MULTIVARIATE CONTROLLER PERFORMANCE ASSESSMENT WITHOUT INTERACTOR MATRIX — A SUBSPACE APPROACH

Ramesh Kadali and Biao Huang [1]

Department of Chemical and Materials Engineering, University of Alberta, Edmonton, AB, Canada T6G 2G6

Abstract: Several methods for multivariate control performance assessment (MPA) with or without using the interactor matrix have been proposed in the literature. They are all equivalent, one way or other, by certain transformations. In this paper a subspace framework for MPA is proposed for the estimation of MVC-benchmark variance for feedback multivariate systems. The merit of the new approach is that we start straight from data, and a performance index is calculated directly from subspace matrices without relying on a parametric dynamic model. In addition, a proof that the proposed solution is exactly the same as that of the conventional approaches is provided.

1. INTRODUCTION

Periodic performance assessment of the controllers is important for maintaining normal process operation and to sustain the performance of controllers achieved when the controllers are commissioned. Minimum variance control is theoretically the best possible control (Astrom and Wittenmark, 1984). Controller performance assessment using MVC-benchmark involves comparing the current process output variance with the output benchmark variance if a minimum variance controller were implemented on the process. Although the intention of many industrial controllers is not minimum variance control, MVC-benchmark is nevertheless used as a first step in the controller performance assessment (Harris, 1989). Calculation of the MVC-benchmark variance for univariate systems from routine closed loop data requires *a priori* knowledge of only the process time delay (Harris, 1989; Huang and Shah, 1999). Calculation of the MVC-benchmark variance for multivariate systems involves calculation of the interactor matrix (Huang and Shah, 1999) for the system from the first few process Markov parameters. Furthermore, the concept of the interactor is not well known in practice. Hence, estimation of the MVC-benchmark without the interactor matrix has been an active area of research.

(Ko and Edgar, 2001) proposed a method for the estimation of the multivariate MVC-benchmark using closed loop data, which does not require the intermediate interactor matrix calculation. It is shown that their result is equivalent to the result of (Huang and Shah, 1999). Recent progress in the subspace approach to closed loop identification (Kadali and Huang, 2002) inspires an alternative approach for the estimation of multivariate MVC-benchmark. In the current paper we will show the estimation of the multivariate MVC-benchmark with *neither* the interactor matrix calculation *nor* the Markov parameters. The only *a priori* knowledge required is the deterministic subspace matrix directly calculated from data. The important difference between the "calculation of the subspace

[1] The author to whom all correspondence should be addressed. Phone: (780)492-9016, FAX: (780)492-2881. E-mail:bhuang@ualberta.ca

matrix" and subspace identification is that the former does not extract an explicit "model" and is also known as model-free approach in the literature. This will further simplify the procedure for the calculation of the multivariate performance index. No concepts such as interactor matrix, Markov parameter, multivariate transfer function matrix, state space model etc. are needed to apply this technique and this will make the multivariate controller performance assessment technique more applicable in practice. (McNabb and Qin, 2001) have also proposed another subspace approach to multivariate performance monitoring by projecting delay-matrix filtered output data onto past data.

In the subspace method, certain subspace matrices are identified as a first step in the subspace identification methods. The minimum variance controller can be designed directly using these intermediate subspace matrices, without a parametric model such as the state space model or transfer function model. The closed loop subspace identification method proposed in (Kadali and Huang, 2002) allows a convenient identification of subspace matrices from the closed loop data with external excitations. The MVC-benchmark variance can be calculated with the knowledge of only the deterministic subspace matrix and eliminates the intermediate step of estimating the unitary interactor matrix or extracting the Markov parameters.

We do not claim that the subspace approach as proposed in this paper requires less a priori knowledge than other methods. In fact, the equivalent information of the interactor matrix or Markov parameters is implicitly buried in the subspace matrices. However, avoiding direct use of the interactor matrix and/or Markov parameter matrices does have an advantage of easier acceptance by practitioners and reduces unnecessary intermediate modeling step. Another merit of this paper lies in the direct data based approach, i.e. from process experiment data, a multivariate performance index is directly calculated. Comparing with the conventional methods such as that proposed by (Huang and Shah, 1999), our method is different in the sense of subspace approach versus conventional transfer function approach. The method proposed by (Ko and Edgar, 2001), even though without using the interactor matrix, is nevertheless following the transfer function matrix approach and is an extension of (Harris *et al.*, 1996; Huang and Shah, 1999). Our approach, which may be considered an extension to (Ko and Edgar, 2001), adopts the subspace framework, and further avoids the use of the transfer function matrix and Markov parameters.

2. SUBSPACE MATRICES DESCRIPTION

Consider the following innovations state space representation of a linear time-invariant system with l-inputs (u_k), m-outputs (y_k) and n-states (x_k) as:

$$x_{k+1} = Ax_k + Bu_k + Ke_k \tag{1}$$

$$y_k = Cx_k + e_k \tag{2}$$

where the state space system matrices A, B, C and K^f are $(n \times n)$, $(n \times l)$, $(m \times n)$ and $(n \times m)$ matrices respectively. K^f is the Kalman filter gain and e_k is an unknown innovation sequence.

The matrix input-output equations used in subspace identification (Overschee and Moor, 1994; Overschee and Moor, 1995; Overschee and Moor, 1996) expressed using certain subspace matrices L_w, L_u and L_e (Overschee and Moor, 1996) as

$$y_f = L_w w_p + L_u u_f + L_e e_f \tag{3}$$

where

$$y_f = \begin{bmatrix} y_{t+1} \\ \cdots \\ y_{t+N} \end{bmatrix}; y_p = \begin{bmatrix} y_{t-N+1} \\ \cdots \\ y_t \end{bmatrix}; e_f = \begin{bmatrix} e_{t+1} \\ \cdots \\ e_{t+N} \end{bmatrix};$$

$$u_f = \begin{bmatrix} u_{t+1} \\ \cdots \\ u_{t+N} \end{bmatrix}; u_p = \begin{bmatrix} u_{t-N+1} \\ \cdots \\ u_t \end{bmatrix}; w_p = \begin{bmatrix} y_p \\ u_p \end{bmatrix}$$

The subspace matrices are estimated as an intermediate step by data projections (Overschee and Moor, 1996). L_u and L_e are dynamic matrices containing the estimated Markov parameters corresponding to the process and noise respectively.

Recent results in subspace closed-loop identification (Kadali and Huang, 2002) allow the direct estimation of two of the subspace matrices, L_u and L_e, from the closed loop data with set point excitation. Note that although the deterministic subspace matrix and closed loop noise matrix contain process Markov parameters and noise Markov parameters respectively, the two matrices are directly calculated from closed-loop data by a projection method and one never needs to know what are inside these two matrices in order to apply our algorithms. The only reason to mention Markov parameters here and in the sequel is to analytically compare our results with conventional results available in the literature.

3. DESIGN OF MINIMUM VARIANCE CONTROL USING SUBSPACE MATRICES

The minimum variance controller (MVC) is designed to minimize the following quadratic cost function J over the horizon N, as $N \longrightarrow \infty$:

$$J = \frac{1}{N} E\{ \sum_{k=1}^{N} [(r_{t+k} - y_{t+k})^T \ (r_{t+k} - y_{t+k})]\} \quad (4)$$

$$= \frac{1}{N} \sum_{k=1}^{N} [(r_{t+k} - \hat{y}_{t+k})^T \ (r_{t+k} - \hat{y}_{t+k})] \quad (5)$$

where E is the expectancy operator, r_t is the reference for output trajectory. $\hat{y}_{t+k}$ is the k-step ahead predicted output given the past inputs and outputs upto time t.

Using equation (3), the optimal predictor equation can be written as $\hat{y}_f = L_w w_p + L_u u_f$. The notation in the cost function can be simplified for regulatory control, by letting $r_{t+k} = 0$ as:

$$J = \min_{u_f^2} \ \frac{1}{N} [\hat{y}_f^T \ \hat{y}_f]$$

$$= \min_{u_f^2} \frac{1}{N} [(L_w w_p + L_u u_f)^T (L_w w_p + L_u u_f)] \quad (6)$$

To obtain the minimum variance control law, we differentiate J with respect to u_f and set it to zero to obtain the control law as:

$$u_f = -L_u^{\dagger} (L_w w_p) \quad (7)$$

where, † represents pseudo-inverse. The above control law is the minimum variance control law as the number of block-rows in the subspace matrices L_w and L_u tend to infinity.

4. ESTIMATION OF THE MULTIVARIATE MVC-BENCHMARK

From the very first block-element of Y_f in equation (3) we can write

$$y_{t+1} = L_{y_p} \begin{bmatrix} y_{t-N+1} \\ ... \\ y_t \end{bmatrix} + L_{u_p} \begin{bmatrix} u_{t-N+1} \\ ... \\ u_t \end{bmatrix} + L_0 e_{t+1} \quad (8)$$

where $L_{y_p} = L_w(1:m, 1:mN)$ and $L_{u_p} = L_w(1:m, mN+1:(l+m)N)$. Equation (8) can be transformed for an equivalent expression of y_{t+1} in terms of the past inputs and the past noise as

$$y_{t+1} = \left[G_1 \ ... \ G_N \right] \begin{bmatrix} u_t \\ ... \\ u_{t-N+1} \end{bmatrix} + \left[L_1 \ ... \ L_N \right] \begin{bmatrix} e_t \\ ... \\ e_{t-N+1} \end{bmatrix} + L_0 e_{t+1} \quad (9)$$

where G_i and L_i are the i-th impulse response coefficients (Markov parameters for multivariate systems) of the process and noise models respectively. In other words, we can express the past (state) contribution term, $L_w w_p$, as

$$L_w w_p = \begin{bmatrix} G_1 & ... & G_{N-1} & G_N \\ G_2 & ... & G_N & 0 \\ ... & ... & ... & ... \\ G_N & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} u_t \\ ... \\ u_{t-N+1} \end{bmatrix} + \begin{bmatrix} L_1 & ... & L_{N-1} & L_N \\ L_2 & ... & L_N & 0 \\ ... & ... & ... & ... \\ L_N & 0 & ... & 0 \end{bmatrix} \begin{bmatrix} e_t \\ ... \\ e_{t-N+1} \end{bmatrix} \quad (10)$$

However, the controller output, u_{t+1} is calculated using all the data available at time '$t+1$', i.e., $\{u_t, y_{t+1}, u_{t-1}, y_t, ...\}$. Hence the original subspace predictor expression in equation (3) and the subspace based minimum variance control law in equation (7) have to be modified to obtain the closed loop expressions for u_f and y_f. First, define

$$L_G = \begin{bmatrix} G_1 & G_2 & ... & G_{N-1} & G_N \\ G_2 & G_3 & ... & G_N & 0 \\ ... & ... & ... & ... & \\ G_N & 0 & 0 & ... & 0 \end{bmatrix}; \tilde{u}_p = \begin{bmatrix} u_t \\ u_{t-1} \\ ... \\ u_{t-N+1} \end{bmatrix}$$

$$L_H = \begin{bmatrix} L_0 & L_1 & ... & L_{N-1} & L_N \\ L_1 & L_2 & ... & L_N & 0 \\ ... & ... & ... & ... & ... \\ L_{N-1} & 0 & 0 & ... & 0 \end{bmatrix}; \tilde{e}_p = \begin{bmatrix} e_{t+1} \\ e_t \\ ... \\ e_{t-N+1} \end{bmatrix}$$

$$\tilde{L}_e = \begin{bmatrix} 0 & 0 & ... & 0 \\ L_0 & 0 & ... & 0 \\ ... & ... & ... & ... \\ L_{N-2} & L_{N-3} & ... & 0 \end{bmatrix}; \tilde{e}_f = \begin{bmatrix} e_{t+2} \\ e_{t+3} \\ ... \\ e_{t+N+1} \end{bmatrix}$$

Since L_G and L_H contain the process and noise model Markov parameters, they can be formed from the subspace matrices L_u and L_e respectively. Therefore the equation based on the first column of Y_f in equation (3) can be alternatively written as

$$y_f = L_G \tilde{u}_p + L_H \tilde{e}_p + L_u u_f + \tilde{L}_e \tilde{e}_f \quad (11)$$

Substituting equation (10) in equation (7), we can write

$$u_f = -L_u^{\dagger} \ \{L_w w_p\} = -L_u^{\dagger} \{L_G \tilde{u}_p + L_H \tilde{e}_p\} \quad (12)$$

The closed loop expression for y_f can be written as

$$y_f = (I - L_u L_u^{\dagger}) \ (L_G \tilde{u}_p + L_H \tilde{e}_p) + \tilde{L}_e \tilde{e}_f \quad (13)$$

Now that we have derived closed-loop expressions for both u and y, the next step is to calculate their variance expressions which are actually the H_2 norm of the closed-loop expressions weighted by the variance of e. A simple method to derive the variance expression is given below.

Let a disturbance enter the process at time $= t+1$, i.e., $u_t = u_{t-1} = ... = u_{t-N+1} = 0$; $e_t = e_{t-1} = ... = e_{t-N+1} = 0$; and $e_{t+2} = e_{t+3} = ... = e_{t+N} = 0$. Then the cumulative effect of the noise e_{t+1} on the process output variance can be obtained from equation (13), which simplifies to

$$y_f = (I - L_u L_u^{\dagger}) L_h \;\; e_{t+1} \;\; = \begin{bmatrix} \psi_0 \\ \psi_1 \\ ... \end{bmatrix} e_{t+1} \quad (14)$$

where $L_h = \begin{bmatrix} L_0 \\ ... \\ L_{N-1} \end{bmatrix}$, the vector of noise model Markov parameters, and ψ_i represents the Markov parameter of i-th lag of the closed loop noise model if a minimum variance controller were implemented on the system described in equations (1)-(2). The variance of the closed-loop system can be calculated from the Markov parameters/impulse response of the closed-loop system and the minimum variance control variance expression for the process output is given by

$$var[y_t]_{MVC} = \sum_{i=0}^{\infty} \psi_i var[e_t] \psi_i^T \quad (15)$$

Note that estimation of the interactor matrix is *not* required for obtaining the MVC-benchmark variance. However the above result requires the knowledge of $L_h = \begin{bmatrix} L_0 \\ ... \\ L_{N-1} \end{bmatrix}$, and hence it appears that estimation of the noise model in the Markov parameters model is necessary.

However, we will show that the estimation of $\begin{bmatrix} L_0 \\ ... \\ L_{N-1} \end{bmatrix}$ is *not* required. The closed loop noise model Markov parameters $L_h^{CL} = \begin{bmatrix} L_0^{CL} \\ ... \\ L_{N-1}^{CL} \end{bmatrix}$ (the vector of closed-loop noise model which can be estimated from the routine operating data) can be used in the place of $\begin{bmatrix} L_0 \\ ... \\ L_{N-1} \end{bmatrix}$ and we can still be able to obtain the MVC-benchmark variance, where

$$L_h^{CL} = (I + L_u L_c)^{-1} L_h \quad (16)$$

and L_c represents the dynamic matrix containing the Markov parameters of the controller.

Lemma 1: *Ψ can be obtained using the vector of Markov parameters of the closed loop noise model, L_h^{CL}, in place of the L_h in equation (14) .*

Proof: The above statement is equivalent to saying that $(I - L_u L_u^{\dagger}) L_h$ and $(I - L_u L_u^{\dagger}) L_h^{CL}$ yield the same result. Now, $(I - L_u L_u^{\dagger}) L_h^{CL} = (I - L_u L_u^{\dagger})(I + L_u L_c)^{-1} L_h$. Therefore on observation, we need to show that

$$(I - L_u L_u^{\dagger}) = (I - L_u L_u^{\dagger})(I + L_u L_c)^{-1} \quad (17)$$

to prove the lemma, which is equivalent to showing

$$(I - L_u L_u^{\dagger})(I + L_u L_c) = (I - L_u L_u^{\dagger}) \quad (18)$$

Expanding the left hand side term in the above equation

$$(I - L_u L_u^{\dagger}) \;\; (I + L_u L_c) = I - L_u L_u^{\dagger} \quad (19)$$

The last equation follows since $L_u L_u^{\dagger} L_u = L_u$.

Lemma 1 is essentially the subspace version of the invariance property of the first few Markov parameters of the interactor-filtered noise model under the transfer function framework originally derived in Huang and Shah (1999)(Huang *et al.*, 1997). This invariance property has also been proved in (Ko and Edgar, 2001).

Hence the Markov parameters of the closed loop noise model can be used in place of Markov parameters of the open loop noise model and we can still get the same benchmark variance. Therefore, we need the subspace matrix L_u (which contains Markov parameters of the process and is estimated from data) for the calculation of the minimum variance control benchmark. The subspace matrix L_u can be estimated from closed loop data with set point excitation as explained in (Kadali and Huang, 2002). The Markov parameters of the closed loop noise model (or noise subspace matrix) can be easily estimated from the routine operating data (Kadali and Huang, 2002).

5. EQUIVALENCE OF SUBSPACE APPROACH AND THE CONVENTIONAL TRANSFER FUNCTION APPROACH IN OBTAINING THE MVC-BENCHMARK

In the transfer function approach,

$$D(z) = \left[D_d z^d + ... + D_1 z \right]$$

represents the interactor matrix for a process represented by the transfer function matrix $G(z^{-1}) = \left[G_0 + G_1 z^{-1} + G_2 z^{-2} + ... \right]$, then the condition for the interactor matrix from theorem 3.2.1 in (Huang and Shah, 1999) is

$$\lim_{z^{-1} \longrightarrow 0} DG = K \quad (20)$$

where K is a full rank matrix. The above expression can be alternatively expressed as two matrix conditions

Condition-1

$$\left[\, D_1 \;...\; D_d \,\right] \begin{bmatrix} G_0 & ... & 0 \\ ... & ... & 0 \\ G_{d-1} & ... & G_0 \end{bmatrix} = \left[\, 0 \;...\; 0 \,\right] \quad (21)$$

Condition-2

$$\left[\, D_1 \;...\; D_d \,\right] \begin{bmatrix} G_1 \\ ... \\ G_d \end{bmatrix} = K \quad (22)$$

with $rank(K) = min\{m, l\}$.

We need to show that the coefficients obtained in the subspace approach are same as those obtained in the transfer function domain approach, i.e. the above two conditions are satisfied by using the matrix $\left(I - L_u \;\; L_u^{\dagger}\right)$. Therefore we have to prove the following theorem for the subspace approach:

Theorem 1: $\left(I - L_u \;\; L_u^{\dagger}\right)$ *contains interactor matrix for the process. An interactor matrix can be constructed directly from this expression. The subspace approach for the calculation of the minimum variance control benchmark is equivalent to that of the conventional transfer function approach.*

Proof:

$$\left(I - L_u \;\; L_u^{\dagger}\right) \begin{bmatrix} G_0 & 0 & ... & 0 \\ G_1 & G_0 & 0 & ... \\ ... & ... & ... & ... \end{bmatrix} = \left(I - L_u \;\; L_u^{\dagger}\right) L_u$$
$$= L_u - L_u \;\; L_u^{\dagger} L_u = 0 \quad (23)$$

since $L_u \;\; L_u^{\dagger} L_u = L_u$. From the above equation condition-1 expressed in equation (21) is satisfied.

Next, consider the transformed Markov parameter matrix $\begin{bmatrix} \hat{G}_1 \\ ... \\ \hat{G}_d \end{bmatrix} = (I - L_u L_u^{\dagger}) \begin{bmatrix} G_1 \\ ... \\ G_d \end{bmatrix}$. Note that the matrices L_u and $\begin{bmatrix} G_1 \\ ... \\ G_d \end{bmatrix}$ are *essentially disjoint* (see appendix A). Following the corollary 17.2.10 in ref.(Harville, 1997) :

$$rank \begin{bmatrix} \hat{G}_1 \\ ... \\ \hat{G}_d \end{bmatrix} = rank \begin{bmatrix} G_1 \\ ... \\ G_d \end{bmatrix} \quad (24)$$

Now let

$$K = \hat{G}_1 + ... + \hat{G}_d = \left[\, I_m \;...\; I_m \,\right] \begin{bmatrix} \hat{G}_1 \\ ... \\ \hat{G}_d \end{bmatrix} \quad (25)$$

The matrix $\left[\, I_m \;...\; I_m \,\right]$ is $(m \times dm)$ dimensional with rank m. Consider $A = \left[\, I_m \;...\; I_m \,\right]_{m \times dm}$ and $B = \begin{bmatrix} \hat{G}_1 \\ ... \\ \hat{G}_d \end{bmatrix}_{dm \times l}$. Using the corollary 17.5.2 from (Harville, 1997) we can write

$$rank\{K\} = rank[A] + rank[B] - (dm)$$
$$+ rank\left[(I_{dm} - BB^{\dagger})(I_{dm} - A^{\dagger}A)\right] \quad (26)$$

We can expand

$$(I - BB^{\dagger})(I - A^{\dagger}A)$$
$$= (I - A^{\dagger}A) - B^{\dagger}B(I - A^{\dagger}A) \quad (27)$$

For using the item (3) in appendix B, we take,

$$R = (I - A^{\dagger}A); \;\; S = -B; \;\; T = B^{\dagger}; \;\; U = (I - A^{\dagger}A)$$

Using equations (26) and (B.2), we write,

$$rank[K] = rank[A] + rank[B] - dm - rank[B^{\dagger}]$$
$$+ rank[(I - A^{\dagger}A)] + rank[(AB)^{\dagger}(AB)B^{\dagger}]$$
$$= rank[(AB)^{\dagger}(AB)B^{\dagger}] \quad (28)$$

In the above equation we used $rank[B] = rank[B^{\dagger}]$ and $rank[(I - A^{\dagger}A)] = (d-1)m$. Consider the two cases,

(i) $m \geq l$: In this case $(AB)^{\dagger}(AB) = I_l$. Therefore $rank[(AB)^{\dagger}(AB)B^{\dagger}] = rank[B^{\dagger}] = l$ and $rank[K] = l$.

(ii) $m < l$: In this case $rank[(AB)^{\dagger}(AB)] = m$. Since B is a full rank matrix $rank[(AB)^{\dagger}(AB)B^{\dagger}] = m$. Therefore $rank[K] = m$.

Hence K is a full rank matrix and condition-2 expressed in equation (22) is satisfied. Hence the theorem is proved.

To summarize, the matrix $\left(I - L_u \;\; L_u^{\dagger}\right)$ performs the same function as an interactor matrix in the transfer function domain. But the calculation of interactor matrix is not required in deriving the MVC-benchmark variance of the process output for controller performance analysis. Therefore *a priori* knowledge of only the subspace matrix L_u or equivalently the first 'd' process Markov parameters is the requirement for obtaining the MVC-benchmark, the same conclusion as obtained in the previous literature, but expressed here in a much more simplified notation..

6. CONCLUSIONS

Calculation of the multivariate performance index without using the interactor matrix is an important step toward practical application of multivariate performance assessment techniques. It is shown in this paper the design of multivariate minimum variance controller can be done using subspace matrices. Using the subspace matrices the MVC-benchmark variance for the process outputs is obtained from closed loop data without having to first calculate the unitary interactor matrix or knowing the first few Markov parameters of the noise model. The equivalence of the subspace approach to the conventional transfer function approach for obtaining the MVC-benchmark variance is also proved.

REFERENCES

Astrom, K.J. and B. Wittenmark (1984). *Computer Control Systems: Theory and Design*. Prentice-Hall.

Harris, T. (1989). Assessment of closed loop performance. *Can. J. Chem. Eng.* **67**, 856–861.

Harris, T.J., F. Boudreau and J.F. MacGregor (1996). Performance assessment of multivariable feedback controllers. *Automatica* **32**(11), 1505–1518.

Harville, D.A. (1997). *Matrix Algebra From a Statistician's Perspective*. Springer-Verlag, New York.

Huang, B. and S.L. Shah (1999). *Control Loop Performance Assessment: Theory and Applications*. Springer Verlag, London.

Huang, B., S.L. Shah and K.Y. Kwok (1997). Good, bad or optimal? performance assessment of multivariable processes. *Automatica* **33**(6), 1175–1183.

Kadali, R. and B. Huang (2002). Estimation of dynamic matrix and noise model for model predictive control using closed loop data. *Industrial and Engineering Chemistry Research* **41**(4), 842–852.

Ko, B.S. and T.F. Edgar (2001). Performance assessment of multivariate feedback control systems. *Automatica* **37**(5), 899–905.

McNabb, C.A.R. and J.S. Qin (2001). MIMO control performance monitoring based on subspace projections. In: *AIChE Annual Conference, Reno*.

Overschee, P. Van and B. De Moor (1994). N4SID: Subspace algorithm for the identification of combined deterministic-stochastic systems. *Automatica* **30(1)**, 75–93.

Overschee, P. Van and B. De Moor (1995). A unifying theorem for three subspace system identification algorithms. *Automatica* **31(12)**, 1877–1883.

Overschee, P. Van and B. De Moor (1996). *Subspace Identification for Linear Systems: Theory implementation applications*. Kluwer Academic Publishers.

Appendix A. ESSENTIALLY DISJOINT CONDITION

From (Harville, 1997)
Lemma 17.1.4. Let U and V represent subspaces of $R^{m\times n}$, then

(1) U and V are essentially disjoint if and only if, for matrices $\mathbf{U}\in U$ and $\mathbf{V}\in V$, the only solution to the matrix equation

$$\mathbf{U}+\mathbf{V}=0 \tag{A.1}$$

is $\mathbf{U}=\mathbf{V}=0$; and

(2) U and V are essentially disjoint if and only if, for every non-null matrix $\mathbf{U}\in U$ and every non-null matrix $\mathbf{V}\in V$, $\mathbf{U}$ and $\mathbf{V}$ are linearly independent.

We assume that the process transfer function $G(z^{-1})$ is full rank with proper and stable transfer functions. Therefore, the matrices $L_u = \begin{bmatrix} G_0 & 0 & ... \\ G_1 & G_0 & ... \\ ... & ... & ... \end{bmatrix}$ and $\begin{bmatrix} G_1 \\ G_2 \\ ... \end{bmatrix}$ are essentially disjoint.

Appendix B. COROLLARIES

(1) Corollary 17.5.2 Let A represent an $m\times n$ matrix and B an $n\times p$ matrix. Then,

$$\begin{aligned} rank(AB) &= rank(A)+rank(B)-n \\ &+rank\left[(I-BB^{\dagger})(I-A^{\dagger}A)\right] \end{aligned} \tag{B.1}$$

(2) Corollary 17.2.10 Let A represent an $m\times n$ matrix, B an $m\times p$ matrix. Then $\text{rank}\left[(I-AA^{\dagger})B\right]=\text{rank}\,(B)$ if and only if C(A) and C(B) are essentially disjoint.

(3) From chapter 18 Let R represent an $n\times q$ matrix, S an $n\times m$ matrix, T an $m\times p$ matrix, and U a $p\times q$ matrix. Then,

$$\begin{aligned} rank(R+STU) &= rank(R)+rank(Q)+rank(M) \\ &+rank(N)-rank(T) \\ &+rank\left[(I-MM^{\dagger})XQ^{\dagger}Y(I-N^{\dagger}N)\right] \end{aligned} \tag{B.2}$$

where $E_R=I-RR^{\dagger}$; $F_R=I-R^{\dagger}R$; $X=E_RST$; $Y=TUF_R$; $M=X(I-Q^{\dagger}Q)$ and $N=(I-QQ^{\dagger})Y$. Refer to (Harville, 1997) for proofs of the above corollaries.