (in one case globally) to a stable equilibrium was shown in all cases.

  A compartmental description of the three tanks would typically consist of three compartments, each linked to one tank. The difference to the phase notion is apparent. In the one control input cases, the total mass is controlled meaning that the single phase consists of the masses of the three tanks. In the two control input case the masses in one tank and the two other tanks, respectively, define the phases.

  The "interconnection structure" $\Phi(x)$ includes the internal flows between the tanks in the one control input case while it includes the internal flow between the two tanks within one phase in the two control input case. The uncontrolled external flows $\Psi(x)$ include the flow between the two phases, i.e. between the two first tanks or the two last tanks, in the two control input case. In the one control input case $\Psi(x)$ consists of the inflow to the first tank when the control input is defined by the outflow from the third tank, or vice versa. The robustness with respect to modeling errors in $\Phi(x)$ and $\Psi(x)$ is obviously important since these terms will contain errors.
- An 2-dimensional food-chain (prey-predator) system (Ortega *et al.*, 1999) with one control input corresponding to the creation of prey has been examined applying the controller (2). The (one) phase was defined by the total mass, of prey and predator, in the system. Global convergence to an asymptotically stable equilibrium was shown. The system can be generalized to an n-dimensional food-chain (prey-predators) system again using the total mass, of prey and predators, in the system as the controlled variable. The controller guarantees convergence to Ω, and simulations show convergence to the single desired equilibrium on Ω, but Lyapunov-based analysis of the dynamics of Ω did, however, not succeed in this case.
- Gas-lifted wells are important as a means to produce oil and gas from hydrocarbon reservoirs with low reservoir pressure (Golan and Whitson, 1991). The well system consist of two volumes: volume 1 holding gas, and volume 2 holding oil and gas. The system is divided into two phases, the mass of gas in the two volumes and the mass of oil in volume 2. The two control inputs are the gas inflow to volume 1, and the gas and oil outflow from volume 2. Analysis on a 3-dimensional model showed local convergence to an asymptotically stable equilibrium on Ω. Further, simulations using the controller on an industry-standard simulator (Bendiksen *et al.*, 1991), gave nice results.
- It should be noted that convergence to Ω does not necessarily imply convergence to an equilibrium. This was shown on a synthetic 3-dimensional system where the analysis indicated

and simulations showed convergence to a periodic orbit in Ω.

The theory has also been applied to a standard test case in process control, the Van der Vusse reactor. Details on this are given in the next section.

5. EXAMPLE: VAN DER VUSSE REACTOR

We consider the van der Vusse reaction kinetic scheme

$$A \to B \to C$$
$$2A \to D$$

taking place in a CSTR. Application of the controller on this reactor based on a mass balance model was demonstrated in Imsland (2002). Here, we will use a model consisting of both mass and energy balance (the two phases) taken from Chen *et al.* (1995), and control heat removal, and inflow rate of substance A.

The first phase consist of a mass balance of substance A and B, on a concentration (c_A and c_B) basis. The second phase consists of an energy balance that describes the cooling that is caused by the cooling jacket. The states are the temperatures in the reactor, T and in the cooling jacket, T_K. Energy is removed from the cooling jacket by means of a heat exchanger. The rate of energy removal is the second input to the system. The mass and energy balance constitutes the two phases according to the setup in Section 2, the first phase being inflow controlled, the second outflow controlled. The model taken from Chen *et al.* (1995) is

$$\dot{c}_A = -k_1(T)c_A - k_3(T)c_A^2 + u_1(c_{Af} - c_A) \tag{6a}$$
$$\dot{c}_B = k_1(T)c_A - k_2(T)c_B - u_1 c_B \tag{6b}$$
$$\dot{T} = u_1(T_0 - T) - \frac{\Delta E_R(x)}{\rho C_p} + \frac{k_w A_R}{\rho C_p V_R}(T_K - T) \tag{6c}$$
$$\dot{T}_K = \frac{1}{m_K C_{PK}}(-u_2 + k_w A_R(T - T_K)), \tag{6d}$$

where

$$\begin{aligned} \Delta E_R(x) &= k_1(T)c_A \Delta H_{R_{AB}} \\ &\quad + k_2(T)c_B \Delta H_{R_{BC}} + k_3(T)c_A^2 \Delta H_{R_{AD}} \end{aligned} \tag{6e}$$

and the reaction kinetics are given from the Arrhenius law

$$k_i(T) = k_{i0} e^{E_i/T},\ i = 1, 2, 3. \tag{6f}$$

Nominal values of the physical and chemical parameters in the model (6) can be found in Chen *et al.* (1995).

Since the reactor and the cooling jacket have different heat capacities, the transfer of energy between them leads to asymmetric temperature changes. This means that the energy transfer does not fulfill the interconnection assumption A1. This is remedied by taking the energies $\rho C_p V_R T$ and $m_K C_{PK} T_K$ as states, in stead of the temperatures.

The control problem (from Chen *et al.* (1995)) is to stabilize the system at the working point $c_A = 2.14\frac{mol}{l}$, $c_B = 1.09\frac{mol}{l}$, $T = 387.2K$ and $T_K = 386.1K$.

The input is then defined in terms of (2) and

$$\begin{aligned} \tilde{u}_1 &= \frac{1}{c_{Af} - c_A - c_B}(k_3(T)c_A^2 \\ &\qquad + k_2(T)c_B + \lambda_1(M_1^* - M_1(x))) \\ \tilde{u}_2 &= u_1(T_0 - T)\rho C_p V_R \\ &\qquad - V_R \Delta E_R(x) - \lambda_2(M_2^* - M_2(x))), \end{aligned}$$

with saturations at $\bar{u}_1 = 35\frac{1}{h}$ and $\bar{u}_2 = 9000\frac{kJ}{h}$. The phase masses are $M_1(x) = c_A + c_B$ and $M_2(x) = \rho C_p V_R T + m_K C_{PK} T_K$.

The "controllability Assumption" A3 for the first phase holds (at least) for $0 \le c_A + c_B \le 7$, for a reasonable temperature range. For the second phase, A3 [3] holds only for a rather small operating range around the desired equilibrium. The reason for this is related to the exothermic nature of the reaction - for some initial conditions close to the desired equilibrium, the energy produced by the reaction is larger than the cooling jacket capacity, such that the total energy is increasing. A remedy for getting a larger guaranteed region of attraction could be to choose another equilibrium, with lower temperatures. This could also be seen as choosing an operating point with better controllability. However, simulations indicate that the controller still works well even outside the region where the controllability assumptions for the second phase holds, since the system dynamics take the states into a region where the assumption holds. This illustrates the sufficient nature of A3.

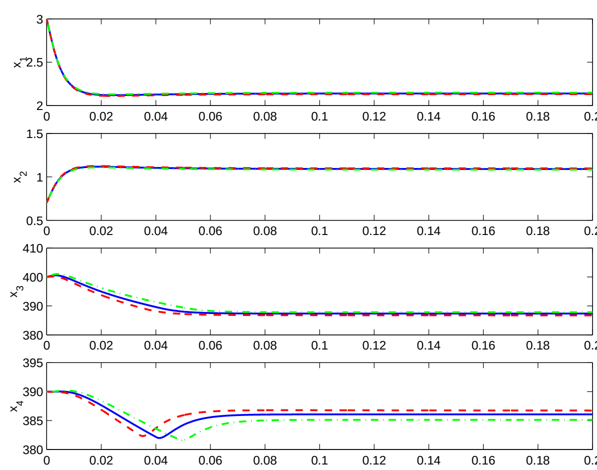

Fig. 1. Simulation of Van der Vusse reactor showing the states, from initial condition $c_A = 3.0$, $c_B = .70$, $T = 400$ and $T_K = 390$. Nominal parameter set is shown with whole lines, set 1 is dashed, set 2 is dash-dotted.

The simulations in Figures 1-3 show that the controller is robust to the two "extreme" cases of parameter uncertainty taken from Chen *et al.*

[3] Note that the input u_1 in (6c) is taken as a function of state while checking A3 for phase 2.

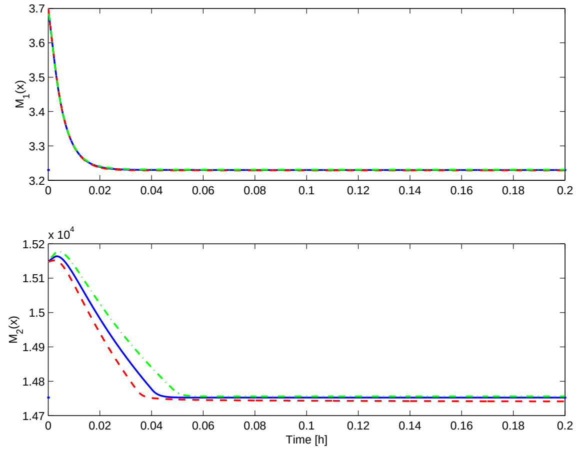

Fig. 2. Simulation of Van der Vusse reactor showing the masses of the phases. Nominal parameter set is shown with whole lines, set 1 is dashed, set 2 is dash-dotted.

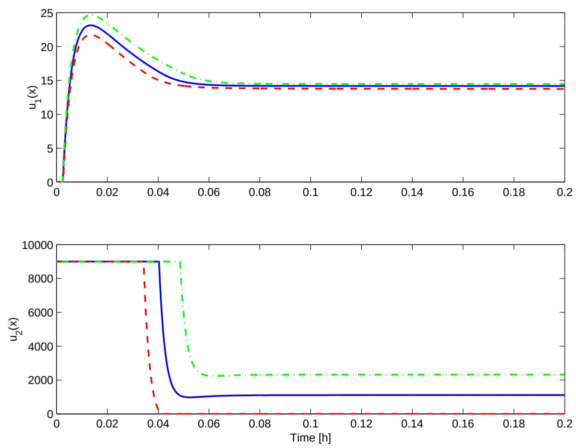

Fig. 3. Simulation of Van der Vusse reactor showing the inputs. Nominal parameter set is shown with whole lines, set 1 is dashed, set 2 is dash-dotted.

(1995) in the sense that stability and convergence to close to Ω is preserved. However the desired equilibrium is only approximately preserved. Note that for parameter set 2, the second input reaches its upper saturation at convergence, such that the theory does not really cover this case. Physically, the saturation says that heat removal is not necessary at this working point, for these parameters. Also the controller in Chen *et al.* (1995) saturates at the equilibrium for this parameter set.

Since the "mass" of phase 2 is increasing initially (Figure 3), the controllability assumptions A3 are not fulfilled for these initial conditions. The controller still works well, as discussed above.

6. DISCUSSION AND CONCLUDING REMARKS

The system class is potentially advantageous to systems with positive state variables. Positive state variables are common in dynamic model based on first principles. The advantage is pronounced for systems with an internal structure that is susceptible to the presented system class and that are hard to model accurately. We have presented several quite different examples of such systems.

An obvious limitation of the this paper is the requirement of state feedback control. The natural approach to the output feedback problem is in this case to use observers to estimate the state. Design of observers that can exploit positivity and system structure in a similar manner as the feedback design, is an interesting area for further research.

To conclude we have presented a system class for positive systems, an accompanying state feedback controller with robust stability guarantees, and argued that the theory has potentially wide applicability.

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STABILIZATION OF GAS LIFTED WELLS BASED ON STATE ESTIMATION

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Abstract: This paper treats stabilization of multiphase flow in a gas lifted oil well. Two different controllers are investigated, PI control using the estimated downhole pressure in the well, and nonlinear model based control of the total mass in the system. Both control structures rely on the use of a state estimator, and are able to stabilize the well flow with or without a downhole pressure measurement available. In both cases stabilization of gas lifted wells increases total production significantly.

Keywords: State estimation with Kalman filter, control, multiphase flow, positive systems.

1. INTRODUCTION

The use of control in multiphase flow systems is an area of increasing interest for the oil and gas industry. Oil wells with highly oscillatory flow are a significant problem in the petroleum industry. Several different instability phenomena related to oil and gas wells exist, in this study unstable gas lifted wells will be the area of investigation.

Gas lift is a technology to produce oil and gas from wells with low reservoir pressure by reducing the hydrostatic pressure in the tubing. Gas is injected into the tubing, as deep as possible, and mixes with the fluid from the reservoir, see Figure 2. The gas reduces the density of the fluid in the tubing, which reduces the downhole pressure, DHP, and thereby increases the production from the reservoir. The lift gas is routed from the surface and into the annulus, the volume between the casing and the tubing. The gas enters the tubing through a valve, an injection orifice.

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The dynamics of highly oscillatory flow in a gas lifted well can be described as follows:

(1) Gas from the casing starts to flow into the tubing. As gas enters the tubing the pressure in the tubing falls. This accelerates the inflow of gas.
(2) The gas pushes the major part of the liquid out of the tubing.
(3) Liquid in the tubing generates a blocking constraint downstream the injection orifice. Hence, the tubing gets filled with liquid and the annulus with gas.
(4) When the pressure upstream the injection orifice is able to overcome the pressure on the downstream side, a new cycle starts.

This type of oscillation is described as casing-heading instability and is shown in the first part of Figure 5 and 6. More information can be found in Xu and Golan (1989).

There are in principle two approaches to eliminate highly oscillating well flow in gas lifted wells: The first approach is to increase the pressure drop caused by friction; either by increasing the gas flow rate, reducing the opening of the production choke or reducing the size of the gas orifice. The second method is the use of active control to stabilize the well flow, which is the subject of this study.

Figure 1 shows a conceptual gas lift production curve. The produced oil and gas rate is a function of the flow rate of gas injected into the well. The curve shows under which conditions the well exhibits stable or highly oscillatory flow. It is important to note that the average production rate may be significantly lower with unstable, see the line "open loop production", compared to stable well flow, see the line "theoretical production". The region of optimum lift gas utilization may lie in the unstable region.

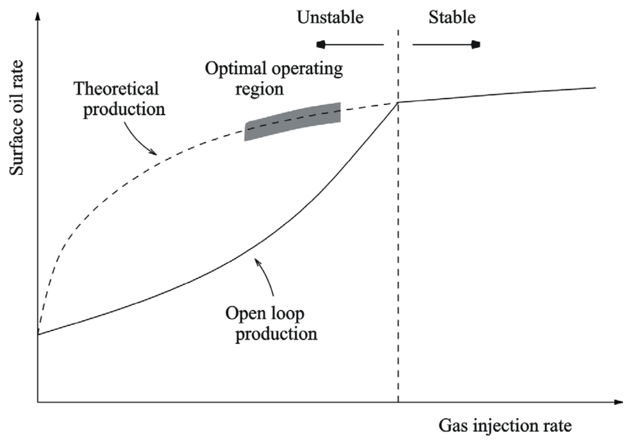

Fig. 1. The gas lift curve with the region of optimum lift gas utilization.

Large oscillations in the flow rate from the well causes lower total production, poor downstream oil/water separation, limits the production capacity and causes flaring. A reduction of the oscillations gives increased processing capacity because of the reduced need for buffer capacity in the process equipment.

Control has to a limited degree been studied for single well systems, see Jansen *et al.* (1999), Kinderen and Dunham (1998) and Dalsmo *et al.* (2002). In addition a two-well simulation study was investigated in Eikrem *et al.* (2002).

The scope of this paper is to study the use of state estimation and control as a tool for stabilizing highly oscillatory well flow in gas lifted wells. Further, earlier work with state feedback for nonlinear positive systems is extended to a realistic output feedback case.

This paper is structured as follows: The system and models are described in Section 2 and 3. A brief theoretical basis is outlined in Section 4 and 5, while the results are shown in Section 6. The paper ends with a discussion and some concluding remarks.

2. SYSTEM DESCRIPTION

2.1 Single Well System

The basis for this study is a realistic gas lifted well, see Figure 2. Reservoir fluid flows through a perforated well, into the wellbore, upwards through the tubing, through the production choke, before it enters downstream equipment which typically will be a manifold and an inlet separator. Gas is injected into the annulus and enters the tubing close to the bottom of the well. The gas mixes with the reservoir fluid to reduce the density of the fluid in the tubing.

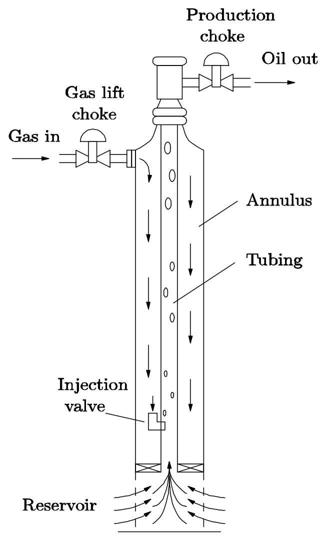

Fig. 2. A gas lifted oil well

The well is described by the following parameters:

- Well parameters
  - · 2048 m vertical well
  - · 5 inch tubing
  - · 2.75 inch production choke
  - · 0.5 inch injection orifice
- Reservoir parameters
  - · P_R = 160 bara
  - · T_R = 108 ^{o}C
  - · PI = 2.47E-6 kg/s/Pa
- Separator inlet pressure
  - · 15 bara
- Gas injection into annulus
  - · 0.8 kg/s
  - · 160 bara
  - · 60 ^{o}C

The productivity index, PI, is defined by

$$PI = \frac{\dot{m}}{\Delta P}$$

where $\dot{m}$ is the total mass flow rate from the reservoir to the well and ΔP is the pressure difference between the reservoir and the bottom of the well. This index relates the mass flow from the reservoir and into the well to the corresponding pressure drop. The PI is assumed constant.

It is assumed that there is no water in the produced fluids, only oil and gas. The gas/oil ratio, GOR, is 80 Sm^3/Sm^3. GOR is defined by:

$$GOR = \frac{\dot{q}_{gas}}{\dot{q}_{oil}}$$

Hence the GOR is defined as the ratio between the volumetric gas rate and the volumetric oil rate at standard temperature and pressure.

The valve model for the production choke includes limitation for the actuator speed, closing time for the valve is 420 sec.

2.2 Simulator

The transient multiphase flow simulator OLGA 2000 [2], commonly used in the petroleum industry, is selected as a platform for the simulations. The state estimator and the controllers are implemented in Matlab [3]. OLGA 2000 and Matlab are connected using a Matlab-OLGA link [4].

OLGA 2000 is a modified two-fluid model, i.e. separate continuity equations for the gas, liquid bulk and liquid droplets are applied. Two momentum equations are used, one for the continuous liquid phase and one for the combination of gas and possible liquid droplets. Entrainment of liquid droplets in the gas phase is given by a slip relation. One mixture energy equation is applied. This yields six conservation equations to be solved in each volume (Scandpower, 2001).

The OLGA 2000 model developed for the gas lifted well is built upon the description given in Section 2. The OLGA 2000 model consists of an annulus divided into 25 volumes, and a tubing divided into 25 volumes. The fluid used in the simulations consists of two phases, oil and gas. The inflow of oil and gas from the reservoir is modelled by use of the productivity index, as defined in section 2. The injection rate of lift gas to the annulus is fixed, a fast and well tuned flow controller is assumed used. Fixed boundary conditions for the tubing is assumed, i.e. a fixed reservoir pressure and a fixed separator pressure downstream the production choke.

3. A SIMPLE GAS LIFT MODEL

To be able to develop a state estimator, a simplified model of the gas lifted well is required. This model uses the same boundary conditions as the OLGA 2000 model, but has no mass transfer between the phases, only one volume for the annulus and only one volume for the tubing. The flow between the volumes, into the system and out of the system is controlled by general valve models:

$$w = \begin{cases} C\sqrt{\rho(p_2 - p_1)} & \text{if } \ p_2 \geq p_1 \\ 0 & \text{else} \end{cases} \tag{1}$$

where w is the mass flow, C is the valve parameter, ρ is the density, while the $p_2 - p_1$ is the pressure drop across the restriction. C takes on different values for each restriction.

The pressures in the system are calculated from the mass in the volumes and the pressure drop through the volumes. The pressure at the top of the annulus is calculated by use of the ideal gas law. The pressure at the bottom of the annulus is given by adding the pressure drop from the gas column to the pressure at the top of the annulus. The pressure at the top of the tubing is calculated by the ideal gas law. The volume of the gas in the tubing is given by the volume which is not occupied by oil. The pressure at the bottom of the tubing is given by adding the pressure drop from the fluid column to the pressure at the top of the tubing. Based upon the pressure calculations of the system, the mass flows in and out of the volumes are given by the valve equation (1). The model parameters are tuned based upon OLGA simulations.

To summarize, the following mass balances are assumed to describe the dynamics of the gas lifted well:

$$\begin{aligned} \dot{x}_1 &= w_{iv}(x) - w_{gc}(x) && \text{Mass of gas in annulus} \\ \dot{x}_2 &= w_{gc}(x) - w_{pg}(x,u) && \text{Mass of gas in tubing} \\ \dot{x}_3 &= w_r(x) - w_{po}(x,u) && \text{Mass of oil in tubing} \end{aligned}$$

The symbols are described in Table 1.

Table 1. Symbols

Symbol	**Description**
$w_{iv}(x)$	Gas flow from source into annulus
$w_{gc}(x)$	Gas flow from annulus into tubing
$w_{pg}(x,u)$	Gas flow out of tubing
$w_r(x)$	Oil flow from reservoir into tubing
$w_{po}(x,u)$	Oil flow out of tubing
u	Production choke
M	Total mass in system
λ	Mass control parameter
w_{ref}	Setpoint for flow controller

The simplified model herein is a modified version of the simplified gas lift well model given in Imsland (2002).

[2] Scandpower AS, Norway
[3] The Mathworks Inc
[4] ABB AS, Norway

4. THEORETICAL BASIS

4.1 State Estimation

A standard extended Kalman filter based on the simplified model is developed. Numerical derivation of the simplified model is used to derive a linear model at each time step, corresponding to the current operating point. The covariance matrices for the process and measurement noise are diagonal matrices. The measurement noise matrix is designed based upon the uncertainty of the measurement devices. This matrix is scaled to account for differences in the range of the measurements. The process noise matrix is tuned to obtain a reasonable bandwidth for the state estimator.

4.2 Positive Systems and Feedback Control

Positive systems are dynamical systems which are described by ODEs where the state variables are non-negative. Since mass is an inherently positive quantity, systems modelled by mass balances are a natural example of positive systems, see e.g. Bastin (1999). In Imsland (2002) and Imsland *et al.* (2003) a state feedback controller that exploits positivity is developed. Further, it is shown that the controller exhibits robust stability properties. This work is extended by applying this method in a realistic output feedback setting.

The purpose of the controller is to stabilize the total mass in the system. This is achieved by linearizing the total mass dynamics and exploiting the positivity of the system. The controller calculates the setpoint for an "inner" PI mass flow control loop, and this setpoint is given by:

$$w_{ref} = \max\{0, w_r(x) + w_{iv}(x) + \lambda[M^* - M(x)]\} \tag{2}$$

where M^* is the total mass setpoint. The symbols are described in Table 1.

5. CONTROL STRUCTURES

Several control structures for stabilization of gas lifted wells are available. The possibilities of stabilizing a gas lifted well by use of the measured downhole pressure or the measured casing head pressure have been shown in e.g. Eikrem *et al.* (2002).

5.1 Kalman Filter and Measurements

The Kalman filter uses the available process measurements for correction of the states in the simplified model, in this case the masses in the system. The selected measurements are the pressure at the top of the tubing, the pressure at the top of the casing and the pressure at the bottom of the well. These are realistic measurements from an industrial point of view. Since the downhole pressure measurement is located in a harsh and quite inaccessible location, the effect of failure of this measurement will be investigated.

The Kalman filter includes a check on positivity of the state variables in the sense that state estimates always will be positive.

5.2 Pressure Control and DHP Measurement Failure

The first control structure uses the opening of the production choke as the manipulated variable and the estimated downhole pressure as the controlled variable. The PI-controller is tuned on the basis of process knowledge and iterative simulations. The controller, including the state estimator, is shown in Figure 3.

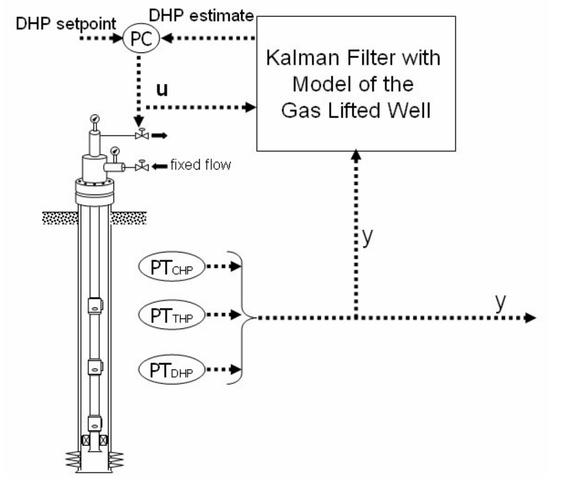

Fig. 3. Control structure for stabilization of a gas lifted well, by controlling the estimated downhole pressure.

5.3 Mass Control and DHP Measurement Failure

The second control structure uses the opening of the production choke as the manipulated variable and the total mass in the system as the controlled variable. In a cascade-manner, the setpoint for the "inner" flow control loop is given by w_{ref}, see (2). The controller including the state estimator is shown in Figure 4.

5.4 Simulation Scenario

The simulations follow the same scenario:

- Timeslot 1, (0-4 h): The well simulator is run in open loop with 50% choke opening.

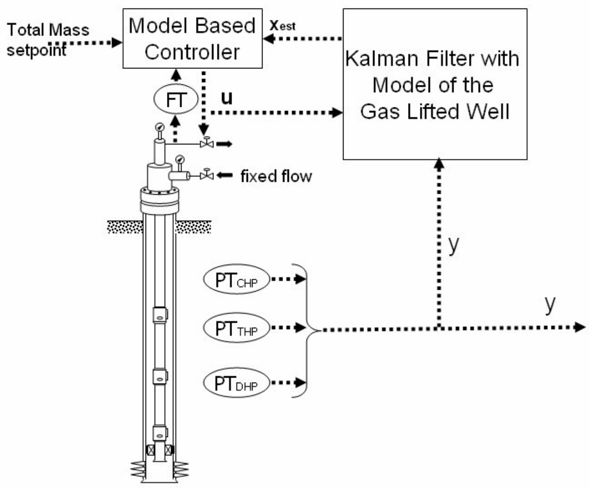

Fig. 4. Control structure for stabilization of gas lifted well, by controlling the total mass in the system.

- Timeslot 2, (4-10 h): The well is stabilized by use of a small choke opening, 20%.
- Timeslot 3: (10-19h): Control using estimated variable, with DHP measurement available.
- Timeslot 4: (19-25h): Control using estimated variable, without DHP measurement available.
- Timeslot 5 (25-30 h): Open loop simulation.

6. SIMULATION RESULTS

6.1 Pressure Control and DHP Measurement Failure

The result from the stabilization of the gas lifted well based upon estimated downhole pressure is given in Figure 5.

The highly oscillatory behaviour is clearly observed during Timeslot 1. The flow is stabilized by closing the valve to 20%, i.e. by increasing the pressure drop caused by friction. The flow is well behaved during Timeslot 3. There is a major disturbance due to the loss of the DHP measurement at 19 hours. This is reasonable since the DHP estimate is heavily influenced by the DHP measurement. The important issue, however, is the fact that the flow is stable. Moreover, the flow becomes highly oscillatory after the controller has been deactivated during Timeslot 5. It should be mentioned that the controller showed a close-to-identical behaviour during Timeslot 3, when the DHP estimate was replaced by the DHP measurement.

The values for the controller parameters for the PI controller are K_p = -0.1 and T_i = 7200 sec. The pressure measurement is given in bara.

The production of oil and gas is given in Figure 6. The stabilization of the gas lifted oil well gives a significant increase in the produced amount of oil and gas. This is particularly pronounced by comparing the production during Timeslot 4 and 5, and this agrees with Figure 1. In the unstable region, the average production rate is 6 kg/s, while the stabilized region gives a production of 15 kg/s.

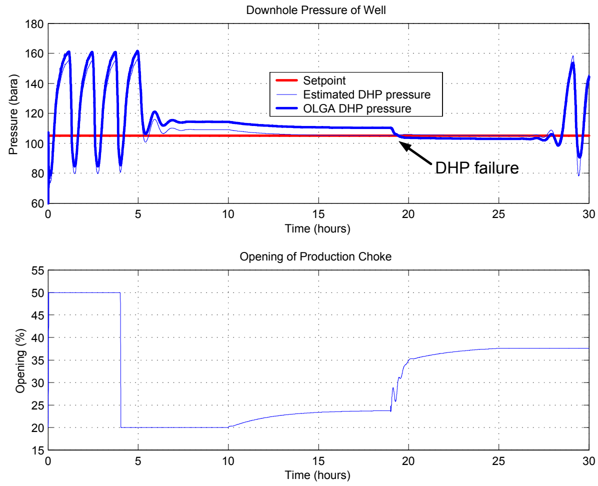

Fig. 5. The estimated and the OLGA downhole pressure for the well. The OLGA well is stabilized by stabilizing the estimated downhole pressure.

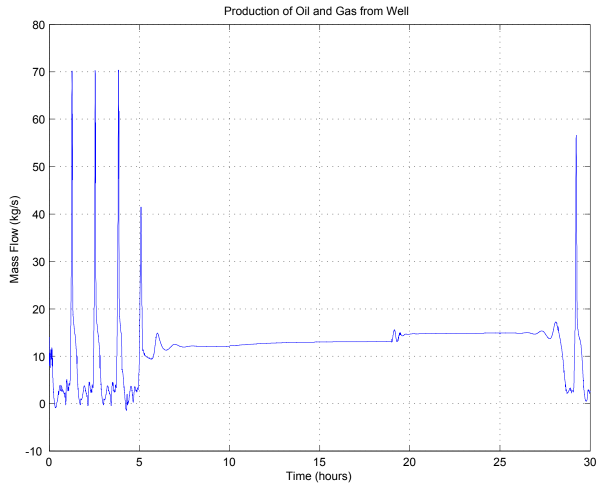

Fig. 6. The oil production from the well when the well is unstable and when it is stabilized.

6.2 Mass Control and DHP Measurement Failure

The result from the stabilization of the gas lifted well based upon mass estimation is given in Figure 7. The downhole pressure measurement fails after 19 hours. The description of the simulation results is identical to the description in the previous case, see Section 6.1. Note again the disturbance when the DHP measurement fails.

Figure 7 reveals that the controller quickly takes the system mass to the desired value, or close to, due to model and estimation error. It can be observed that the input continues to move afterwards. This can be explained by the fact that the point on the "constant mass"-manifold which the system initially converges to, is not the closed loop equilibrium. The slow dynamics on the manifold takes the system to this equilibrium.

The controller parameters for the "inner" mass flow control loop are $K_p = 0.004$ and $T_i = 5$ sec, while the parameter for the "outer" total mass control loop is $\lambda = 0.003$. The mass is given in kg.

The production of oil and gas for the system in open and closed loop is similar to the results given in Figure 6.

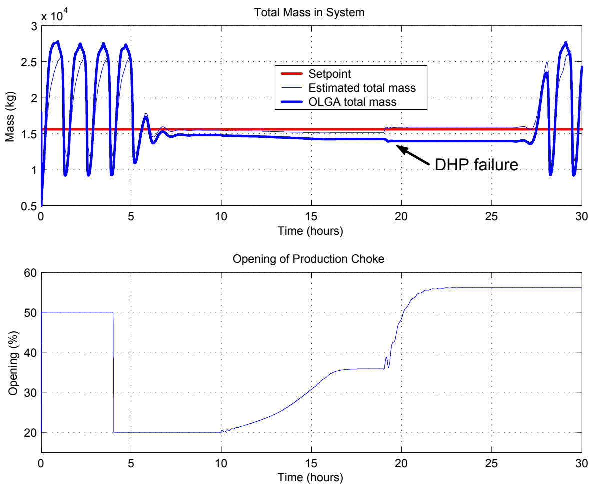

Fig. 7. The estimated and the OLGA total mass for the well. The system is stabilized by controlling the estimated total mass.

7. DISCUSSION AND CONCLUDING REMARKS

This study shows how a low order model in a Kalman filter can provide useful information for control purposes. In particular it is shown how a state estimator can alleviate the common situation in which a difficult accessible downhole measurement fails. It is further shown that stabilization of gas lifted wells is important since it gives an increased production, in this case the production of oil and gas is more than doubled, see Figure 6.

The state estimator functions well both with the traditional PI-controller and the nonlinear controller for positive systems. In this paper the controllers are not pushed to the limit to assess the potential of the nonlinear controller. The possible merit of the nonlinear controller has been showed in a realistic output feedback application.

The simplified model needs to be well tuned to reflect the dynamics of the real system as the DHP measurement fails. The main challenge is related to the estimation of downhole conditions upon the loss of the DHP measurement. In practice the tuning is done by adjusting the valve parameters, see (1). Typically they have been changed $\pm 25\%$ compared to their original values. It should be mentioned that the low order model is observable at all times.

An alternative to the current approach is the use of an augmented Kalman filter in which model parameters are tuned online.

To re-iterate there are alternative control structures, that do not involve downhole measurements nor estimation, that are able to stabilize the highly oscillatory flow for this particular well. There is still considerable value in the problem addressed in this paper since other more complex well completions may require measurements or estimates of downhole conditions.

8. ACKNOWLEDGEMENT

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H_∞ CONTROL OF DESCRIPTOR SYSTEMS: AN APPLICATION FROM BINARY DISTILLATION CONTROL

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Abstract

In this paper the H_∞ control problem for descriptor systems is considered. This problem can efficiently be solved by specialization of a recent solution of the general quadratic performance control problem to the H_∞ case. The solution is given in terms of strict linear matrix inequality (LMI) conditions. Contrary to previous solutions of the descriptor H_∞ control problem, these synthesis conditions can easily be evaluated by standard LMI solvers. The presented synthesis result is applied to a S/KS H_∞ control problem from binary distillation control. The process model of the underlying separation process is given by means of a phenomenological descriptor model which describes the movement of concentration profiles in rectifying and stripping section of the distillation column.
Keywords: Descriptor control; Mixed sensitivity problem; distillation control; Linear matrix inequalities; Generalized Bounded Real Lemma

1. Introduction

Descriptor systems (sometimes also referred to as singular, semistate or differential-algebraic equation (DAE) systems) describe a broad class of systems which are not only of theoretical interest but also have great practical significance. Models of chemical processes for example typically consist of differential equations describing the dynamic balances of mass and energy while additional algebraic equations account for thermodynamic equilibrium relations, steady-state assumptions, empirical correlations, etc. [3]. In mechanical engineering descriptor systems result from holonomic and non-holonomic constraints [11]. Also in electronics and even in economics modeling in terms of descriptor systems is frequently encountered [5].

Descriptor systems are able to describe system behaviors, that cannot be captured by "non-descriptor" systems (i.e. systems governed only by differential equations) [1]. Therefore index reduction techniques (i.e. reduction of a descriptor system to an ODE) necessarily are connected to a loss of information for high index systems. Due to this fact in recent years much work has focused on analysis and design techniques for high index descriptor systems (see [4] for an overview). For linear systems many of the standard design techniques for state-space systems have been extended to descriptor systems. Especially there has been a focus on LMI synthesis techniques which guarantee bounds on induced vector norms (e.g. H_2, H_∞-norm) for input-output descriptions of the form

$$\begin{aligned} E\dot{\boldsymbol{\xi}}(t) &= A\boldsymbol{\xi}(t) + B\boldsymbol{w}(t), \;\; t \geq 0, \; \boldsymbol{\xi}(0^-) = \boldsymbol{\xi}_0^- \\ \boldsymbol{z}(t) &= C\boldsymbol{\xi}(t) + D\boldsymbol{w}(t). \end{aligned} \tag{1}$$

Here $\boldsymbol{\xi}(t) \in I\!R^{n_\xi}$ denote the descriptor variables, $\boldsymbol{w}(t) \in I\!R^{n_w}$ the external input variables, and $\boldsymbol{z}(t) \in I\!R^{n_z}$ the external output variables. E, A, B, C, D are constant system matrices of appropriate dimensions with E being a possibly singular $n_\xi \times n_\xi$ matrix with $n_\xi \geq \operatorname{rank}(E) =: r$. Usually the LMI approaches to this kind of problems (e.g. [6, 9]) assume an E-matrix in SVD form, i.e.

$$E = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}, \quad \Sigma = \Sigma^{\mathrm{T}} \in I\!R^{r \times r}. \tag{2}$$

Theoretically there is no loss of generality connected to this assumption since a transformation to an E matrix of the form (2) is always possible. However, this transformation may be ill conditioned. This is especially the case for mechanical descriptor descriptions where point masses of extremely different magnitudes are involved. Furthermore the approaches based on (2) result in synthesis LMIs with all occurring system matrices partitioned according to (2). This is not only notational inconvenient but means in fact that the standard case (regular E matrix) is not included. These shortcomings are overcome for the general quadratic performance (GQP) output feedback control problem for descriptor systems in [10].

In this paper the GQP synthesis result is specialized to the most important subproblem, namely the descriptor H_∞ control problem. The solution of the controller synthesis problem is based on congruence transformation of a corresponding analysis result in descriptor form. The analysis result basically is an LMI based test (the generalized bounded real lemma) which allows for a given closed loop system to decide whether or not a prescribed H_∞ norm bound is met or not. This test is given here for convenience of the reader. The transition to the controller synthesis solution is only briefly outlined. Details can be found in [10]. The focus here is to show the applicability of the descriptor H_∞ controller synthesis result to realistic control problems in process control. To our knowledge, this is the first application of a descriptor H_∞ controller synthesis result to a realistic control setup.

2. The Generalized Bounded Real Lemma

In contrast to state space system descriptions a descriptor system may allow non-unique solutions which possibly contain impulses. This certainly does not fit into the internal stability requirement which goes along with the H_∞-norm bound requirement in the standard H_∞ control problem. As a generalization one therefore considers regular (i.e. descriptor systems with a unique solution) and impulse-free descriptor systems. Descriptor systems which additionally are stable are termed *admissible* [6]. An LMI based characterization of admissible descriptor systems (E, A, B, C) (i.e. descriptor systems (1) with $D = 0$) which are H_∞-norm bounded is given in the following proposition:

Proposition 2..1 *(Generalized bounded real lemma, GBRL) A system (E, A, B, C) is a stable index one system with*

$$\|G\|_\infty < \gamma, \quad G(s) := C(sE - A)^{-1}B \quad (3)$$

iff there exists a matrix X with

$$E^{\mathrm{T}}X = X^{\mathrm{T}}E \geq 0 \quad (4)$$

$$\mathcal{B}(\gamma, X) := \begin{bmatrix} A^{\mathrm{T}}X + XA & X^{\mathrm{T}}B & C^{\mathrm{T}} \\ B^{\mathrm{T}}X & -\gamma I & 0 \\ C & 0 & -\gamma I \end{bmatrix} < 0. \quad (5)$$

Proof. See [10]. □

Remark 1. The consideration of the case $D = 0$ in the previous proposition is not restrictive since every descriptor system (1) can be reformulated as a descriptor system with $D = 0$ if additional descriptor variables with $\boldsymbol{\xi}_{add}(t) := D\boldsymbol{w}(t)$ are introduced.

Remark 2. The LMI (4) is non-strict. The key towards a strict inequality is the symmetry constraint $E^{\mathrm{T}}X = X^{\mathrm{T}}E$ expressed in (4). All X fulfilling this constraint can be parameterized in terms of the fundamental subspaces of E as

$$X = \tilde{X}E + E^{\perp}W, \quad \tilde{X} = \tilde{X}^{\mathrm{T}} \quad (6)$$

with $E^{\perp}$ denoting a full rank matrix such that $E^{\mathrm{T}}E^{\perp} = 0$ and $\tilde{X}$, U being matrices of appropriate dimensions. The parameterization (6) in $\tilde{X}$, W is valid since we may write (4) as $VE^{\mathrm{T}}U^{\mathrm{T}}U^{-\mathrm{T}}XV^{\mathrm{T}} = VX^{\mathrm{T}}U^{-1}UEV^{\mathrm{T}}$ with $E_{svd} := UEV^{\mathrm{T}}$ being a SVD decomposition of E. With $X' := U^{-\mathrm{T}}XV^{\mathrm{T}}$ we get $E_{svd}^{\mathrm{T}}X' = X'^{\mathrm{T}}E_{svd}$, i.e. $X' = \begin{bmatrix} X_1' & 0 \\ X_3' & X_4' \end{bmatrix}$ with a block structure corresponding to E_{svd}. This X' clearly can be parameterized as in (6). Finally we observe that the (1,1)-element in (5) implies the regularity of X. In view of (4) the parameterization (6) can be strengthen by $\tilde{X} > 0$. A strict inequality characterization of a H_∞-norm bound γ then can be derived by substituting (6) into (5) and replacing (4) by $\tilde{X} > 0$.

$$
\begin{bmatrix}
AY_1 + Y_1^{\mathrm{T}}A^{\mathrm{T}} + B_2\hat{C}_K + \left(B_2\hat{C}_K\right)^{\mathrm{T}} & \left(A + B_2\hat{D}_K C_2\right) + \hat{A}_K^{\mathrm{T}} & B_1 & Y_1^{\mathrm{T}}C_1^{\mathrm{T}} \\
\left(A + B_2\hat{D}_K C_2\right)^{\mathrm{T}} + \hat{A}_K & A^{\mathrm{T}}X_1 + X_1^{\mathrm{T}}A + \hat{B}_K C_2 + C_2^{\mathrm{T}}\hat{B}_K^{\mathrm{T}} & X_1^{\mathrm{T}}B_1 & C_1^{\mathrm{T}} \\
B_1^{\mathrm{T}} & B_1^{\mathrm{T}}X_1 & -\gamma I & 0 \\
C_1 Y_1 & C_1 & 0 & -\gamma I
\end{bmatrix} < 0, \tag{7}
$$

$$
\begin{array}{rcl}
Y_1 & := & RE^{\mathrm{T}} + E^{\mathrm{T}\perp}W_Y, \quad R > 0, \\
X_1 & := & SE + E^{\perp}W_X, \quad S > 0,
\end{array}
\quad \begin{bmatrix} R & E^{+} \\ E^{\mathrm{T}+} & S \end{bmatrix} > 0 \tag{8}
$$

Note that the matrix X is over-parameterized by (6) with respect to the variables not affected by the positive definiteness requirement in (4). This may be used to put further constraints on $\tilde{X}$ in (6).
The previous remark shows how to check H_∞-norm bounds with standard strict LMI solvers as e.g. the LMI toolbox in MatLab. However, the main importance of this remark will become clear in the context of the corresponding H_∞ controller synthesis problem for DAE systems which is addressed in the next section.

3. The H_∞ Control Problem for Linear Descriptor Systems

Consider a generalized plant Σ_E that is a descriptor system

$$
\Sigma_E : \quad
\begin{array}{rcl}
E\dot{\boldsymbol{x}}(t) & = & A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t) + B_2\boldsymbol{u}(t) \\
\boldsymbol{z}(t) & = & C_1\boldsymbol{x}(t) \\
\boldsymbol{y}(t) & = & C_2\boldsymbol{x}(t)
\end{array} \tag{9}
$$

where $\boldsymbol{x}(t) \in I\!R^{n_x}$ denotes the descriptor variables, $\boldsymbol{u}(t) \in I\!R^{n_u}$ the control input, $\boldsymbol{w}(t) \in I\!R^{n_w}$ the external input, $\boldsymbol{z}(t) \in I\!R^{n_z}$ the external output, and $\boldsymbol{y}(t) \in I\!R^{n_y}$ the measured output. A, B_i, C_i are constant matrices of appropriate dimension and E is a possibly singular matrix having the same dimension as A. Notice that there is no loss of generality in the descriptor setup in neglecting a direct fed-through of control/external input to the measured/external output since such a dependency also can be expressed by means of an augmented descriptor vector $\boldsymbol{x}$ [6].
The control problem is to find a linear output feedback controller such that the undisturbed closed loop ($\boldsymbol{w} \equiv \boldsymbol{0}$) is an admissible system and such that the transfer matrix from the external input $\boldsymbol{w}$ to the external output $\boldsymbol{z}$ is H_∞-norm bounded by a prescribed number $\gamma > 0$.

With a controller K_E,

$$
K_E : \quad
\begin{array}{l}
E\dot{\boldsymbol{\zeta}}(t) = A_K\boldsymbol{\zeta}(t) + B_K\boldsymbol{y}(t) \\
\boldsymbol{u}(t) = C_K\boldsymbol{\zeta}(t) + D_K\boldsymbol{y}(t), \quad \boldsymbol{\zeta}(t) \in I\!R^{n_x}
\end{array} \tag{10}
$$

parametrized by A_K, B_K, C_K, D_K the closed loop system is given by

$$
\begin{array}{rcl}
E_{cl}\dot{\boldsymbol{\xi}}(t) & = & A_{cl}\boldsymbol{\xi}(t) + B_{cl}\boldsymbol{w}(t) \\
\boldsymbol{z}(t) & = & C_{cl}\boldsymbol{\xi}(t), \qquad \boldsymbol{\xi}(t) \in I\!R^{2n_x},
\end{array} \tag{11}
$$

$$
E_{cl} = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}, \; A_{cl} = \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix},
$$
$$
B_{cl} = \begin{bmatrix} B_1 \\ 0_{n_x \times n_w} \end{bmatrix}, \; C_{cl} = \begin{bmatrix} C_1 & 0_{n_z \times n_x} \end{bmatrix}. \tag{12}
$$

Then all controllers K_E solving the H_∞ control problem for descriptor systems are characterized by the following theorem:

Theorem 3..1 *Consider a plant (9) and a controller (10). There exists a controller parameterization A_K, B_K, C_K, D_K such that the undisturbed closed loop system (11) is admissible with $\|G_{cl}\|_\infty < \gamma$ (with $G_{cl}(s) := C_{cl}(sE_{cl} - A_{cl})^{-1}B_{cl}$) if and only if the LMIs (7), (8) at the top of the page[1] admit a solution $\{R,\, S,\, W_Y,\, W_X,\, \hat{A}_K,\, \hat{B}_K,\, \hat{C}_K,\, \hat{D}_K\}$.*

Proof. The Theorem is a special case of the GQP result in [10]. Here only a brief sketch of the proof is imparted.

Application of the generalized bounded real lemma (Proposition 2..1) to the closed loop

[1]Here E^+ denotes any generalized inverse with the property $EE^+E = E$.

system matrices (12) renders the necessary and sufficient LMI/BMI conditions

$$E_{cl}^{\mathrm{T}} X = X^{\mathrm{T}} E_{cl} \geq 0, \tag{13}$$

$$\begin{bmatrix} A_{cl}^{\mathrm{T}} X + X A_{cl} & X^{\mathrm{T}} B_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}} X & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} < 0. \tag{14}$$

This matrix inequality is clearly nonlinear due to products of unknown controller matrices with the matrix X. The idea in the following is to introduce new matrix variables ("linearizing change of variables") such that (13), (14) can be replaced by LMIs. This is not possible directly but with an intermediate step, i.e. a congruence transformation of (13), (14). Then, new variables can be introduced such that we get synthesis LMIs. These LMIs are constructive since the new variables parameterize a system of linear equations which uniquely can be solved for the controller matrices. With $Y := X^{-1}$ and

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}, \\ X_i, Y_i \in I\!R^{n_x \times n_x}. \tag{15}$$

non-singular transformation matrices

$$\Pi_1 := \begin{bmatrix} Y_1 & I \\ Y_3 & 0 \end{bmatrix}, \quad \Pi_2 := \begin{bmatrix} I & X_1 \\ 0 & X_3 \end{bmatrix} \tag{16}$$

can be defined such that $X\Pi_1 = \Pi_2$ holds true. Since Π_1 is non-singular, a non-singular congruence transformation

$$\Pi_1^{\mathrm{T}} E_{cl}^{\mathrm{T}} X \Pi_1 = \Pi_1^{\mathrm{T}} X^{\mathrm{T}} E_{cl} \Pi_1 \geq 0 \tag{17}$$

$$\Psi_{\Pi_1}^{\mathrm{T}} \begin{bmatrix} A_{cl}^{\mathrm{T}} X + X A_{cl} & X^{\mathrm{T}} B_{cl} & C_{cl}^{\mathrm{T}} \\ B_{cl}^{\mathrm{T}} X & -\gamma I & 0 \\ C_{cl} & 0 & -\gamma I \end{bmatrix} \Psi_{\Pi_1} < 0 \\ \text{with } \Psi_{\Pi_1} := \operatorname{diag}(\Pi_1, I, I) \tag{18}$$

of (13), (14) is possible. The matrix inequality (18) together with the linearizing changes of variables

$$\begin{aligned} \hat{D}_K &:= D_K \\ \hat{C}_K &:= C_K Y_3 + D_K C_2 Y_1 \\ \hat{B}_K &:= X_3^{\mathrm{T}} B_K + X_1^{\mathrm{T}} B_2 D_K \\ \hat{A}_K &:= X_1^{\mathrm{T}} (A + B_2 D_K C_2) Y_1 + X_3^{\mathrm{T}} A_K Y_3 + \\ &\quad + X_3^{\mathrm{T}} B_K C_2 Y_1 + X_1^{\mathrm{T}} B_2 C_K Y_3 \end{aligned} \tag{19}$$

leads to (7). Inequality (17) becomes

$$\begin{bmatrix} E & 0 \\ 0 & E^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} R & E^{+} \\ E^{\mathrm{T}+} & S \end{bmatrix} \begin{bmatrix} E^{\mathrm{T}} & 0 \\ 0 & E \end{bmatrix} \geq 0, \tag{20}$$

with $R > 0$, $S > 0$. The strict inequality in (8) can be ensured by means of the degrees of freedom in R, S (see Remark 2).
To show sufficiency an inversion of the congruence transformation (17), (18) has to be established. More precisely the validity of $X\Pi_1 = \Pi_2$ with non-singular matrices Π_1, Π_2 as in (16) has to be shown. Some lengthy calculations show that this condition always can be established if

$$X_1 Y_1 + X_2 Y_3 = I \tag{21}$$

$$X_3 Y_1 + X_4 Y_3 = 0 \tag{22}$$

hold true with non-singular matrices X_3, Y_3 (these equations correspond to the block matrices of X, Y in (15) together with the symmetry constraints

$$E^{\mathrm{T}} X_2 = X_3^{\mathrm{T}} E, \;\; E Y_2 = Y_3^{\mathrm{T}} E^{\mathrm{T}}, \;\; E^{\mathrm{T}} X_4 = X_4^{\mathrm{T}} E. \tag{23}$$

A detailed analysis shows that (21), (22) always can be established provided the synthesis LMIs (7), (8) admit an solution. □
The preceding (conceptual) proof is constructive: with a solution of the LMIs (7), (8) it is possible to establish (21), (22) by simple factorization techniques. Then the linear equations (19) can be solved for the controller matrices D_K, C_K B_K, A_K.

4. Descriptor Control of a Binary Distillation Column

We consider separation of a binary mixture in a 40 tray distillation column with one feed stream. A schematic representation of the process is given in Fig. 1 (a). Exemplary we consider the separation of two alcohols (Methanol,n-Propanol). The mixture is fed in the column with the feed flow rate F. Feed flow rate F and feed composition x_F (molar fraction) are determined by upstream processes. The stationary feed flow rate and feed composition are corrupted by disturbances. The feed stream separates the column into rectifying- (upper part of the column) and stripping section (lower part of the column). Separation is achieved due to intensive heat and mass transfer between liquid flow and countercurrently rising vapor flow. At the bottom of the column the liquid flow splits up into a liquid product stream which is removed with flow rate B from the column and

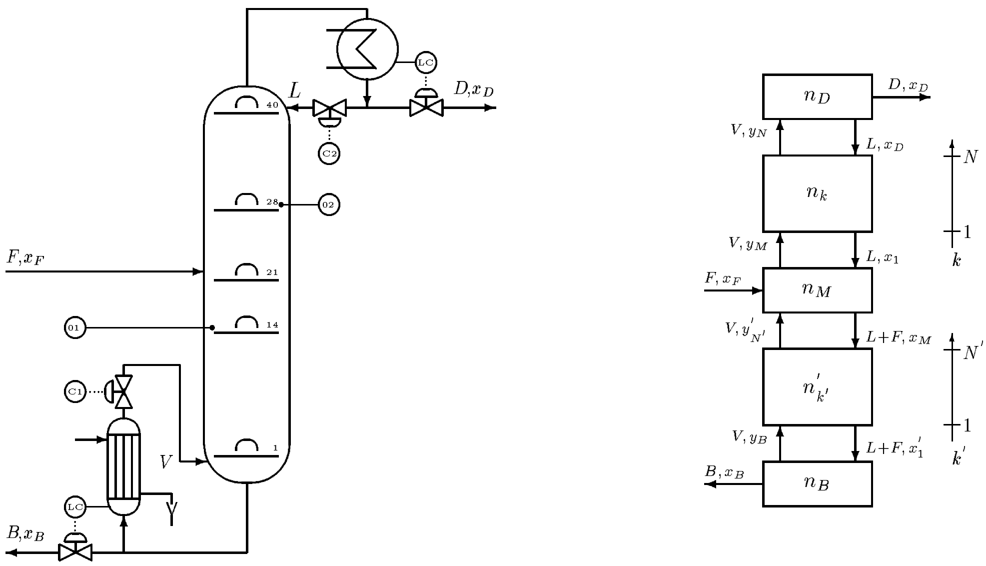

Figure 1: (a) Distillation column (scheme) (b) Subsystems of the column

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} \frac{d\Delta x_B}{dt} \\ \frac{d\Delta s_r}{dt} \\ \frac{d\Delta x_M}{dt} \\ \frac{d\Delta s_s}{dt} \\ \frac{d\Delta x_D}{dt} \end{bmatrix} = \begin{bmatrix} * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{bmatrix} \begin{bmatrix} \Delta x_B \\ \Delta s_r \\ \Delta x_M \\ \Delta s_s \\ \Delta x_D \end{bmatrix} + \begin{bmatrix} 0 & * \\ 0 & * \\ * & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_F \\ \Delta F \end{bmatrix} + \begin{bmatrix} * & * \\ * & * \\ * & * \\ * & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \Delta L \\ \Delta V \end{bmatrix}$$

$$\begin{bmatrix} \Delta x_{14} \\ \Delta x_{28} \end{bmatrix} = \begin{bmatrix} * & * & * & 0 & 0 \\ 0 & 0 & * & * & * \end{bmatrix} \left[\Delta x_B, \Delta s_r, \Delta x_M, \Delta s_s, \Delta x_D\right]^{\mathrm{T}} \tag{24}$$

a stream which is, after being heated in the reboiler, recirculated back to the column as vapor flow with flow rate V. At the top of the column the vapor flow with the accumulated more volatile product is completely condensed in the condenser. The condensate is partly pumped back in the column with a flow rate L (reflux stream) and is partly removed as the distillate product with a flow rate D [2]. We consider the distillation column in "LV" configuration, that is: liquid flow rate L and vapor flow rate V are considered to be control inputs. Measured variables are the concentrations on trays 14 and 28.

The control objective is to stabilize the product concentrations at the top and bottom of the column at their stationary values. The control relevant dynamics of the process can be captured by a reduced model of the distillation column [8]. This model assumes that the concentrations of the lighter component (molar fractions, denoted by x in the following) in the rectifying and stripping section can be described by the movement of a concentration profile. A descriptor model with concentration x_B in the reboiler, position of profile s_r in the rectifying section, concentration x_M for the feed tray, position of profile s_s in the stripping section, and concentration x_D in the condenser as descriptor variables is given in (24). Here "$*$" denotes numerical entries. A detailed derivation of the model and numerical values are given in [7].

4.1. S/KS Mixed Sensitivity Problem Setup

The control problem is solved in terms of a mixed sensitivity problem depicted in Fig. 2 with G representing the plant, K the controller, and W_1, W_2, V frequency dependent weighting matrices. Controller design by "loop shaping" requires a selection of the weighting matrices such that the solution of the H_∞ control problem

$$\left\| \begin{matrix} W_1(I+GK)^{-1}V \\ -W_2K(I+GK)^{-1}V \end{matrix} \right\|_\infty \overset{!}{<} \gamma \tag{25}$$

results in a well behaved closed loop system. In this setup V can be interpreted as a filter

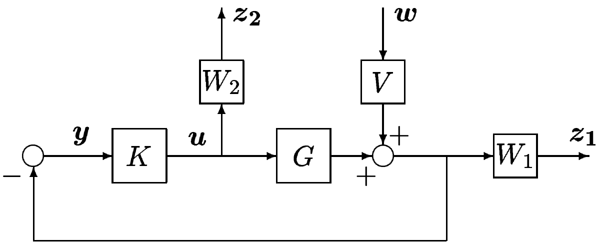

Figure 2: A mixed sensitivity configuration

which models the disturbance considered to be relevant for the problem at hand. With $S(s) := (I + GK)^{-1}$ being the sensitivity matrix of the closed loop the expression (25) with $\gamma = 1$ suggests to choose W_1 to be approximately the inverse of the wanted behavior for $S(s)$ and analogously W_2 to be the inverse of $K \cdot S$. General indications on selecting these weighting matrices can be found in [12].

In case of the distillation control problem at hand an indirect approach is taken: with stabilizing the measured concentrations x_{14}, x_{28} also the stationary profiles are fixed and thus approximately also the product concentrations. In order to realize this idea the descriptor S/KS H_∞ control problem depicted in Figure 2 (with G being the descriptor model (24)) is solved by the outlined descriptor H_∞ synthesis procedure. The synthesis LMIs are jointly optimized with respect to γ. A final value of $\gamma = 1.01$ shows that the control objectives are approximately met. The resulting controller is tested in simulation with a first principles model of the distillation process and shows a good control performance even for large input disturbances.

5. Conclusions

We presented a constructive solution to the descriptor H_∞ control problem. Synthesis conditions are given as numerically feasible strict LMI conditions. The resulting controller computation is successfully applied to a realistic control problem from chemical process control. To our knowledge this is in fact the first application of descriptor H_∞ control to a control problem with real physical background. Future work is concerned with robustness considerations in case of descriptor models of the distillation column with norm bounded uncertainties.

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LOWER LIMIT ON CONTROLLER GAIN FOR ACCEPTABLE DISTURBANCE REJECTION

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Abstract: The objective of almost all controller tuning rules found in the literature, going back to the classic PID rules of Ziegler an Nichols (1942), is to get the "fastest" possible closed-loop response, subject to maintaining stability with reasonable robustness margins. This gives a maximum limit on the controller gain. In practice, however, we often want control to be as smooth and "slow" as possible, subject to satisfying some minimum performance requirements. This gives a minimum limit on the controller gain, and the goal of this paper is to derive this minimum limit, when the performance requirements is to achieve a specified level of disturbance rejection. Together with the more traditional tunings rules this results in a *range* for the acceptable controller gain.

Keywords: Process control, PID tuning, averaging control, measurement noise

1. INTRODUCTION

The objective of almost all PID tuning rules found in the literature, e.g., (Ziegler and Nichols, 1942) (Cohen and Coon, 1953) (Astrom and Hagglund, 1995), (Rivera *et al.*, 1986), is to get the "fastest" possible closed-loop response, subject to maintaining stability with reasonable robustness margins. The model-based direct synthesis approaches of Rivera *et al.* (1986) and Smith and Corripio (1985) contain the closed-loop time constant τ_c as a tuning parameter, but also in these works the emphasis is to obtain a lower bound on τ_c (fast response). To obtain stability and robustness, the value of τ_c is limited by the effective time delay θ of the process, and typically a value $\tau_c = \theta$ is selected (Skogestad, 2003). For processes with a small effective delay this may lead to an unnecessary fast response, and a larger value of τ_c (slower response) should be used. However, the response cannot be too slow, because otherwise we do not achieve acceptable performance. In this paper we assume that the main performance specification is that the disturbance effect on the output should be bounded.

In summary, the goal of this paper is the to derive conditions for the "slowest possible" response (upper bound on closed-loop time constant τ_c; lower bound on controller gain K_c), subject to achieving acceptable disturbance rejection. There has been work along these lines in the literature on controllability analysis and decentralized control (Hovd and Skogestad, 1992) (Hovd and Skogestad, 1994) (Skogestad and Postlethwaite, 1996), but the implications of these results on controller tuning have not been considered.

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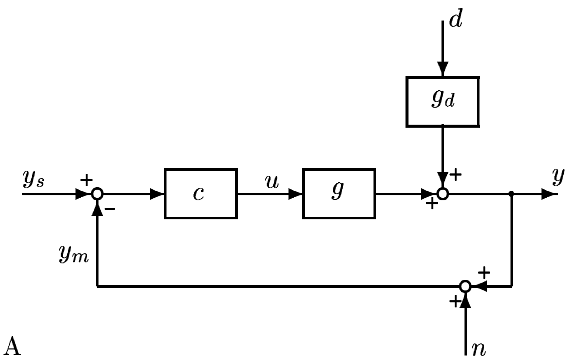

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Fig. 1. Block diagram of feedback control system.

2. DERIVATION OF LOWER LIMIT ON CONTROLLER GAIN

The linear transfer function model in deviation variables is written (Figure 1)

$$y = g(s)u + g_d(s)d \tag{1}$$

where u is the manipulated input (controller output), d the disturbance, y the controlled output, $g(s)$ the process transfer function, and $g_d(s)$ the disturbance transfer function model. The Laplace variable s is often omitted to simplify notation.

With feedback control we have $u = c(s)(y_s - y)$, where $c(s)$ is the feedback controller and we in the following do not consider setpoint changes, i.e. $y_s = 0$. The effect of the disturbance d on the control output y under closed-loop control is then

$$y = \frac{g_d(s)}{1+L(s)}d = S(s)g_d(s) \cdot d \tag{2}$$

where $S = 1/(1+L)$ is the sensitivity function and $L(s) = g(s)c(s)$ is the loop gain.

We consider the following performance requirement:

- The (steady-state) output variation y in (2) should be less than $|y_{max}|$ in response to any sinusoidal disturbance of magnitude $|d_0|$.

For simplicity we assume that the values of $|y_{max}|$ and $|d_0|$ are constant, independent of frequency. From (2) the performance requirement $|y(j\omega)| \leq y_{max}$ then gives

$$|S(j\omega)| \cdot |g_d(j\omega)| \cdot |d_0| \leq |y_{max}|$$

or equivalently

$$|1 + L(j\omega)| \geq \frac{|g_d(j\omega)| \cdot |d_0|}{|y_{max}|} = |G_d(j\omega)| \tag{3}$$

where we have introduced the scaled disturbance gain

$$G_d \overset{\text{def}}{=} g_d \cdot \frac{|d_0|}{|y_{max}|} \tag{4}$$

The requirement (3) is illustrated in Figure 2.

We define the bandwidth ω_B as the frequency where $|S| = 1/|1+L|$ first crosses 1 from below, and ω_d as the highest frequency where $|G_d|$ crosses 1, i.e. $|G_d(j\omega_d)| = 1$. From (3) and Figure 2 we must require $\omega_B \geq \omega_d$, that is, ω_d provides a lower limit on the closed-loop bandwidth for acceptable disturbance rejection.

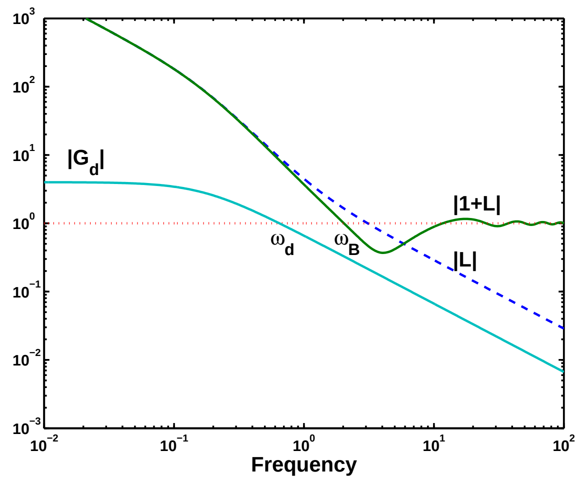

Fig. 2. Performance requirement $|1 + L| \geq |G_d|$ (3) is satisfied at all frequencies.
Data: $g_d = g = 4\frac{e^{-0.25s}}{6s+1}$, $|y_{max}| = 1, |d_0| = 1$, PI-control with $K_c = 4.313$ and $\tau_I = 0.82$.

At low frequencies $\omega \leq \omega_B$ [rad/s], within the closed-loop bandwidth, we have $|L| \gg 1$ and (3) gives $|L| \geq |G_d|$, which gives the following lower limit on the frequency-dependent controller gain for acceptable disturbance rejection

$$|c(j\omega)| \geq \frac{|g_d(j\omega)| \cdot |d_0|}{|g(j\omega)| \cdot |y_{max}|}; \quad \omega < \omega_B \tag{5}$$

which may be rewritten as

$$|c(j\omega)| \geq \frac{|u_0(j\omega)|}{|y_{max}|}; \quad \omega < \omega_B \tag{6}$$

where $|u_0(j\omega)| \overset{\text{def}}{=} \frac{|g_d(j\omega)\cdot|d_0|}{|g(j\omega)|}$ is the magnitude of the input change needed to reject the disturbances at frequencies where $|L| \gg 1$. This interpretation follows since at low frequencies $y \approx 0$ and from (1) the required input to reject the disturbance is $u_0 = -(g_d/g)d_0$. From (6) we derive the following useful rule at lower frequencies where control is effective:

- The minimum controller gain at a given frequency is approximately equal to input change required for disturbance rejection divided by the allowed output variation.

As expected, tight control (with $|y_{max}|$ small) requires a large controller gain $|c|$, as does a large disturbances (with $|u_0|$ large).

2.1 Load disturbance

For the special (and very common) case of an input (load) disturbance ($g_d = g$) the required input change equals the disturbance magnitude, $|u_0| = |d_0|$, and the bound (5) becomes

$$\text{Load disturbance}: |c(j\omega)| \geq \frac{|d_0|}{|y_{max}|};\ \omega < \omega_B \quad (7)$$

where $|d_0|$ is the magnitude of the input (load) disturbance. This bound is illustrated in Figure 3 for a PI- and PID-controller.

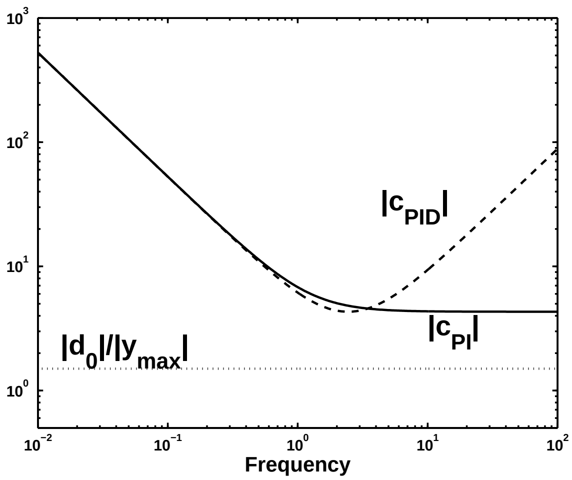

Fig. 3. Controller gain $|c|$ as a function of frequency for PI- and PID-controller.
Data PI-controller: $K_c = 4.31, \tau_I = 0.82$; PID-controller: $K_c = 4.31, \tau_I = 0.82, \tau_D = 0.20$.

Both for a PI-controller and for a PID-controller [2]

$$c_{PIDS}(s) = K_c\left(1 + \frac{1}{\tau_I s} + \tau_D s\right) \quad (8)$$

the minimum value of the controller gain $|c(j\omega)|$ as a function of frequency is always equal to K_c (independent of the values of τ_I and τ_D) (see also Figure 3):

$$\min_w |c_{PID}(j\omega)| = K_c$$

For a well-tuned PI- and PID-controller, ω_B is about at the frequency where the controller gain reaches it minimum, and from (7) we then get the following bound in order to achieve acceptable disturbance rejection with PI- and PID-control:

$$\text{Load disturbance}: \quad K_c \geq \frac{|d_0|}{|y_{max}|} \quad (9)$$

For PID tuning rules that are parameterized in terms of a single tuning parameter, like IMC-PID(Rivera *et al.*, 1986) or SIMC-PID(Skogestad, 2003), we can from the value of K_c obtain the tuning parameter (e.g. τ_c) and from this obtain the remaining controller parameters (τ_I and τ_D). For example, the SIMC PI-tunings(Skogestad, 2003) for a first-order delay process

$$g(s) = k\,\frac{e^{-\theta s}}{\tau_1 s + 1} \quad (10)$$

are

$$K_c = \frac{1}{k} \cdot \frac{\tau_1}{\tau_c + \theta} \quad (11)$$

$$\tau_I = \min\left(\tau_1, 4(\tau_c + \theta)\right) \quad (12)$$

and with a given value of K_c, we can obtain τ_c from (11) and then obtain τ_I from (12).

3. PI-EXAMPLE

Consider a first-order with delay process with time constant $\tau_1 = 6$ and time delay $\theta = 0.25$:

$$g(s) = 4\,\frac{e^{-0.25s}}{6s + 1} \quad (13)$$

The performance requirement is that the output deviation should stay within $\pm|y_{max}| = 1$ in response to a step input (load) disturbance of magnitude $|d_0| = 1$, which from (9) requires $K_c \geq |d_0|/|y_{max}| = 1$ (for a sinusoidal disturbance). It is also desirable that control is as smooth as possible, which means that we want K_c as small as possible.

Tuning for fast response. The "closed-loop" Ziegler-Nichols (ZN) settings for this process are

$$K_c = 4.313,\ \tau_I = 0.82 \quad (14)$$

We note that K_c is 4.3 times the minimum required value, so we expect that the output response is much better than the requirement. This is confirmed both by the frequency plot in Figure 2, as well as the time response to a unit step input disturbance in Figure 4. The output deviation in Figure 4 is less than 0.2, well below $|y_{max}| = 1$. However, because of the high controller gain, the input usage and also the output response is sensitive to measurement noise n on y (dashed line in Figure 4).

Tuning for smooth response. The above response is unnecessary fast so the controller gain may be reduced. We choose $K_c = |d_0|/|y_{max}| = 1$. With $k = 4, \tau_1 = 6, \theta = 2.5$ we get from (11) that $K_c = 1$ corresponds to $\tau_c = 1.25$, and from (12) we obtain the following SIMC-settings:

$$K_c = 1; \quad \tau_I = 6 \quad (15)$$

[2] In this paper we consider the "ideal" PID controller in (8) and the ZN-settings are assumed to be given for this form.

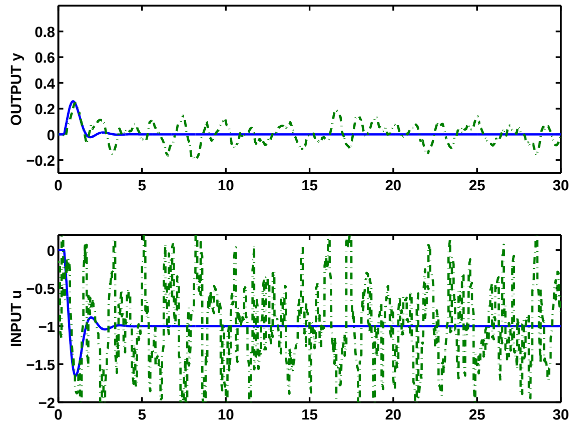

Fig. 4. Response to step load disturbance with "fast" ZN-PI controller (14).
Dashed line: With measurement noise. Solid line: No noise

The corresponding disturbance response in Figure 5 has a maximum output deviation of about 0.7, which is below $|y_{max}| = 1$, and input usage is smooth with no sensitivity to noise. Thus, this tuning is preferred in practice.

Remark: We may reduce K_c further below 1 and still achieve an output deviation less than $y_{max} = 1$. The reason why (9) is not tight in this case, is mainly that the expression is derived for a sinusoidal disturbance whereas we here consider a step disturbance.

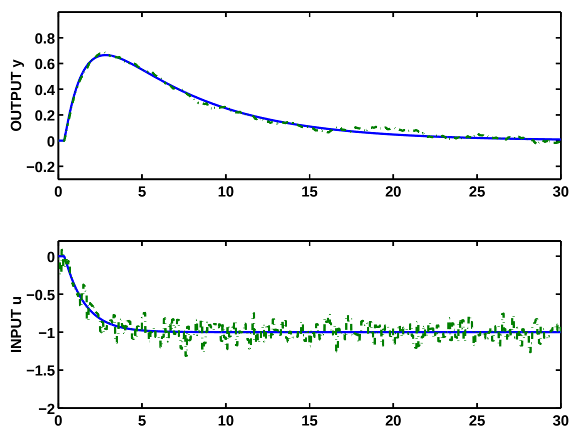

Fig. 5. Response to step load disturbance with "smooth" SIMC PI controller (15).
Dashed line: With measurement noise. Solid line: No noise

4. DISCUSSION

4.1 Averaging level control

A well-known case where a low controller gain is desired is for "averaging level control" where we use a tank in order to smoothen flow disturbances. Here the main control objective is to have smooth input usage (smooth flow variations), subject to the requirement of stabilizing the system and keeping the level within bounds when there are flow disturbances. In (9), $|d_0|$ is the magnitude of the flowrate change ($|\Delta q|$) and $|y_{max}|$ is the allowable level change ($|\Delta h_{max}|$). From (9) the minimum controller gain for averaging level control is

$$K_c \geq \frac{|\Delta q|}{|\Delta h_{max}|} \tag{16}$$

which agrees with the value normally recommended (e.g. (Marlin, 2000)). The process transfer function $g(s)$ from u (flowrate q) to y (level h) is close to integrating (with τ_1 in (10) very large) and can be written

$$g(s) = \frac{k'}{s} e^{-\theta s}$$

where $k' = k/\tau_1$ is the slope of the response. From the SIMC-rule for the controller gain in (11) we get $\tau_c + \theta = 1/(K_c k')$, which upon substitution into (12) gives the integral time

$$\tau_I = \frac{4}{K_c k'} \tag{17}$$

which agrees with the industrially recommended value in Fruehauf *et al.* (1994).

4.2 Controllability implications

An approximate maximum value of the controller gain is achieved by selecting the desired closed-loop response time τ_c in (11) equal to zero. This gives the "maximum" controller gain

$$K_{c,max} = \frac{1}{k}\,\frac{\tau_1}{\theta} = \frac{1}{k'\theta} \tag{18}$$

If the "maximum" controller gain in (18) is smaller than the "minimum" controller gain computed above, then the process is not controllable – at least not with PID control with reasonably robust tunings. In words, the speed of response required for disturbance rejection is faster than what can be achieved with the given time delay. For example, for a load disturbance the minimum controller gain $K_{c,min}$ is given by (9), and requiring $K_{c,max} \geq K_{c,min}$ for controllability gives an upper bound on the allowed delay

$$\theta \leq \frac{|y_{max}|}{|d_0|}\frac{\tau_1}{k}$$

The right hand side represents the minimum response time, and we note, as expected, that a small response time is required if we have a tight performance requirement ($|y_{max}|$ small), a large disturbance ($|d_0|$ large), or a "fast-acting" disturbance ($k' = k/\tau_1$ large).

4.3 Generalization to multivariable systems

The results in this paper can be directly generalized to decentralized control of multivariable systems by introducing the closed-loop disturbance gain (Hovd and Skogestad, 1992) (Hovd and Skogestad, 1994) (Skogestad and Postlethwaite, 1996).

5. CONCLUSION

The requirement of acceptable disturbance rejection (output deviation less than $|y_{max}|$ in response to a sinusoidal disturbance of magnitude $|d_0|$), results in a lower limit (5) on the controller gain. In words, the minimum controller gain at a given frequency is approximately equal to input change required for disturbance rejection divided by the allowed output variation. For a load disturbance and PI or PID control this requirement becomes $K_c \geq |d_0|/|y_{max}|$ (9).

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