DISCRETE CONTROL OF NEARLY INTEGRABLE TWO-DIMENSIONAL CONTINUOUS SYSTEMS

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Abstract: Many manufacturing processes involve an interplay of logical and continuous objectives. Hybrid Control Systems are well-suited for studying interactions between continuous and logical control goals. Here, a technique for generating the interfaces of a family of Hybrid Control Systems is presented. This amounts to extracting (untimed and specification-independent) abstractions for a class of nonlinear continuous systems satisfying an integrability property. A two-dimensional example illustrates an extension of the technique to a larger class of systems.

Keywords: Nonlinear systems, geometric properties, hierarchical control, state-space methods, system analysis.

1. INTRODUCTION

A Hybrid Control System (HCS) consists of a discrete-event controller for a continuous plant as illustrated in Figure 1. The y/s_i interface converts the plant output y into a controller symbolic input s_i, and the s_o/u interface transforms the controller symbolic output s_o into the plant input u.

Two distinct problems arise with HCSs. The first problem focuses on control issues in the presence of known interfaces. The second problem amounts to developing the interfaces. Herein emphasis is put on the second problem in the presence of the following situations: (i) the process has an actuation taking discrete values ("On", "Off", etc.), and (ii) the process has specifications of the type "if tank #1 overflows, then close valve #2". Notice that such a setup still allows continuous control objectives.

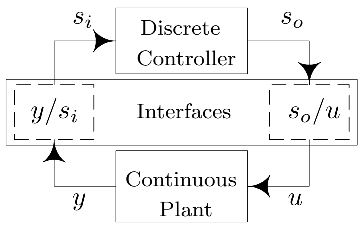

Fig. 1. Hybrid Control System

2. MOTIVATIONS

The interfaces of an HCS can be obtained by generating a finite-state machine (FSM) abstraction of the continuous system. Abstractions can be tailored to a specification (called *s-abstractions*) or independent of specifications (simply called *abstractions*). We favor abstractions because continuous control tasks are not easily expressible in terms of s-abstractions. An example of stabilization with abstractions is given in (Lunze, 1995).

The abstraction technique presented in (Lunze *et al.*, 1999) applies to linear systems. The approach presented in (Raisch, 2000) requires discrete-time

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(linear or linearized) models. Since linearization can destroy controllability (Isidori, 1995), abstractions based on a linearized model may possess fewer control capabilities than the original system. Among the approaches applying to nonlinear systems and concerned with s-abstractions, (Broucke, 1998) and (Stiver *et al.*, 2001) use the notion of dynamical invariants whereas other approaches are proposed in (Zhao, 1994) (Stursberg *et al.*, 2000). Even though nonlinear systems are considered in (Caines and Lemch, 1998), no abstraction technique is proposed. In summary, there exists no systematic technique for generating abstractions of nonlinear systems.

3. BACKGROUND

The continuous plant is modelled by

$$\dot{x}(t) = f(x(t), u(t)),\ y(t) = x(t),\ t \in \mathbb{R}_{\geq 0} \quad (1)$$

where $x(t) \in D$, $u(t) \in U$, and $y(t)$, represent a coordinate function, an input map, and an output function, respectively. The vector field f is defined on D, a connected subset of $\mathbb{R}^n$, and it is assumed to be analytic in x, C^1 in u and complete (Isidori, 1995). The set of input values $U = \Sigma$ is known *a priori* and

$$\Sigma := \{\sigma_k := (\sigma_k^1, \ldots, \sigma_k^m) \in \mathbb{R}^m\}_{k \in \mathcal{I}_\Sigma}, \quad (2)$$

with $\mathcal{I}_\Sigma \subset \mathbb{N}$ the index set of Σ and m the number of inputs involved. The map $u : \mathbb{R}_{\geq 0} \to \Sigma$ generates piecewise-constant (from the right) input signals, which we write $u(\cdot) \in \mathcal{U}_d \subset \mathcal{U}$ with $\mathcal{U}$ the class of admissible controls (Isidori, 1995). The *flow* of the vector field $f(\cdot, \sigma_k)$ is a mapping $\phi_k : I \times D \to D$ satisfying $\phi_k(t = 0, p) = p$ for any $p \in D$ and $\partial\phi_k(t, x)/\partial t = f(\phi_k(t, x), \sigma_k)$, for each $t \in I \subset \mathbb{R}$ where I is a time interval. We denote by $\phi_k^*(I_a, \cdot)$ the flow over the time interval I_a.

Definition 1. (First Integral). Let $f : D \to \mathbb{R}^n$ be a vector field. A C^k ($k \geq 1$) real-valued function $\gamma : D' \to \mathbb{R}$ defined on D', an open subset of D, is said to be a time-independent C^k *first integral* for the vector field f on D' if it satisfies

$$d\gamma(x) \cdot f(x)|_{x=p} = 0, \quad (3)$$

for all $p \in D'$ with $d := [\partial/\partial x_1, \ldots, \partial/\partial x_n]$.$\diamond$

Namely, a first integral is a function whose tangent space is parallel to f everywhere in D'. There exists no generic procedure for extracting the first integrals of a nonlinear continuous system, in general (Goriely, 2001). For vector fields as in (1), a "first integral" means a non-trivial first integral ($d\gamma(x, u)|_{x=p} = 0$ only for some $p \in D'$) that is analytic in x, C^k in u with $k \geq 1$, and defined for any set $\Sigma \subset \mathbb{R}^m$. For the following definition, we require the $n + m$ dimensional space $D'_\sigma = D' \times (\sigma - \epsilon, \sigma + \epsilon)$, where $\sigma \in \Sigma$ and $\epsilon \in \mathbb{R}^m \setminus \{0\}$ is a vector whose norm can be made arbitrarily small.

Definition 2. (Near Integrability). A system characterized by (1) and with an input map $u(\cdot) \in \mathcal{U}_d$ is *nearly integrable* over D' if for any input value $\sigma \in \Sigma$ there exist a vector $\epsilon \in \mathbb{R}^m \setminus \{0\}$ and $n - 1$ first integrals $\gamma_j(\cdot, \sigma)$, $j \in \{1, \ldots, n-1\}$, satisfying the condition

$$rank([d\gamma_1(x, \sigma), \ldots, d\gamma_{n-1}(x, \sigma)]^T) = n - 1 \quad (4)$$

on an open and dense subset of D'_σ.$\diamond$

Definition 3. (ICSS). A system satisfying the property of Definition 2 is an *Integrable Controlled Switched System* (or ICSS). $\diamond$

An ICSS is such that integrability (Isidori, 1995) holds on some subset of D' despite a small perturbation in the input values. A first integral $\gamma_j(\cdot, \sigma_k)$ with $\sigma_k \in \Sigma$ is now written in a compact form as γ_j^k. To the input value set Σ is associated the set of $m \times (n - 1)$ first integrals

$$\Gamma := \{\gamma_j^k \mid k \in \mathcal{I}_\Sigma,\ j \in \{1, \ldots, n-1\}\}. \quad (5)$$

We now present some necessary concepts from differential topology. A smooth mapping between differentiable manifolds $g : M \to N$ with a surjective derivative at p is a submersion at p. The mapping g is a *submersion* if it is a submersion for all $p \in M$. Given $c \in g(M)$, the preimage of a submersion g is the set $g^{-1}(c) := \{p \in M \mid g(p) = c\}$. A point $c \in N$, is a *regular value* of g if $dg|_p$ is surjective for all values of p such that $g(p) = c$; otherwise c is a *critical value*. The set of regular values for g is denoted by $\mathcal{R}^g$. A point $p \in M$ is a *critical point* if $dg|_p$ is not surjective. By Sard's Theorem, the set $\mathcal{R}^{j,k}$ that contains regular values for $\gamma_j^k \in \Gamma$, is dense and open in $\mathbb{R}$. Let $\widetilde{\mathcal{R}}^{j,k} \subseteq \mathcal{R}^{j,k}$ be the largest open subset of regular values that are in the image of $\gamma_j^k(D')$. Given $\sigma_k \in \Sigma$ and $\gamma_j^k \in \Gamma$,

$$L_{j,k}^c := \{p \in D' \mid \gamma_j^k(p) = c \in \widetilde{\mathcal{R}}^{j,k}\} \quad (6)$$

is referred to as a *leaf* and c is called a *first integral constant* (or FIC). The partitions induced by a set of FICs are called *leaf-partitions.*

An FSM is a triple (Q, Σ, δ) where Q, Σ, and δ represent the set of discrete states, the alphabet, and the transition map, respectively. The transition map $\delta : \Sigma \times Q \to 2^Q$ provides the state transitions of an FSM under some input values. An FSM transition structure is *deterministic* if $\delta : \Sigma \times Q \to Q$, or said otherwise, if $(\forall i \in Q)(\forall \sigma \in$

Σ) $\#\delta(\sigma, i) = 1$ whenever $\delta(\sigma, i)$ exists, where $\#A$ stands for the cardinality of set A.

4. PROBLEM STATEMENT

For FSM abstractions, the state set Q is defined by a partition while continuous dynamics (1) determine the transition map δ. Prior to formulating the problem, we explain some phenomena characterizing the transition structure.

Any partitioning of a continuum results in subsets, called *cells*, containing an infinite number of points. A finiteness issue arises due to the infinite number of points to consider in the treatment of each cell. This represents a challenge for obtaining the transition map δ because the task must be completed in a finite number of operations.

The vector field f is *transversal* to the cell boundary ∂ if $N(\partial, x) \cdot f(x)|_{x=p} \neq 0$ for all $p \in \partial$, where $N(\partial, x)|_{x=p}$ stands for the normal to the boundary ∂ evaluated at p. Namely, the vector field f is transversal to ∂ if it is nowhere tangent to it. Given an arbitrary partition and a boundary ∂^i, a trajectory initiated at a point p and intersecting with ∂^i can be (a) tangent to ∂^i at a point, (b) transversal to ∂^i, or (c) travelling along ∂^i, as represented in Figure 2. Transversality is a nice property since it provides a clear delineation of trajectories. The presence of non-transversal trajectories is problematic because small perturbations may alter the partition cells being encountered.

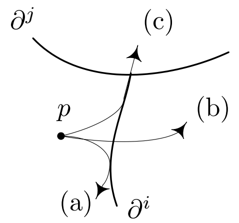

Fig. 2. Trajectory types

Given a partition cell $q_i \in Q$, the set of neighboring cells contains those cells sharing a common boundary with q_i, i.e., $N_i := \{q_{i'} \in Q \setminus \{q_i\} \mid \partial^i \cap \partial^{i'} \neq \emptyset\}$. Formally, $\mathcal{A} := (Q, \Sigma, \delta)$ is an FSM abstraction for (1) if for each pair $q_i, q_j \in Q$ and each input value $\sigma_k \in \Sigma$ satisfying $q_j \in \delta(\sigma_k, q_i)$,

$$\begin{aligned} &(i) \quad q_i = q_j \text{ and } (\exists p \in q_i)\, \phi_k(t, p) = p, \forall t \in \mathbb{R}, \text{ or} \\ &(ii) \quad q_j \in N_i \text{ and } (\exists p \in q_i)(\exists p' \in q_j)(\exists t \in \mathbb{R}_{>0}) \\ &\qquad \phi_k^*([0, t), p) \in \overline{q}_i \text{ and } \phi_k(t, p) = p', \end{aligned} \quad (7)$$

where $\overline{q}_i$ denotes the set closure of q_i, i.e., $\overline{q}_i := q_i \cup \partial^i$. Thus a transition induced by σ_k from q_i to q_j exists if (i) there is an equilibrium point in q_i, or (ii) for at least one point $p \in q_i$ there is a (positive time) trajectory leading to a point $p' \in q_j$.

Consistency is a property that ensures that a sequence of transitions in the abstraction has at least one corresponding trajectory in the underlying continuous system (Kokar, 1995) (Stursberg *et al.*, 2000). As shown in (Lunze *et al.*, 1999), consistency has strong ties with the existence of deterministic transition structures. Moreover, in (Blouin *et al.*, 2003) the authors demonstrated that the lack of transversality may induce nondeterministic transitions.

We are interested in finding, in a finite number of steps, a consistent FSM abstraction $\mathcal{A}$ for (1). Due to space limitations, only transversality is treated.

5. TRANSVERSALITY

Given a leaf-partition, a boundary ∂^i reduces to a subset of a leaf, i.e., $\partial^i \subseteq L^c_{j,k}$ for some $k \in \mathcal{I}_\Sigma$, and $c \in \widetilde{\mathcal{R}}^{j,k}$. By definition a first integral $\gamma^k_j \in \Gamma$ satisfies $d\gamma^k_j(x) \cdot f(x, \sigma_k)|_{x=p} = 0$ for all $p \in D'$. Thus a trajectory initiated on ∂^i travels along the leaf $L^c_{j,k}$ as long as σ_k remains active (case (c) of Figure 2). With two input values $\sigma_k, \sigma_{k'} \in \Sigma$, $\sigma_k \neq \sigma_{k'}$ the characterization of nontransversality is performed by using

$$d\gamma^k_j(x) \cdot f(x, \sigma_{k'}) = \psi_{j,k,k'}(x), \quad (8)$$

where the analyticity of the real-valued function $\psi_{j,k,k'}$ follows from that of γ^k_j and $f(\cdot, \sigma_{k'})$. In the presence of leaf-partitions, equation (8) detects the transversality for the whole family of leaves and flows generated by input values σ_k and $\sigma_{k'}$, respectively. Thus the characterization of the trajectory (a) in Figure 2 corresponds to points $p \in D'$ where $\psi_{j,k,k'}(x)|_{x=p} = 0$, whereas the trajectory (b) coincides with $\psi_{j,k,k'}(x)|_{x=p} \neq 0$. If $\psi_{j,k,k'}$ is a submersion, then $\psi^{-1}_{j,k,k'}(0)$ is referred to as a *non-transversality submanifold* (NTSM). Whenever a leaf $L^c_{j,k}$ does not intersect with a critical point of $\psi_{j,k,k'}$ and if the NTSM exists, then the NTSM divides the leaf in two regions $R_1, R_2 \subset L^c_{j,k}$, such that $R_1 \cap R_2 = \emptyset$, $\overline{R}_1 \cup \overline{R}_2 = L^c_{j,k}$ and $d\gamma^k_j(x) \cdot f(\sigma_{k'}, x)|_{x=p\in R_1} < 0$, and $d\gamma^k_j(x) \cdot f(\sigma_{k'}, x)|_{x=p\in R_2} > 0$. Namely, in one region the flow goes in one direction with respect to the normal $d\gamma^k_j$ while for the other region it circulates in the opposite direction.

Let us identify the set of critical points of $\psi_{j,k,k'}$ in D' with $\mathcal{C}^p_{j,k,k'} \subseteq D'$. Given the set Σ let

$$\mathcal{C}^p_{j,k} := \bigcup_{\sigma_{k'} \in \Sigma \setminus \{\sigma_k\}} \mathcal{C}^p_{j,k,k'} \subseteq D' \quad (9)$$

represent the set of all critical points in D' for the transversality of the flows generated by input values in Σ with respect to the jth leaf induced by the input value $\sigma_k \in \Sigma$. The task of characterizing non-transversal intersections is facilitated if no leaf induced by γ^k_j meets with $\mathcal{C}^p_{j,k}$.

Proposition 1
Let $\gamma_j^k \in \Gamma$ be a first integral and let $\sigma_k' \in \Sigma \setminus \{\sigma_k\}$ be arbitrary. For each point $p \in L_{j,k}^c \subset D' \setminus \mathcal{C}_{j,k}^p$ such that $\psi_{j,k,k'}(x)|_{x=p} = 0$, the set $\psi_{j,k,k'}^{-1}(0)$ forms a submanifold of D' containing p. $\diamond$

Let $\mathcal{C}_{j,k}^v := \{p \in \widetilde{\mathcal{R}}^{j,k} \mid \gamma_j^k(a) = p,\ a \in \mathcal{C}_{j,k}^p\}$ be the set of critical values for the transversality test and let $\hat{\mathcal{R}}^{j,k} = \widetilde{\mathcal{R}}^{j,k} \setminus \mathcal{C}_{j,k}^v$. Under certain conditions on $\hat{\mathcal{R}}^{j,k}$ and on the ICSS it can be shown that there exist approximate (possibly exact) leaf-partitions with well-characterized transversality. This is in part due to near integrability and the fact that under a small perturbation $\Delta u \neq 0$ in the input value u the transversality with input value $u' := u + \Delta u$ can be investigated.

6. BOUNDED LEAF-PARTITIONS

So far, we have assumed that the cells of a leaf-partition were bounded. Even though a visual inspection is possible for systems in $\mathbb{R}^{n=2}$, the task of verifying this assumption becomes extremely complex when $n > 2$. This section investigates when a leaf-partition generated from first integrals taken in $\hat{\Gamma} \subseteq \Gamma$ has all its cells bounded by leaves.

A nonempty open set $P \subset D'$ is *L-bounded* if there exists "a closed box made out of leaves" bounding P. An *L-partition* is a leaf-partition for which each cell is L-bounded. One can interpret L-boundedness as a homogeneity requirement among the cells of a leaf-partition.

Definition 4. (L$^\epsilon$-boundedness). Given an open set $P \subset D'$, a point $p \in P$ is *L$^\epsilon$-bounded* if there exist a neighborhood of p, $N_p \subset P$, and an open subset of N_p containing p and whose boundary is made of leaves. $\diamond$

Let A be a set and define its closure as $\overline{A} = A \cup \partial A$ with ∂A the boundary points of A. A *boundary point* $p \in \partial A$ is a point whose neighborhoods contain a point in A and a point in A^c, the complement of A. An *internal boundary point*, $p \in \partial_{int} A$, is a boundary point of A for which any neighborhood N_p satisfies $N_p \subset \overline{A}$. The external boundary $\partial_{ext} A$ complements $\partial_{int} A$ in ∂A.

Proposition 2
Let R_a and R_b be two open, distinct, and nonempty intersecting L-bounded sets with bounding boxes made out of leaves $B_a = \partial R_a$ and $B_b = \partial R_b$, respectively. Then $R_a \cup R_b \cup \partial_{int}(R_a \cup R_b)$ is a L-bounded set. $\diamond$

Proof: (Blouin *et al.*, 2003). $\square$

Partition cells normally form a collection of adjacent L-bounded sets. The above result implies that any adjacent L-bounded cells can be "glued" together to form a larger L-bounded cell. This construct enables one to link L$^\epsilon$-boundedness to L-boundedness in the following manner.

Lemma 1. Consider an ICSS with a set of first integrals $\hat{\Gamma} \subseteq \Gamma$ as well as an open set $P \subset D'$. If each point $p \in P$ is L$^\epsilon$-bounded, then there exists a connected and nonempty open subset $P' \subset P$ that is L-bounded. $\diamond$

Proof: (Blouin *et al.*, 2003). $\square$

Lemma 1 provides a sufficient condition for the L-boundedness of a subset of D' that relies on the notion of L$^\epsilon$-boundedness. A sufficient condition for L$^\epsilon$-boundedness at a point is to encounter a leaf while travelling from that point in all possible directions. Rather than testing all directions, one can exploit the information about the non-transversality of a set of leaves. Given a set of first integrals $\hat{\Gamma} \subseteq \Gamma$, the set of points in D' where nontransversality occurs is

$$\Omega(\hat{\Gamma}) := \{p \in D' \mid rank(d\hat{\Gamma}(x)|_{x=p}) < n\}, \quad (10)$$

where $rank(d\hat{\Gamma}(x)|_{x=p})$ stands for the rank of the codistribution formed by the differential of all first integrals in $\hat{\Gamma}$. Thus $\Omega(\hat{\Gamma})$ forms a closed set of measure zero, which may fail to be convex (Sussman, 1973). The tangent cone to $\Omega(\hat{\Gamma})$ at a point p in $\Omega(\hat{\Gamma})$ is given by

$$C_{\Omega(\hat{\Gamma})}(p) := \{v \in \mathbb{R}^n \mid d^0_{\Omega(\hat{\Gamma})}(p, v) = 0\}, \quad (11)$$

where $d^0_{\Omega(\hat{\Gamma})}(p, v)$ is the *generalized directional derivative* of $d_{\Omega(\hat{\Gamma})}(p)$, the distance function from a point $p \in D'$ to the set $\Omega(\hat{\Gamma})$ (Clarke, 1990).

Lemma 2. Consider an ICSS with a set of first integrals $\hat{\Gamma} \subseteq \Gamma$ and its set of nontransversality points $\Omega(\hat{\Gamma})$. Let P be an open subset of D'. A point $p \in P$ is L$^\epsilon$-bounded if $p \notin \Omega(\hat{\Gamma})$ or $p \in \Omega(\hat{\Gamma})$ and for any nonzero vector $v \in C_{\Omega(\hat{\Gamma})}(p)$, there exists a first integral $\gamma_j^k \in \hat{\Gamma}$ such that

$$d\gamma_j^k(x) \cdot v|_{x=p} \neq 0. \diamond \quad (12)$$

Proof: (Blouin *et al.*, 2003). $\square$

Lemma 2 requires that $C_{\Omega(\hat{\Gamma})}$ is everywhere transversal to some leaf induced by the first integrals of $\hat{\Gamma}$. As (12) involves the differential of first integrals in $\hat{\Gamma}$, it does not depend on FICs, and is, therefore, invariant to further refinements or aggregations of the partition. In (Blouin *et al.*, 2003), various examples show how the L$^\epsilon$-boundedness property is affected by the set of first integrals constituting $\hat{\Gamma}$, the set of input values Σ, and the selection of P.

Theorem 6.1
Consider an ICSS with the set of first integrals $\hat{\Gamma} \subseteq \Gamma$ and its set of nontransversality points $\Omega(\hat{\Gamma})$. Let $P \subseteq D'$ be an open set. If the conditions of Lemma 2 hold for all $p \in P$ then there exists a nonempty L-bounded connected subset $P' \subset P$.$\diamond$

Proof: By Lemma 1 and Lemma 2.$\square$

As L$^\epsilon$-boundedness characterizes the family of leaves induced by $\hat{\Gamma}$, Theorem 6.1 provides insight into the existence of *L-partitions*.

7. DISCUSSION

The material presented so far holds for systems of arbitrary dimension n. In (Blouin *et al.*, 2003) it was shown that $n = 2$ represents a special case of the theory. An algorithm providing, in finite steps, the leaf-partition and the corresponding FSM abstraction of two-dimensional systems was developed. In what follows, the theory is extended to non-integrable systems evolving in $D \subseteq \mathbb{R}^2$.

In practice many chemical processes do not have known first integrals. In this category one finds nonisothermal continuous stirred-tank reactors (CSTR). An instance of a two-dimensional model involving a first-order irreversible reaction $A \rightarrow B$ with constant hold-up is given by

$$\begin{aligned} \frac{dC}{dt} &= -DC - k_0 e^{(-E/RT)}C + DC_{in} \\ \frac{dT}{dt} &= -DT + \beta k_0 e^{(-E/RT)}C + DT_{in} - u \end{aligned} \tag{13}$$

where C (mol/L) and T (^oC) represent the concentration of A and the temperature of the mixture, respectively (Ogunnaike and Ray, 1994). The input u corresponds to the heat transferred through the coil. The following parameters are used: $k_0 = 1.287e{+}12$, $E/R = 9758.3$, $\beta = 1.4936$, $D = 14.19\ h^{-1}$, $C_{in} = 0.5$ and $T_{in} = 100$.

For two-dimensional systems modeled by $dx_1/dt = f_1(x_1, x_2)$ and $dx_2/dt = f_2(x_1, x_2)$, the corresponding one-form is $\omega = \omega_1 d(x_1) + \omega_2 d(x_2) = f_2 d(x_1) - f_1 d(x_2)$. Any function V satisfying $dV = \omega$ is a first integral. An approximate first integral $\hat{V}$ is such that $d\hat{V} - \omega = \epsilon$ with ϵ an error term. If $\hat{V}$ is expressed as $\hat{V} = \sum_{i=1}^{k} a_i \phi_i(x_1, x_2)$, where ϕ_i's represent approximating functions, then the identification of the weight vector $a = [a_1, \ldots, a_k]$ can be performed by solving

$$\int_\Theta \sum_{j=1}^{n} \left(\omega_j - \sum_{i=1}^{k} a_i \frac{\partial \phi_i}{\partial x_j} \right) \frac{\partial \phi_h}{\partial x_j}\, d\Theta = 0, \tag{14}$$

for $h = 1, \ldots, k$ over a domain Θ. Thus one gets $A\, a^T = b$ where

$$b = \begin{bmatrix} \beta^1 \\ \vdots \\ \beta^k \end{bmatrix},\ A = \begin{bmatrix} \alpha_1^1 & \cdots & \alpha_k^1 \\ \vdots & & \vdots \\ \alpha_k^1 & \cdots & \alpha_k^k \end{bmatrix}, \tag{15}$$

with

$$\begin{aligned} \alpha_i^h &= \int_\Theta \left(\frac{\partial \phi_i}{\partial x_1}\frac{\partial \phi_h}{\partial x_1} + \frac{\partial \phi_i}{\partial x_2}\frac{\partial \phi_h}{\partial x_2} \right) d\Theta \\ \beta^h &= \int_\Theta \left(f_2 \frac{\partial \phi_h}{\partial x_1} - f_1 \frac{\partial \phi_h}{\partial x_2} \right) d\Theta \end{aligned} \tag{16}$$

as $\omega_1 = f_2$ and $\omega_2 = -f_1$. Here, Θ is a region defined to be $[C_l, C_h] \times [T_l, T_h]$ with $C_l = 0.3$, $C_h = 0.65$, $T_l = 350$ and $T_h = 450$. Figure 3 shows trajectories of (13) induced by the input value set $\Sigma := \{-5000, 5000\}$ over Θ. The ϕ_i's are elements of a polynomial of order r. Thus $i = 1, \ldots, r^2$ and

$$\hat{V} = \sum_{a=1}^{r}\sum_{b=1}^{r} a_i \phi_i, \text{ with } \phi_i = C^a T^b. \tag{17}$$

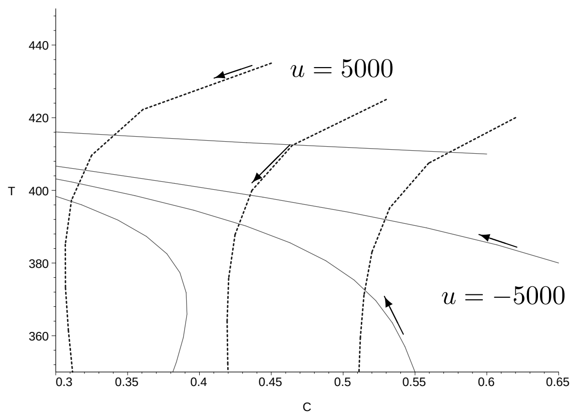

Fig. 3. Trajectories over Θ for $u \in \Sigma$.

In Figure 4, some approximate first integrals $\hat{V}$ for $u \in \Sigma$ are shown. The FICs for $u = -5000$ and $u = 5000$ are $(-2500, -1200, -500, 100, 500, 800)$ and $(3200, 3800, 4400, 5000)$, respectively. The circled integers are labels for the L-bounded cells of the leaf-partition. That is, the FSM abstraction has the state set $Q := \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Polynomials of order 5 and 3 were used for $u = -5000$ and $u = 5000$, respectively. The corresponding FSM abstraction is presented in Figure 5. The transition structure of this FSM abstraction is deterministic because at each state an input value induces an unique transition. Transitions marked by an "$\times$" are disabled because they lead outside the set of L-bounded cells.

Even though the FSM abstraction was obtained in an *ad hoc* manner, the algorithm could be modified so that the abstractions are obtained systematically from exact or approximate first integrals. In the above example the selection of a smaller region Θ or the choice of some other ϕ_i's

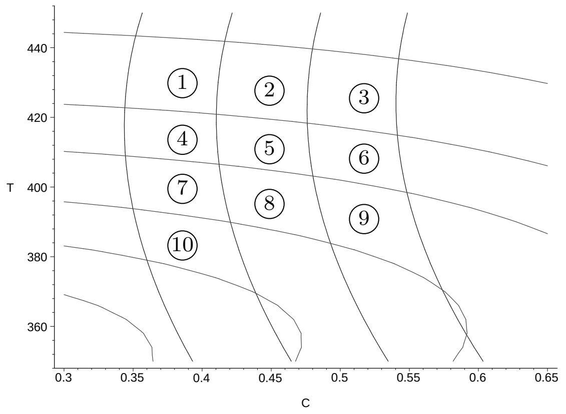

Fig. 4. Approximate leaf-partition.

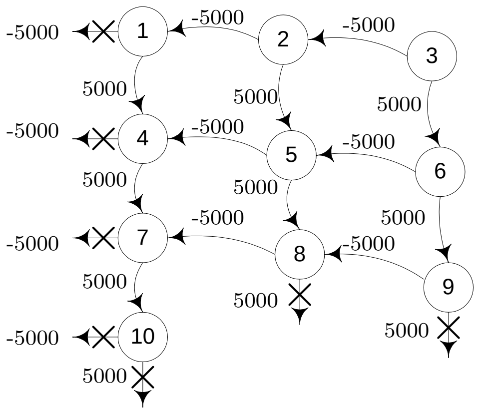

Fig. 5. Approximate FSM abstraction

could provide a better fit between the trajectories and the approximate first integrals.

An extension of the algorithmic procedure to handle higher-dimensional systems is the topic of future research.

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