EFFECT OF PROCESS NONLINEARITY ON LINEAR QUADRATIC REGULATOR PERFORMANCE [1]

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Abstract: In this paper, a new local measure of linear controller performance is introduced for linear controllers operating on a nonlinear plant. The measure, called the performance sensitivity measure, quantifies the departures from optimality of a locally linear quadratic regulators. The measure applies to nonlinear systems that admit a controllable and observable linearization. It is shown that the measure can be related to standard minimum variance benchmarking techniques and can therefore be assessed using closed-loop process data in an operating region of interest.

Keywords: Nonlinearity, Linear controller performance, Performance benchmaring

1. INTRODUCTION

The control of linear systems has been extensively studied and the literature provides a very complete and well–characterized collection of tools for their analysis, monitoring, optimization, and control. As a result, process control engineers focus on linear system representations to solve a wide range of control problems. Unfortunately, the reality is that few processes are linear, and therefore the effectiveness of using linear control strategies can be questioned. Nonlinear control strategies have advanced greatly, and are becoming more widely accepted; however, their implementation is impeded by a considerable degree of mathematical sophistication or computational requirement. As a result local linear approximations of the nonlinear system are often used to develop a control law. In order to test the effectiveness of this approach, it would desirable to develop an index that measures the effect of process nonlinearity on linear controller performance. From a design point of view, such a measure would indicate whether sufficient benefit is available to warrant investment in a nonlinear controller.

Many authors (e.g., (Desoer and Wang 1981), (Allgöwer 1995*a*), (Allgöwer 1995*b*), (Stack and Doyle III 1997), (Haber 1985), (Ogunnaike *et al.* 1993), (Guay *et al.* 1995)) have considered the assessment of process nonlinearity as means of justifying the need for nonlinear control techniques. However such measures provide admittedly open–loop assessment of nonlinearity that are difficult to relate to controller performance. The objec-

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tive of this paper is to introduce a new local measure of linear controller performance for linear controllers operating on a nonlinear plant. The measure, called the performance sensitivity measure, quantifies the departures from optimality of a locally linear quadratic regulators. The measure applies to nonlinear systems that admit a controllable and observable linearization. It is shown that the measure can be related to standard minimum variance benchmarking techniques and can therefore be assessed using closed-loop process data in an operating region of interest. The paper is structured as follows. The proposed performance sensitivity measure is presented in Section 2. In Section 3, we draw a parallel between the proposed measure and standard minimum variance benchmarking techniques. This is followed by brief conclusions in Section 6.

2. PERFORMANCE SENSITIVITY MEASURE

In this section, an alternative control–relevant nonlinearity measure, the "performance sensitivity measure", is introduced. The performance sensitivity measure (PSM) attempts to characterize the extent of performance degradation expected when a nonlinear system is regulated by a linear quadratic regulator (LQR).

Consider the nonlinear time-invariant system,

$$\begin{aligned} \dot{x} &= f(x, u(t)) \\ y &= h(x(t)) \end{aligned} \tag{1}$$

$u(t) \in \mathbb{R}^p$ is the available control input, $y(t) \in \mathbb{R}^m$ is the observed process output, and $x(t) \in \mathbb{R}^n$ represents the internal states of the system. The linearization of the system eq.(1) about the origin is given by the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{2}$$

where A, B, and C are system matrices of appropriate dimension. It is assumed that the triple (A, B, C) is both observable and controllable. By letting C be the identity matrix, full state information is available for use in the control strategy.

For the linear system eq.(2), the linear quadratic regulator given by

$$u(t) = -R^{-1}B^TPx(t) \tag{3}$$

minimizes, for every initial condition $x(0) = x_0$, the quadratic objective function,

$$\eta = \int_0^\infty \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \tag{4}$$

where $Q \in \mathbb{R}^{nxn}$ and $R \in \mathbb{R}^{pxp}$ are problem–specific, non–negative definite state– and input–penalty matrices, and where P is the positive–definite, symmetric solution matrix of the algebraic Riccati equation

$$A^TP + PA - PBR^{-1}B^TP + C^TQC = 0 \tag{5}$$

The cost to regulate about the origin when the system starts at any point at any time t can be approximated by the value function,

$$J^* = x^T(t)Px(t) \tag{6}$$

The level sets of this value function describe ellipses in the state space (ellipsoids in systems with more than two states) from which the system can be moved to the origin for a given cost. For linear systems, the Riccati equation solution matrix is constant throughout the entire state space, and therefore the size and orientation of these level sets is constant. If one implements the LQR to control the nominal nonlinear plant eq.(1), the degree to which the intended linear controller performance is realized depends on the extent of nonlinearity of the process. One way to assess this change in performance due to nonlinearity is to add a perturbation term, denoted $\nu(t)$, to the control law,

$$u^*(t) = -R^{-1}B^TPx(t) + \nu(t) \tag{7}$$

in the closed-loop system:

$$\frac{dx}{dt} = f\left(x(t), u^*(x(t), \nu(t))\right) \tag{8}$$

The perturbation may be considered as a means of incorporating knowledge of the process nonlinearities in the control law to account for setpoint or load changes. To ascertain the effect of $\nu(t)$ on the performance of the closed–loop system, (6) is differentiated with respect to $\nu(t)$. When the optimal linear controller with perturbation, (7) is applied to the linear system, (2), the resultant closed–loop model is

$$\begin{aligned}\dot{x} &= \left(A - BR^{-1}B^TP\right)x(t) + B\nu(t) \\ y(t) &= Cx(t)\end{aligned} \tag{9}$$

The system (9) is a linear system where $\nu(t)$ is an input which is known to enter the solution $x(t)$ linearly:

$$\begin{aligned}x(t) = &x(t_0)e^{(A-BR^{-1}B^TP)(t-t_0)} \\ &+ \int_{t_0}^{t}\left(e^{(A-BR^{-1}B^TP)(t-\tau)}B\nu(\tau)\right)d\tau\end{aligned} \tag{10}$$

If we consider only constant perturbation, $\nu(t) = \nu$, and we assume that the system starts from the origin, $x(t_0) = 0$, J^* can be evaluated as

$$J^* = \nu^T\tilde{P}\nu$$

where $\tilde{P}$ is formed from the coefficient matrix of $\nu(t)$ in the integrand of (10) and the Riccati equation solution matrix P. J^* may be represented about $\nu = 0$ as a Taylor series polynomial,

$$\begin{aligned}J^*(\nu)|_0 = J^*(0) + \left.\frac{\partial J^*(\nu)}{\partial\nu}\right|_{\nu=0}\nu + \frac{1}{2!}\left.\frac{\partial^2 J^*(\nu)}{\partial\nu^2}\right|_{\nu=0}\nu^2 \\ + \frac{1}{3!}\left.\frac{\partial^3 J^*(\nu)}{\partial\nu^3}\right|_{\nu=0}\nu^3 + \mathcal{O}(4)\end{aligned} \tag{11}$$

where $\mathcal{O}(4)$ is a fourth–order truncation error term. Since J^* is a quadratic function of ν for linear systems, the third–order term, and the truncation error, is exactly zero. Thus, it is possible to assess the effect of nonlinearity on local controller performance by assessing the magnitude of the third-order term in eq.(11). Considering only the magnitude of the third derivative of the value function with respect to the input perturbation is wrought with scaling and dimensionality issues, as $\frac{^3 J^*}{\nu^3}$ has units from η and the inputs. In order to assess the magnitude of the third order term, we propose the following dimensionless quantity, called the performance sensitivity measure (PSM):

$$PSM = \frac{\frac{\partial^3 J^*}{\partial\nu^3}}{\|\frac{\partial^2 J^*}{\partial\nu^2}\|^{\frac{3}{2}}}\sqrt{J^*_{min}} \tag{12}$$

where J^*_{min} the minimal (quadratic) cost attainable in the particular region of interest. The PSM considers how the cost J^* changes as the process moves along the closed loop locus normalized by the largest cost contour completely contained within the operating region chosen. A small value of the PSM indicates that the nominal linear controller performance is not sensitive to the effect of the process nonlinearity. In that case, the linear controller provides uniform performance over the region of interest. If the PSM is large then the nonlinearity has a drastic impact of the performance of the linear controller. In general, a PSM value of 1.5 is deemed important as it leads to an average departure of 30$\%$ from the nominal linear controller performance.

2.1 PSM of a Nonlinear System

For a nonlinear system, $\nu(t)$ does not enter the solution $x(t)$ linearly, even for control–affine systems, and therefore J^* is not a quadratic value function. Consequently, the Taylor series expansion of J^* given in (11) has a nontrivial third–order coefficient.

For the case where $\nu(t)$ is a constant, the higher order derivatives of $J^*(\nu)$ are computed as follows. The states are assumed to be scaled to nominal operating regions to permit the identity matrix to be employed for Q in the objective function, and R chosen according to the desired control attenuation level. As described above, the optimal linear controller may be found, and the perturbed input, (7), employed. The approximation of the value function is then

$$\begin{aligned}J^* &= \begin{pmatrix}x_1 & \cdots & x_n\end{pmatrix}\begin{pmatrix}P_{1,1} & \cdots & P_{1,n} \\ \vdots & \ddots & \vdots \\ P_{n,1} & \cdots & P_{n,n}\end{pmatrix}\begin{pmatrix}x_1 \\ \vdots \\ x_n\end{pmatrix} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n}x_iP_{i,j}x_j\end{aligned} \tag{13}$$

where x_i, $1 \leq i \leq n$, represents the solution of the perturbed closed-loop system under constant input ν. Differentiating J^* with respect to ν, we obtain

$$\begin{aligned}\frac{\partial J^*}{\partial\nu_l} &= \sum_{i,j=1}^{n}\left(\frac{\partial x_i}{\partial\nu_l}P_{i,j}x_j + x_iP_{i,j}\frac{\partial x_j}{\partial\nu_l}\right) \\ &= 2\sum_{i,j=1}^{n}P_{i,j}\frac{\partial x_i}{\partial\nu_l}x_j,\end{aligned}$$

$$\frac{\partial^2 J^*}{\partial\nu_l\partial\nu_m} = 2\sum_{i,j=1}^{n}\left(\frac{\partial^2 x_i}{\partial\nu_l\partial\nu_m}P_{i,j}x_j + \frac{\partial x_i}{\partial\nu_l}P_{i,j}\frac{\partial x_j}{\partial\nu_m}\right),$$

and

$$\frac{\partial^3 J^*}{\partial \nu_l \partial \nu_m \partial \nu_k} = 2 \sum_{i,j=1}^{n} P_{ij} \left(\frac{\partial^2 x_i}{\partial \nu_l \partial \nu_k} \frac{\partial x_j}{\partial \nu_m} + \frac{\partial x_i}{\partial \nu_l} \frac{\partial^2 x_j}{\partial \nu_m \partial \nu_k} + \frac{\partial^3 x_j}{\partial \nu_k \partial \nu_l \partial \nu_m} x_i + \frac{\partial^2 x_j}{\partial \nu_l \partial \nu_m} \frac{\partial x_i}{\partial \nu_k} \right)$$

All the derivatives of J^* are evaluated at $x = 0$ and $\nu = 0$ to obtain a local measure of sensitivity that applies to the closed-loop system operating at its setpoint. The computation of the derivatives of J^* requires the calculation of the 1st, 2nd and 3rd order sensitivity coefficients of $x(t)$ with respect to ν. The sensitivity coefficients are computed by the integration of the sensitivity equations. Due to space restrictions, we omit to list the full set of sensitivities. As an illustration, we consider the derivation of the first order sensitivity equations. Differentiating (1) with respect to ν and inverting the order of differentiation, we obtain

$$\frac{d}{dt} \frac{\partial x}{\partial \nu} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial \nu} + \frac{\partial f}{\partial \nu} \quad (14)$$

Eq.(14) can be integrated along with the perturbed closed-loop system to obtain the first order sensitivity coefficients. In the current development, we consider the trivial solution for $x = 0$ at $\nu = 0$. The same applies to the higher order sensitivity coefficients. Note that the solution of the sensitivity equations yields a time-varying PSM value that we could use to assess the variations in performance as a function of time. Since we focus on the infinite-horizon optimal control problem, it is sufficient to evaluate the steady-state value of the derivatives of J^* with respect to ν at $\nu = 0$. The resulting steady-state PSM provides an estimate of the sensitivity of the infinite horizon cost to small perturbations in the control law. By the local stability of the nominal system under LQR control, the steady-state values of the sensitivity coefficients can be shown to exist and to be finite.

An important consideration is the effect of state scaling on the values of the PSM. Process states with significantly different nominal values affects the PSM through the optimal linear controller gain matrix. It is therefore necessary to scale the states of the system appropriately. Knowledge of the typical range of operation can enable standardization, so that each of the states has zero nominal value and varies within the range $[-1, 1]$:

$$z_i(t) = \frac{x_i(t) - \bar{x}_i(t)}{x_i^{max} - x_i^{min}} \quad (15)$$

In general, such scaling is used to ensure consistency of the analysis over a region of particular interest.

In addition, it is important to note that the current development is not restricted to the LQR. The analysis applies equally to the analysis of sensitivity of an LQG controller or any other linear controller design with quadratic cost performance.

2.2 Example: Chemostat Bioreactor

Consider the model of a chemostat bioreactor (Guay *et al.* 1995):

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{\mu_{max} x_1 x_2}{1 + x_2 + K_i x_2^2} - k_d x_1 - u_1 x_1 \\ \frac{dx_2}{dt} &= -\frac{\mu_{max} x_1 x_2}{1 + x_2 + K_i x_2^2} + (S_0 - x_2)\, u_1 \end{aligned} \quad (16)$$

where x_1 and x_2 are the biomass and substrate concentrations, respectively, in $\mathrm{g/L}$, and u_1 is the dilution rate, in min^{-1}. The model parameters $\mu_{max} = 0.5\ \mathrm{min}^{-1}$, $S_0 = 0.3\ \mathrm{g/L}$, $k_d = 0.05\ \mathrm{min}^{-1}$, and $K_i = 10\ \mathrm{L/g}$ represent the specific growth rate, inlet substrate concentration, death rate and substrate inhibition constant, respectively. The nonlinearity measure proposed in (Guay *et al.* 1995) suggests the process would be the most difficult to control with a linear controller near $(x_1, x_2) = (0.02, 0.2)$.

Consider five points of steady state operation, labelled in Figure 1, chosen by uniformly selecting constant input values in the interval $[0.002, 0.018]$, as shown in Table 2.2.

Point Label	u_{nom}	x_1	x_2
a	0.002	0.00622	0.13826
b	0.006	0.01528	0.15736
c	0.010	0.01982	0.18107
d	0.014	0.01881	0.21403
e	0.018	0.00327	0.28765

Table 2.1. Selected operating points for chemostat bioreactor.

Choosing point "c", the states are scaled about the nominal steady state operating point $(x_1^{nom}, x_2^{nom}) = (0.01982, 0.18107)$, with the ranges chosen as $\tilde{x}_1 \in [x_1^{nom} \pm 0.0025]$, and $\tilde{x}_2 \in [x_2^{nom} \pm 0.025]$. We express

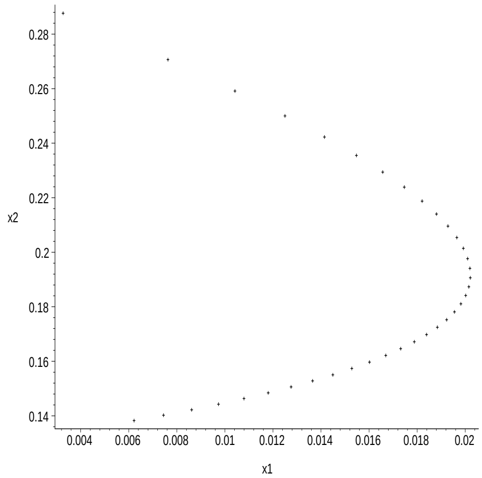

Fig. 1. The steady state locus of the chemostat bioreactor. Points represent the steady states of the system when the input is chosen uniformly in the interval $[0.002, 0.018]$.

the system equations in scaled coordinates and we choose the quadratic performance metric

$$\eta = \int_{t=0}^{\infty} \left(\tilde{x}^T \tilde{x} + (50 u_1)^2 \right) dt$$

Linearizing the system about the origin, corresponding to (2), and solving the corresponding algebraic Riccati equation we obtain the linear quadratic regulator

$$u_1(t) = 0.010 + 0.0069160\tilde{x}_1(t) - 0.024268\tilde{x}_2(t) + \nu(t) \quad (17)$$

For this system, the first and second partial derivatives are found to be

$$\frac{\partial \tilde{x}_1}{\partial \nu} = 29.300 \qquad \frac{\partial \tilde{x}_2}{\partial \nu} = 38.001$$
$$\frac{\partial^2 \tilde{x}_1}{\partial \nu^2} = -6855.5 \qquad \frac{\partial^2 \tilde{x}_2}{\partial \nu^2} = -1304.5$$

If we pick the operating region to be $\left\{ \tilde{x}_1^2 + \tilde{x}_2^2 \leq 1 \right\}$ then the value of J^*_{min} is simply equal to the minimum eigenvalue of the Riccati equation solution matrix, P. The PSM at point "c" is $PSM = -0.94005$.

To provide an indication of whether the PSM value for point "c" is significant, consider the evaluation of the PSM for the other four points previously identified: The relative PSM values are consistent with results that should be expected from the geometry of the steady state locus

Point Label	u_{nom}	x_1	x_2	PSA
a	0.002	0.00622	0.13826	-0.037179
b	0.006	0.01528	0.15736	-0.13989
c	0.010	0.01982	0.18107	-0.94005
d	0.014	0.01881	0.21403	-0.92211
e	0.018	0.00327	0.28765	0.016861

Table 2.2. Computed PSM values for the five selected points of the chemostat bioreactor.

(see Figure 1). From the actual PSM values computed for the chemostat bioreactor, it is expected that a linear controller could be used without significant deviation in performance about any of the five operating points.

3. EMPIRICAL MEASURES OF CONTROLLER PERFORMANCE

Much of the work in the assessment of process control schemes within the last decade can be traced back to (Harris 1989). Minimum variance benchmarking, as proposed in (Harris 1989), is a widely accepted for the assessment of performance in control systems. In this study, we focus on Harris's controller performance measure for single–input, single–output processes. The reader is referred to (Harris 1989) for more details on the evaluation of the performance measure.

Since the PSM indicates the sensitivity of quadratic system performance of a linear controller, it is reasonable to assume that a large PSM value would also indicate significant variations in a minimum variance benchmarking measure over a particular region of interest. Thus if we design a linear control based on a local linear approximation of the process, the large PSM would indicate that the implementation of the linear controller at other setpoints in the region of interest would result in significant deviations in controller performance measures. In order to evaluate this premise, we consider the chemostat bioreactor model operating at point "c" in closed-loop with the LQR control eq.(17) (with $\nu(t) = 0$). In order to proceed with the assessment of controller performance, we consider the biomass concentration as the measured output. Furthermore, we corrupt the measurements with uncorrelated white noise, $a(t)$ passed through the discrete transfer function

$$\frac{1}{1 + 0.4z^{-1}}$$

The noise power is set to 0.001, chosen to ensure that the closed–loop deviation from steady state is less than one in magnitude, meaning the process remains in the region suggested by the scaling of Section 2.2. The process was simulated for 400 minutes, with a fixed step–size of 0.1 minutes. For this process, Harris' minimum variance benchmark $\eta(0)$ was 0.15 for the regulation of the system at point "c". This value indicates that only 15% of current output variance could be eliminated through use of a minimum variance controller. Thus the linear control operates well in this region.

The strategy is to implement the linear controller developed under the conditions at point "c" at different setpoints. To move the process about the operating region along the closed–loop locus, a constant perturbation, $\nu(t)$ is input to the system which is then allowed to reach the new steady state. An equivalent way to handle this problem would be to assign setpoints along the steady-state locus. By evaluating the controller performance measure about each setpoint we obtain an estimate of the sensitivity linear controller performance to the location of the setpoint. It is clear that if the plant is linear then the performance measure remains essentially unchanged over the region of interest. Therefore, this relatively simple exercise provides a potential substitute to the PSM for operating control systems. It remains to show that the interpretation of the PSM provides a good indication of the sensitivity of linear control performance.

Table 3 shows the perturbed steady state values, and the minimum variance performance measure found at each of the points. Although the closed loop gains vary throughout the region considered, there is very little change in performance. In fact, the variability is insignificant given the computations required to compute $\eta(0)$. The operating region about point "c" is unsusceptible to performance degradation from closed loop nonlinearity and therefore a linear, finite gain controller performs well throughout.

Point Label	$\nu(t)$	$\tilde{x}_1$	$\tilde{x}_2$	$\eta(0)$
c+++	0.008	0.072274	0.27352	0.1629
c++	0.004	0.070454	0.14320	0.1630
c+	0.002	0.045610	0.073582	0.1631
c	0	0	0	0.1631
c-	-0.002	-0.074435	-0.078981	0.1631
c–	-0.004	-0.18670	-0.16527	0.1632
c—	-0.008	-0.63025	-0.38039	0.1618

Table 3.3. Perturbed operating points for chemostat bioreactor about point "c".

4. CONCLUSIONS

The performance sensitivity array has been introduced as a closed–loop measure that attempts to quantify the effect of nonlinearity on the performance of a nonlinear system subject to a linear quadratic regulator. The results demonstrate that the PSM can be used to predict the linear controller performance on nonlinear systems. Its impact can be verified by considering a simple minimum variance benchmarking approach.

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