DEVELOPMENTS IN MULTI-RATE PREDICTIVE CONTROL

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Abstract: Much of the work on predictive based multi-rate control has been based on the GPC algorithm (Clarke *et al*, 1987). However academic practitioners in single rate predictive control tend to favour approaches with better stability and performance guarantees. This paper demonstrates how those approaches might be deployed in a multi-rate framework and discusses some issues that arise.

Keywords: Multi-rate control, predictive control, constraint handling, feedforward

1. INTRODUCTION

A system is considered multi-rate (MR) when the inputs and outputs of a system are sampled at different rates. Typically this would be necessary if there were no restrictions on the speed at which the input is updated (denote this the fast rate (FR)), but output measurements are available only at a relatively slow rate (SR), for instance where a laboratory test is needed. MR systems take many forms depending on the system dimensions and the sampling rates used. Although quite common in industry, such systems have recieved relatively little study from process control (Li *et al*, 2001; Sheng, 2002) academics and hence this paper is preliminary work and we will adopt the simplest case of a dual rate (DR) system where only two sample rates are present. Moroever we assume that the output sample period is a simple multiple of the input sample period. The study of more complex cases constitutes future work.

One reason why MR systems may have recieved little attention is that one cannot easily use all the tools of linear control design. Single rate control assumes that an output measurement is available every sampling instant, then using z-transform theory one can analyse the behaviour of the nominal loop. However, such linear theory is not applicable when output measurements are available only periodically and hence at first appearances conventional design approaches cannot be used. There are two popular solutions to this difficulty: (i) inferential control (IC) (Lee *et al*, 1992) and (ii) lifting (Kranc, 1957).

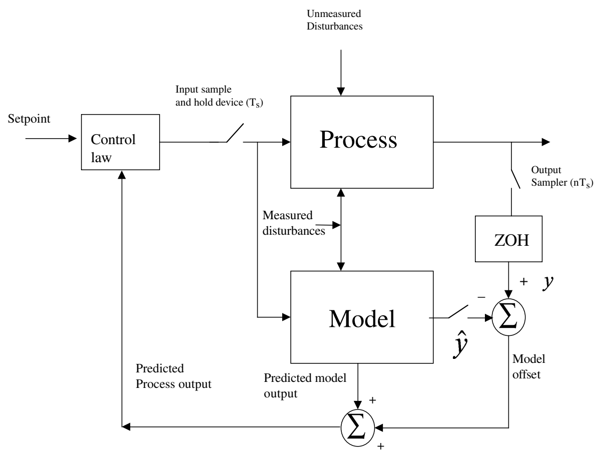

Figure 1: Internal model structure

IC makes use of an internal (see figure 1 (Garcia *et al*, 1982)) process model which operates at the fast rate (FR). This model is used to supply output estimates at the fast sample rate much like a state estimator supplies state values to be used in lieu of the actual (and unknown) state. However, this approach needs more study as there are several obvious weaknesses: (i) the state estimator/internal model receives actual output updates very slowly and this could have repercussions on accuracy and (ii) the approach relies on knowledge of a fast SR model which would have to be identified from MR data; recent work (Li *et al*, 2001) has shown that this is possible in some cases but a clear understanding of the robustness of these models constitutes work in progress.

A more popular alternative (Kranc, 1957) has been to use lifting. In essence this transforms a MR single input single output system (SISO) to a single rate multi input multi output (MIMO) system or if the system were already MIMO it increases the dimension. As the lifted system is SR (the slow rate at which the output is updated) one can use linear design and analysis methods. However: (i) there is the price of working in a significantly increased dimension and hence the design itself maybe far more complex and (ii) there is the so called causality constraint (Chen *et al*, 1994; Sheng *et al*, 2002) whereby one must ensure that the structure of the controller does not make current controls dependent on future controls. This implies a structure constraint that the feedthrough term in the controller is block lower triangular. For both IC and lifting based schemes there is also the issue of intersample ripple (Tangirala *et al*, 2001); to avoid this requires additional constraints in the controller structure.

This paper will contrast two alternative model predictive control (MPC) methods in both the lifted and IC frameworks with the aim of giving the reader a clear summary of what they gain and lose with each scenario. Section 2 will describe the necessary notation and background information. Section 3 will discuss a finite horizon algorithm (denoted FHMPC) and section 4 will develop and discuss an infinite horizon algorithm denoted (IHMPC). Section 5 will discuss the impact of constraints and section 6 presents the conclusions.

2. BACKGROUND AND OBSERVATIONS

2.1 Model predictive control

For simplicity of notation the following is restricted to single input single ouput systems, however the results are equally applicable to MIMO processes. Assume for now a single rate process. Design a finite horizon predictive control (FHMPC) law at the FR along the lines of GPC (Generalised predictive control (Clarke *et al*, 1987)) or Dynamic matrix control (DMC,(Cutler *et al*, 1980)), that is at every sample instant minimise a performance index of the form:

$$\min_{u_0,\ldots,u_{n_c-1}} \quad J = \sum_{i=0}^{n_y}(r-y_{i+1})^2 + \sum_{i=0}^{n_c-1}\lambda(u_i-u_{ss})^2 \qquad s.t. \begin{cases} u_i = u_{ss}, \ i \geq n_c \\ \text{constraints} \end{cases} \tag{1}$$

where u_{ss} is the current estimate of the input required to remove steady-state offset [1] . The signals u, y, x, r are the inputs, outputs, states and set point respectively. The constraints include limits on the input, input rate and states and are assumed affine in the degrees of freedom (d.o.f.).

The weakness of FHMPC is that there are no guaranteed a priori stability results, largely because of the mismatch between the prediction assumption and the closed-loop behaviour. For computational reasons one requires the no. of d.o.f. (n_c) to be small but as a consequence the implied constraint (see (1), $u_i = u_{ss}, \ i \geq n_c$, is not close to the closed-loop evolution that is desired. This inconsistency can result in the performance being poor because the minimisation is ill posed; that is one is minimising predicted performance subject to an artifical prediction constraint that is never invoked. Hence the minimum may lie a good distance from the minimum that would arise without the artificial constraint. The effect is much less marked for larger n_c but can cause a significant degradation when n_c is small.

2.2 Infinite horizon MPC

In order to improve the properties of MPC, many authors have proposed the use of infinite costing horizons. One of the most popular of the IHMPC algorithms is given in (Scokaert *et al*, 1996). It can be summarised as at every sample minimise a cost w.r.t n_c degrees of freedom (d.o.f.),

$$\min_{u_0,\ldots,u_{n_c-1}} \quad J = \sum_{i=0}^{\infty}(r-y_{i+1})^2 + \lambda(u_i-u_{ss})^2 \qquad s.t. \begin{cases} u_i - u_{ss} = -K(x_i - x_{ss}), \ i \geq n_c \\ \text{constraints} \end{cases} \tag{2}$$

K is an optimal state feedback; that is the optimal control minimising J in the absence of constraints.

The strength of IHMPC is that the open-loop predictions match the expected closed-loop behaviour, for the nominal case. Hence the optimisation is well posed and one can guarantee, a priori, stability and good performance. The main issue with this method is a possible inconsistency between the terminal constraint $u_i - u_{ss} = -K(x_i - x_{ss}), \ i \geq n_c$ and constraints, but that is not a topic of this paper.

[1] The control law takes a slightly different form if one uses input increments as the control variables.

2.3 Inferential control and lifting

The above algorithms were summarised for the SR case. However the context of this paper is MR systems or in particular dual rate (DR) processes where the input is updated every T seconds, but a measurement is taken every nT seconds. The algorithms need modifying to fit into this scenario. How this modification can be performed depends upon what model is available.

Inferential control (IC) requires a FR model. This assumption is a weakness but one should also state that if such a model exists, then it is to be expected that a control design using this model should outperform one based on a slow rate model.

Lifting based approaches use a DR model. There is a need to show how the IHMPC algorithm can be reformulated for this scenario and moreover to analyse its behaviour. In particular one should note (Rossiter *et al*, 2003) as discussed in section 2.1 that the restriction to DR models can give quite poorly performing control laws when one uses FHMPC. It will be shown how the move to IHMPC can overcome this weakness.

2.4 Dual rate and single rate models

Consider a FR state space model of the form

$$x_{k+1} = Ax_k + Bu_k; \quad y_k = Cx_k \tag{3}$$

The DR equivalent to this system could be written down as

$$x_{k+n} = \Gamma x_k + \Theta U_k; \quad y_k = Cx_k; \quad U_k = \begin{bmatrix} u_k \\ \vdots \\ u_{k+n-1} \end{bmatrix} \tag{4}$$

where $\Gamma = A^n$, $\Theta = \begin{bmatrix} A^{n-1}B & \cdots & AB & B \end{bmatrix}$. In many scenarios (Li *et al*, 2001) one may be able to identify Γ, Θ (or equivalent model form) from input/output data fairly easily but not A, B. Model (4) will be denoted the lifted model as the input has been lifted from u_k to U_k. Also the output/state is updated only every n samples of the FR. Effectively this gives a SR model with a lifted input.

IC assumes knowledge of the FR model whereas lifted control will make use of the lifted model and assumes the FR model is unknown.

3. FINITE HORIZON MPC

3.1 FHMPC in the lifted environment

This section will illustrate how the FHMPC control laws must be modified to cope with DR signals. First define the performance index to take the form:

$$\min_{u_0,\ldots,u_{n_c-1}} J = \sum_{i=1}^{n_y} (r - y_{k+ni})^2 + \sum_{i=0}^{n_c-1} \lambda(u_{k+i} - u_{ss})^2$$
$$s.t. \begin{cases} u_{k+i} = u_{ss}, \; i \geq n_c \\ \text{constraints} \end{cases} \tag{5}$$

Define the corresponding prediction vectors as:

$$\underset{\rightarrow}{y} = \begin{bmatrix} y_{k+n} \\ y_{k+2n} \\ \vdots \\ y_{k+n_y n} \end{bmatrix}; \; \underset{\rightarrow}{u} = \begin{bmatrix} \underset{\rightarrow}{u}_1 \\ Z \end{bmatrix}; \quad \underset{\rightarrow}{u}_1 = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+n_c-1} \end{bmatrix}$$

where Z is a vector of zeros and it is noted that the output can only be predicted every nth sample due to the limitations of the model (4). However, the input can be updated every sample. Assuming the state x is available (via an observer) the prediction model takes the form

$$\underset{\rightarrow}{y} = \underbrace{[H_1|H_2]}_{H} \begin{bmatrix} \underset{\rightarrow}{u}_1 \\ Z \end{bmatrix} + Px_k;$$
$$H = \begin{bmatrix} \Theta & 0 & 0 & \ldots \\ \Gamma\Theta & \Theta & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}; \quad P = \begin{bmatrix} \Gamma \\ \Gamma^2 \\ \vdots \end{bmatrix} \tag{6}$$

where the partition of H is conformal with that of $\underset{\rightarrow}{u}$. One can now subsitute this prediction into (5) to derive the first n_c steps of the optimal control trajectory as:

$$\underset{\rightarrow}{u}_1 - Lu_{ss} = [H_1^T H_1 + \lambda I] H_1^T P(x - x_{ss}) \tag{7}$$

where L is an n_c vector of ones and u_{ss}, x_{ss} depend upon r and a disturbance estimate.

Remark 3.1. The main weakness of this approach (Rossiter *et al*, 2003) is the assumption that in the predictions $u_{k+i} = u_{ss}$. This assumption ensures the number of d.o.f. (n_c) is small. Where $n_c < n$ in particular the input signal has large discontinuities which are not removed by the usual receding horizon arguments as the receding horizon update takes place only every n samples in the lifted framework.

3.2 FHMPC with inferential control

In inferential control, one assumes that a fast rate model is available. Hence one can update the control optimisation at the fast sample rate, albeit the estimates of u_{ss}, x_{ss} are only updated at the slow rate. The advantage of such a change is that one no longer has to deal with the discontinuites within the input signal. What is not obvious is how to compare IC and lifting based approaches. One would expect IC control to be better simply because the receding horizon update is faster and this will be demonstrated. However this may not be a logical comparison:

- Due to modelling restrictions, lifting based MPC can only cost every nth value of the predicted output (5). No account can be taken of the unknown intersample output behaviour and this may be oscillatory (Tangirala *et al*, 2001).
- With IC one can estimate intersample outputs and hence it would be more appropriate to use the cost function of (1).

For simplicity we compare lifting and IC FH algorithms with the cost of (5). However it is noted that in practice if one were to adopt IC methods, then it would be better to use cost (1).

3.3 Example contrasting lifting and inferential control with FHMPC

Consider an example with a fast rate state space model

$$x_{k,l+1} = \begin{bmatrix} 0.3 & 0.5 \\ 0.1 & 0.9 \end{bmatrix} x_{k|l} + \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} u_{k|l}; \quad y_k = [1 \;\; 0] x_k \tag{8}$$

For the lifted algorithm one would assume that only the equivalent model of form (4) is known. Assume that the output is sampled 4 times slower than the input, i.e. $n = 4$. The FHMPC algorithm of (7) and (5) is implemented for $n_u = 1$ with $n_y = 8, \;\; \lambda = 1$. The simulations are displayed in figures 2a,b for outputs and inputs respectively; circles and dotted lines are used for the IC algorithm and crosses and solid lines are used for the lifted algorithm. The x-axis has units of the fast sample rate so new output measurements are given only every 4th sample. The corresponding closed-loop runtime costs are given in table 1 for $n_u = 1$.

Lifted algorithm	3.18
Inferential Control	2.21

Table 1: Closed-loop runtime costs J

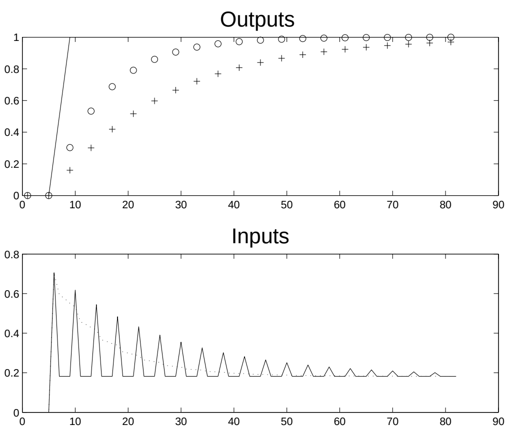

Figure 2: Simulations with $n_u = 1$

It is clear both from the table and the figure that the the use of a fast receding horizon update allowed in IC has given a dramatic improvement in performance, even though there has been not new output measurements. The limitation of the prediction assumption in FHMPC is very clear in figures 2b where it can be seen that the input moves to a *poor* default value, that is u_{ss}, during the later intersample periods. If one uses a fast receding horizon the negative effects of this poor assumption can be alleviated, as the only predicted value actually implemented is the current and the far future is continually updated. In the lifted framework, the first n moves are used and hence one is forced to use a poorly defined input trajectory.

3.4 Summary of FHMPC

FHMPC algorithms typically use input predictions which do not match the expected or desired closed-loop behaviour. This limitation is overcome by the use of the receding horizon concept whereby one updates the predictions at every sample instant so that there is a continual improvement on the initial assumption. Unfortunately in a lifted framework, the receding horizon update only takes place at a slow rate (every n samples) and as a consequence a naive use of FHMPC will cause the control law to inherit a poor input prediction. One obvious solution to this is to use IC, which was popular in some early papers on MR systems (Lee *et al*, 1992). IC allows the use of a fast receding horizon update to improve performance. However it should be emphasised that IC assumes the knowledge of a fast rate model which is not always a realistic assumption. Alternative ways around this are a topic of current research (Rossiter *et al*, 2003).

4. IHMPC IN THE MULTI-RATE ENVIRONMENT

4.1 The motivation for IHMPC

In conventional single rate MPC, there has been a move towards infinite horizons because of the attendant guarantee of stability that can be obtained. However there has been less thought given to understanding what underpins this guarantee as typically it is assumed simply to be a consequence of facilitating the definition of a Lyapunov function. However, there is a more significant change which was made use of in (Scokaert *et al*, 1996) and mentioned in section 2.2.

Ideally one wants the optimised open-loop predictions to match the actual closed-loop behaviour. Then the optimisation is well posed (unlike in FHMPC where one minimises over a class known to be different from the behaviour that will result). The consequence of this change is that the input discontinuities apparent in Fig. 2b should not occur, even in the lifted environment! To rephrase this, in the nominal case, the optimum input trajectory at time k will match exactly the optimum computed at the previous sample (in the absence of constraints). Hence whether one updates the control law at the fast rate or the slow rate, the control inputs will be the same.

We will illustrate this using the example of the previous section and the control implied by the optimisation of (2)[2]. Figure 3 below shows the simulation plots with both a lifted control law (crosses) and an IC control law (circles). Clearly the plots are identical. This implies that if one sets up the infinite horizon algorithm such that only outputs at the same sample rate are costed, then the use of lifting or IC will give the same closed-loop behaviour (in the constraint free case). However this is confusing because one would expect IC control to have more potential due to the faster receding horizon update. This apparent anomaly is discussed in section 5.

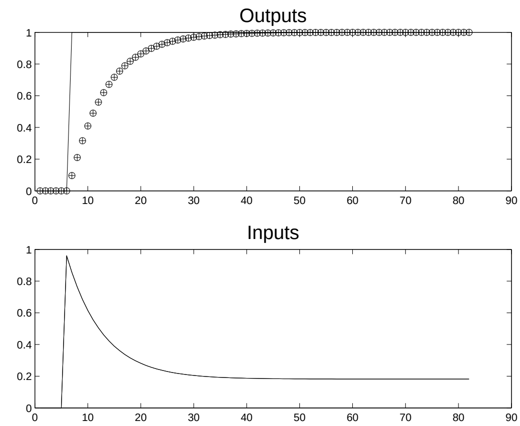

Figure 3: Simulations with optimal control

4.2 Infinite horizons need not imply that IC is equivalent to lifting

The example in the previous section made the assumption that the performance index J was the same for both the IC control and the lifting based control, that is they costed the outputs at the same sample rate, be it fast or slow. Of course in a true multi-rate framework, one does not have access to intermediate output estimates without a fast rate model. So one would use IC if a fast rate model were available and lifting otherwise. These would be based on different performance indices hence giving different control. Logically the lifted approach could not give as tight control over the unmeasured and hence uncontrolled intersample outputs (Tangirala *et al*, 2001). However there is a more noticeable difference which is discussed next.

[2] Assume that the lifted algorithm has access to intersample output estimates. One also gets the same result if both algorithms assume the cost of (5).

5. THE IMPACT OF CONSTRAINTS AND COMPUTATIONAL LOAD ON ALGORITHM SET UP

The conclusions of the previous two sections are contradictory. They imply that if one uses FHMPC then there are significant benefits from using IC. However if one uses IHMPC, then there are no benefits, that is one can obtain just as good control with an algorithm updating the control actions only at the slow rate. But, these conclusions apply to the constraint free case only, that is in the presence of constraints the global optimal input trajectory may not be known.

A popular (Rossiter *et al*, 1998) reparameterisation of the IHMPC optimisation (2) is given as

$$\min_{c_i,\ i=0,\ldots,n_c-1} J = \sum_{i=0}^{n_c-1} c_i^T c_i$$
$$s.t. \begin{cases} u_i - u_{ss} = -K(x_i - x_{ss}) + c_i,\ i < n_c \\ u_i - u_{ss} = -K(x_i - x_{ss}),\ i \geq n_c \\ \text{constraints} \end{cases} \tag{9}$$

Typically the global optimal requires $c_i \neq 0,\ i \geq n_c$ that is the global optimal differs from the unconstrained optimal for p steps where $p > n_c$; this is not allowed for in the prediction class so the global optimal can be reached in the optimisation. In this case it is evident that a fast receding horizon approach will give benefits as the speed of the receding horizon update governs the rate at which new d.o.f., in this case c_i, are introduced into the optimisation. Although no new observations appear at "inter-observation" instants, nevertheless the solution of the optimisation (2) does change, moving closer to the global optimal with each extra d.o.f., and hence there is a major advantage in using IC where that is possible.

5.1 Numerical example

Next a simple simulation study is used to illustrate the point that a IHMPC using IC outperforms a lifting based approach in the presence of constraints. Consider a model represented by the state equation:

$$x_{k+1} = \begin{bmatrix} 1.4 & -0.105 & -0.108 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x_k + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u_k$$
$$y = [0.5\ \ 0.75\ \ 0.5] x_k \tag{10}$$

and n is taken to be 5. For a unit set point change simulations are displayed in Figs. 4a, 4b for the constraint free case and Figs. 4c, 4d with constraints $|u| \leq 0.06$ and $|u_i - u_{i-1}| \leq 0.03$. The solid lines are with a lifted algorithm and the dotted lines represent the IC. The runtime costs J are summarised in table 2.

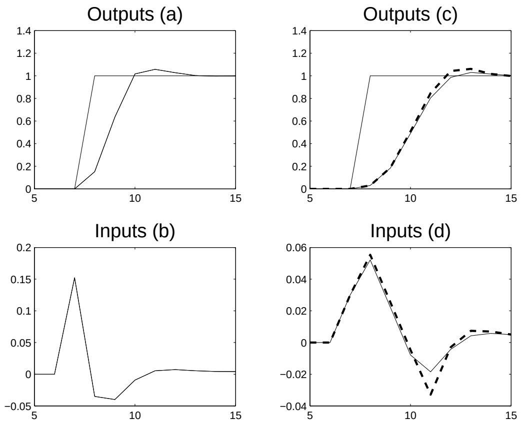

Figure 4: Simulations with optimal control

Algorithm	Figs 4a,b	Figs 4c,d
Lifted	1.11	1.946
IC	1.11	1.926

Table 2: Runtime costs

It is clear that IC has outperformed the lifted algorithm when constraints are active, although in this case by only a small amount. Larger differences will occur when the prediction class available is further from the global optimal, such as may arise with non-minimum phase and unstable systems.

5.2 Conclusion

The differences between FHMPC and IHMPC for MR systems: It was shown that the assumption, usual in FHMPC, that the predicted input move to a fixed value after n_c steps does not mesh well with MR control design. This is because the assumption is made to reduce computation not to improve control and does not match expected closed-loop behaviour well enough. Good control is recovered only by applying the receding horizon concept at a fast enough update rate. Conversely IHMPC techniques are setup to ensure a good match between predictions and expected closed-loop behaviour. Hence in this case the slow rate algorithm moves across to the MR case with a far smaller (zero for some algorithms) deterioration in performance.

The advantages of updating control with a fast receding horizon based on a FR internal model: IHMPC is identical with a FR update or lifting only in the case where the global optimal is in the class of allowable predictions. Usually restrictions to the number of d.o.f. imply this is not the case and hence one can improve performance by introducing more d.o.f. Clearly the faster the rate of receding horizon update the more quickly extra d.o.f. can be introduced to improve performance. Hence IC will always outperform lifting during constraint handling, even for IHMPC in the nominal case.

The weakness of these conclusions is the implicit assumption that one should use IC control as it gives better control for FHMPC and IHMPC. Also there is also an implication that IHMPC should always be prefered. However this is a simplistic. FR models are not always available and there is still study required to analyse their reliability. Also work in progress (Rossiter *et al*, 2003) is looking at means of obtaining control of similar quality to that obtained with IC, but based only on a lifted model. The argument of finite or infinite horizons is well known in the single rate literature of MPC and will not be repeated here.

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