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Numerical Discretization Methods for Linear Quadratic Control Problems with Time Delays

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Abstract: This paper presents the numerical discretization methods of the continuous-time linear-quadratic optimal control problems (LQ-OCPs) with time delays. We describe the weight matrices of the LQ-OCPs as differential equations systems, allowing us to derive the discrete equivalent of the continuous-time LQ-OCPs. Three numerical methods are introduced for solving proposed differential equations systems: 1) the ordinary differential equation (ODE) method, 2) the matrix exponential method, and 3) the step-doubling method. We implement a continuous-time model predictive control (CT-MPC) on a simulated cement mill system, and the objective function of the CT-MPC is discretized using the proposed LQ discretization scheme. The closed-loop results indicate that the CT-MPC successfully stabilizes and controls the simulated cement mill system, ensuring the viability and effectiveness of LQ discretization.

Keywords: Linear Quadratic Optimal Control, Numerical Discretization, Time Delay Systems

1. INTRODUCTION

Time delays are common in many industrial processes, shaping control systems' trajectories by not solely relying on the present state but also integrating their history. These delays can significantly influence the robustness and control performance (Gu et al., 2003; Lee, 1995).The linear-quadratic optimal control problems (LQ-OCPs) with time delays find extensive practical applications and needs tailored algorithms for the system identification (Jørgensen and Jørgensen, 2007a,b,c) as well as in the linear quadratic Gaussian (LQG) control and model predictive control (MPC) implementation (Frison and Jørgensen, 2013; Jørgensen et al., 2012). Therefore, there is a need for discretization methods tailored for LQ-OCPs with time delays.

Although research on discretization methods for time-delay systems is extensive (Franklin et al., 1990; Hendricks et al., 2008; Kassas and Dunia, 2006; Otto and Radons, 2017), studies on discretization methods for continuous-time LQ-OCP, particularly incorporating with time delays, are limited. Hendricks et al. (2008) introduced the discrete-time approximation for LQ-OCPs without time delays, employing the zero-order hold (ZOH) parameterization on system states and inputs. Åström (1970), Åström and Wittenmark (2011) and Franklin et al. (1990) provided analytical expressions for equivalent discrete weighting matrices by extending continuous-time cost functions, which can be solved using the matrix exponential method (Al-Mohy and Higham, 2011; Moler and Van Loan, 1978). The discretization and solution methods for continuous-time linear-quadratic regulator (CLQR) problems are described by Pannocchia et al. (2015, 2010). They used the matrix exponential method to obtain the discrete equivalent and proposed a novel computational procedure for solving the optimal control problem.

On the other hand, stochastic LQ-OCPs can be valuable in some scenarios, such as Conditional Value-at-Risk (CVaR) optimization problems (Capolei et al., 2015; Rockafellar and Uryasev, 2001). Åström (1970) and Åström and Wittenmark (2011) introduced the analytic expressions of continuous-time stochastic LQ-OCPs' cost function and described its expectation. However, as far as we know, the existing literature has not addressed the case of time delays or stochastic systems. In this paper, we thus focus on discretization methods for deterministic and stochastic LQ-OCPs with time delays. The key problems that we address in this paper:

1. Formulation of differential equation systems for LQ discretization with time delays
2. Numerical methods for solving resulting systems of differential equations

This paper is organized as follows. Section 2 introduces the discretization of time-delay systems and deterministic and stochastic LQ-OCPs with time delays. The discrete weighting matrices are described as differential equation systems. Section 3 describes three numerical methods for solving proposed differential equation systems. We test the proposed numerical methods by a numerical experiment in Section 4 and give the conclusions in Section 5.

2. LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEMS

This section describes deterministic and stochastic LQ-OCPs with time delays.

2.1 Certainty equivalent LQ for deterministic time-delay systems

Consider a SISO, time-delay, linear state space model

$$\dot{x}(t) = A_c x(t) + B_c u(t-\tau), \quad (1a)$$
$$z(t) = C_c x(t) + D_c u(t-\tau), \quad (1b)$$

where $x \in \mathbb{R}^{n_x \times 1}$ is the state, $u \in \mathbb{R}^{n_u \times 1}$ is the input and $z \in \mathbb{R}^{n_z \times 1}$ is the output. $\tau \in \mathbb{R}_0^+$ is the input time delay.

Assuming piece-wise constant input $u(t) = u_k$ for $t_k \leq t < t_{k+1}$ and T_s is the sampling time. Note that $m \in \mathbb{Z}_0^+$ and $0 \leq v < 1$ are the integer and fractional time delay constants for $l = \tau/T_s = m - v$. By taking integral on both side, we obtain the solution of (1)

$$x(t) = A(t)x_k + B_o(t)\tilde{u}_k, \quad (2a)$$
$$z(t) = C_c x(t) + D_c u_{k-m} = C_c x(t) + D_o \tilde{u}_k, \quad (2b)$$

where $\tilde{u}_k = [u_{o,k}; u_k]$ is the augmented input vector and $u_{o,k} = [u_{k-m}; \cdots; u_{k-1}]$ is the historical input vector.

The corresponding system matrices are

$$A(t) = e^{A_c t}, \qquad B_1(t) = \int_0^t A(s) ds B_{1c}, \quad (3a)$$
$$A_v(t) = e^{vA_c t}, \qquad B_2(t) = v\int_0^t A_v(s) ds (B_{2c} - B_{1c}), \quad (3b)$$
$$B_{1c} = B_c e^1_{m+1}, \quad B_{2c} = B_c e^2_{m+1}, \quad (3c)$$
$$D_o = D_c e^1_{m+1}, \quad B_o(t) = B_1(t) + B_2(t), \quad (3d)$$

where $e^p_{m+1} = [0 \cdots I \cdots 0]$ for $p = 1, 2, \ldots, m+1$ is an unit vector for selecting $u_{k-(m+1)+p}$ from $\tilde{u}_k$ such that $u_{k-m+1+p} = e^p_{m+1}\tilde{u}_k$.

Set $t = T_s$, and we obtain

$$\overbrace{\begin{bmatrix} x_{k+1} \\ \hline u_{o,k+1} \end{bmatrix}}^{=\tilde{x}_{k+1}} = \overbrace{\begin{bmatrix} A & B_{o,1} \\ \hline 0 & I_A \end{bmatrix}}^{=\tilde{A}} \overbrace{\begin{bmatrix} x_k \\ \hline u_{o,k} \end{bmatrix}}^{=\tilde{x}_k} + \overbrace{\begin{bmatrix} B_{o,2} \\ \hline I_B \end{bmatrix}}^{=\tilde{B}} u_k, \quad (4a)$$

$$z_k = \overbrace{\begin{bmatrix} C_c & D_{o,1} \end{bmatrix}}^{=\tilde{C}} \tilde{x}_k + \overbrace{D_{o,2}}^{=\tilde{D}} u_k, \quad (4b)$$

where the system matrices are

$$B_{o,1} = B_o(:, 1: end - n_u), B_{o,2} = B_o(:, mn_u : end), \quad (5a)$$
$$D_{o,1} = D_o(:, 1: end - n_u), D_{o,2} = D_o(:, mn_u : end), \quad (5b)$$
$$I_A = \begin{bmatrix} 0 & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I \\ 0 & 0 & \ldots & 0 \end{bmatrix}, \qquad I_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}. \quad (5c)$$

To present time delays in one single MIMO state space model, we consider the following $n_z \times n_u$ SISO systems

$$\dot{x}_{ij}(t) = A_{c,ij} x_{ij}(t) + B_{c,ij} u_j(t - \tau_{ij}), \quad (6a)$$
$$z_{ij}(t) = C_{c,ij} x_{ij}(t) + D_{c,ij} u_j(t - \tau_{ij}), \quad (6b)$$

with

$$u = [u_1; u_2; \ldots, u_{n_u}], \quad x = [x_{11}; x_{21}; \ldots; x_{n_z n_u}], \quad (6c)$$
$$z = [z_1; z_2; \ldots; z_{n_z}], \quad z_i = \sum_{j=1}^{n_u} z_{ij}, \quad (6d)$$

where $A_{c,ij}$, $B_{c,ij}$, $C_{c,ij}$, $D_{c,ij}$ and τ_{ij} for $i = 1, 2, \ldots, n_z$ and $j = 1, 2, \ldots, n_u$ are parameters of the $[i, j]$ SISO system. It describes the dynamics from the jth input to ith output. The corresponding time delay constants are denoted as $l_{ij} = m_{ij} - v_{ij} = \tau_{ij}/T_s$.

The historical input vector is $u_{o,k} = [u_{k-\bar{m}}; \ldots; u_{k-1}]$ with $\bar{m} = \max\{m_{ij}\}$. For the augmented input vector $\tilde{u}$, we have the following expression in the MIMO case

$$u_{j,k-(\bar{m}+1)+p} = e_j u_{k-(\bar{m}+1)+p} = e_j E^p_{\bar{m}+1}\tilde{u}_k, \quad (7)$$

where $E^p_{\bar{m}+1}$ is an unit matrix that select $u_{k-(\bar{m}+1)+p}$ from $\tilde{u}_k$ and $e_j = [0 \cdots 1 \cdots 0]$ for $j = 1, 2, \ldots, n_u$ is an unit vector for selecting j^{th} input from $u_{k-(\bar{m}+1)+p}$.

Stacking all SISO systems' solutions, we can obtain the solution of the MIMO time-delay model that has the same expressions as the SISO case (2). The matrices A, A_v, B_1, B_2 have the same expressions introduced in (3a) and (3b), and their coefficients become

$$A_c = \text{diag}(A_{c,11}, A_{c,21}, \ldots, A_{c,n_z n_u}), \quad (8a)$$
$$B_{1c} = [B_{1c,11}; B_{1c,21}; \ldots; B_{1c,n_z n_u}], \quad (8b)$$
$$B_{2c} = [B_{2c,11}; B_{2c,21}; \ldots; B_{2c,n_z n_u}], \quad (8c)$$
$$V = \text{diag}(V_{11}, V_{21}, \ldots, V_{n_z n_u}), \quad (8d)$$

where $V_{ij} = I v_{ij}$, $B_{1c,ij} = B_{c,ij} e_j E^{m_{ij}}_{\bar{m}+1}$ and $B_{2c,ij} = B_{c,ij} e_j E^{m_{ij}+1}_{\bar{m}+1}$.

The matrices C_c and D_o output function matrices become

$$C_c = [\bar{C}_1, \bar{C}_2, \cdots, \bar{C}_{n_u}], \quad (9a)$$
$$D_o = [\bar{D}_1; \bar{D}_2; \cdots; \bar{D}_{n_z}], \quad (9b)$$
$$\bar{C}_j = \text{diag}(C_{c,1j}, C_{c,2j}, \ldots, C_{c,n_z j}), \quad (9c)$$
$$\bar{D}_{c,i} = \sum_{j=1}^{n_u} D_{c,ij} e_j E^{m_{ij}}_{\bar{m}+1}, \quad \text{for } i = 1, 2, \ldots, n_z. \quad (9d)$$

Consequently, set $t = T_s$, we can obtain the discrete-time system of (6) with same expressions introduced in the SISO case (4) with $B_{o,1} = B_o(:, 1: end - n_u)$, $B_{o,2} = B_o(:, \bar{m}n_u : end)$, $D_{o,1} = D_o(:, 1: end - n_u)$ and $D_{o,2} = D_o(:, \bar{m}n_u : end)$.

Proposition 1. The system of differential equations

$$\dot{A}(t) = A_c A(t), \qquad A(0) = I, \quad (10a)$$
$$\dot{A}_v(t) = V A_c A_v(t), \qquad A_v(0) = I, \quad (10b)$$
$$\dot{B}_1(t) = A(t) B_{1c}, \qquad B_1(0) = 0, \quad (10c)$$
$$\dot{B}_2(t) = A_v(t) \bar{B}_{2c}, \qquad B_2(0) = 0, \quad (10d)$$

where

$$\bar{B}_{2c} = V(B_{2c} - B_{1c}), \quad (11)$$

may be used to compute $(A = A(T_s)$, $B_o = B_1(T_s) + B_2(T_s))$ for the discretization of MIMO time-delay systems.

2.2 Deterministic linear-quadratic optimal control problem

Consider the following deterministic LQ-OCP

$$\min_{x,u,z,\tilde{z}} \phi = \int_{t_0}^{t_0+T} l_c(\tilde{z}(t)) dt \quad (12a)$$
$$s.t. \quad x(t_0) = \hat{x}_0, \quad (12b)$$
$$u(t) = u_k, \; t_k \leq t < t_{k+1}, \; k \in \mathcal{N}, \quad (12c)$$
$$\dot{x}(t) = A_c x(t) + B_c u(t-\tau), t_0 \leq t < t_0 + T, \quad (12d)$$
$$z(t) = C_c x(t) + D_c u(t-\tau), t_0 \leq t < t_0 + T, \quad (12e)$$
$$\bar{z}(t) = \bar{z}_k, \; t_k \leq t < t_{k+1}, \; k \in \mathcal{N}, \quad (12f)$$
$$\tilde{z}(t) = z(t) - \bar{z}(t), \; t_0 \leq t < t_0 + T, \quad (12g)$$

with the stage cost function

$$l_c(\tilde{z}(t)) = \frac{1}{2}\|W_z \tilde{z}(t)\|_2^2 = \frac{1}{2}\tilde{z}(t)' Q_c \tilde{z}(t), \quad (13)$$

where $T = NT_s$ and $N \in \mathbb{Z}^+$ for $\mathcal{N} = 0, 1, \ldots, N-1$ is the control interval. $Q_c = W_z' W_z$ is a semi-positive definite weight matrix.

We assume piece-wise constant inputs $u(t) = u_k$ and target variables $\bar{z}(t) = \bar{z}_k$ for $t_k \leq t < t_{k+1}$. Replacing $z(t)$ with the expressions introduced in (2), the discrete-time equivalent of (12) can be defined as

$$\min_{x,u} \quad \phi = \sum_{k \in \mathcal{N}} l_k(x_k, u_k) \tag{14a}$$

$$s.t. \quad x_0 = \hat{x}_0, \tag{14b}$$

$$x_{k+1} = Ax_k + Bu_k, \quad k \in \mathcal{N}, \tag{14c}$$

where the states x and system matrices A, B, C and D are in the augmented form and correspond to $\tilde{x}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ described in (4).

The stage cost function $l_k(x_k, u_k)$ is

$$l_k(x_k, u_k) = \frac{1}{2} \begin{bmatrix} x_k \\ u_k \end{bmatrix}' Q \begin{bmatrix} x_k \\ u_k \end{bmatrix} + q_k' \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \rho_k, \quad k \in \mathcal{N}, \tag{15}$$

and its affine term's coefficients and the constant term are

$$q_k = M\bar{z}_k, \; \rho_k = \int_{t_k}^{t_{k+1}} l_c(\bar{z}_k) dt = l_c(\bar{z}_k) T_s, \; k \in \mathcal{N}. \tag{16}$$

Proposition 2. The system of differential equations

$$\dot{A}(t) = A_c A(t), \qquad A(0) = I, \tag{17a}$$

$$\dot{A}_v(t) = V A_c A_n(t), \qquad A_v(0) = I, \tag{17b}$$

$$\dot{B}_1(t) = A(t) B_{1c}, \qquad B_1(0) = 0, \tag{17c}$$

$$\dot{B}_2(t) = A_v(t) \bar{B}_{2c}, \qquad B_2(0) = 0, \tag{17d}$$

$$\dot{Q}(t) = \Gamma(t)' Q_c \Gamma(t), \qquad Q(0) = 0, \tag{17e}$$

$$\dot{M}(t) = -\Gamma(t)' M_c, \qquad M(0) = 0, \tag{17f}$$

where

$$\bar{B}_{2c} = V(B_{2c} - B_{1c}), \; \Gamma(t) = [C_c \; D_o] \begin{bmatrix} A(t) & B_o(t) \\ 0 & I \end{bmatrix}, \tag{18}$$

may be used to compute ($A = A(T_s)$, $B_o = B_1(T_s) + B_2(T_s)$, $Q = Q(T_s)$, $M = M(T_s)$) for the discretization of deterministic LQ-OCPs with time delays.

2.3 Certainty equivalent LQ control for a stochastic time-delay system

Consider the following linear, stochastic, time-delay system

$$d\boldsymbol{x}(t) = (A_c \boldsymbol{x}(t) + B_c u(t - \tau))\, dt + G_c d\boldsymbol{\omega}(t), \tag{19a}$$

$$\boldsymbol{z}(t) = C_c \boldsymbol{x}(t) + D_c u(t - \tau). \tag{19b}$$

and the initial state $\boldsymbol{x}_0 \sim N(\hat{x}_0, P_0)$ and stochastic variable $d\boldsymbol{\omega}(t) \sim N_{iid}(0, Idt)$.

Based on expressions obtained in the deterministic case (2) and (4), we can define the discrete-time system of (19) as

$$\tilde{\boldsymbol{x}}_{k+1} = \tilde{A} \tilde{\boldsymbol{x}}_k + \tilde{B} u_k + \tilde{\boldsymbol{w}}_k, \tag{20a}$$

$$\boldsymbol{z}_k = \tilde{C} \tilde{\boldsymbol{x}}_k + \tilde{D} u_k, \tag{20b}$$

and $\tilde{\boldsymbol{w}}_k = [\boldsymbol{w}_k; 0]$ is expressed in terms of Itô

$$\boldsymbol{w}_k = \int_{t_k}^{t_{k+1}} A(t) G_c d\boldsymbol{\omega}(t), \quad \boldsymbol{w}_k \sim N_{iid}(0, R_{ww}), \tag{20c}$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are identical to the deterministic case (4) and $R_{ww} = \mathrm{Cov}(\boldsymbol{w}_k)$ is the covariance matrix.

Proposition 3. The system of differential equations

$$\dot{A}(t) = A_c A(t), \qquad A(0) = I, \tag{21a}$$

$$\dot{A}_v(t) = V A_c A_v(t), \qquad A_v(0) = I, \tag{21b}$$

$$\dot{B}_1(t) = A(t) B_{1c}, \qquad B_1(0) = 0, \tag{21c}$$

$$\dot{B}_2(t) = A_v(t) \bar{B}_{2c}, \qquad B_2(0) = 0, \tag{21d}$$

$$\dot{R}_{ww} = \Phi(t) \Phi(t)', \qquad R_{ww}(0) = 0, \tag{21e}$$

where

$$\bar{B}_{2c} = V(B_{2c} - B_{1c}), \quad \Phi(t) = A(t) G_c, \tag{22}$$

may be used to compute ($A = A(T_s)$, $B_o = B_1(T_s) + B_2(T_s)$, $R_{ww} = R_{ww}(T_s)$) for the discretization of stochastic time-delay models.

2.4 Stochastic linear-quadratic optimal control problem

Consider the stochastic LQ-OCP governed by (19)

$$\min_{\boldsymbol{x}, u, \boldsymbol{z}, \tilde{\boldsymbol{z}}} \psi = E\left\{ \phi = \int_{t_0}^{t_0 + T} l_c(\tilde{\boldsymbol{z}}(t)) dt \right\} \tag{23a}$$

$$s.t. \quad \boldsymbol{x}(t_0) \sim N(\hat{x}_0, P_0), \tag{23b}$$

$$d\boldsymbol{\omega}(t) \sim N_{iid}(0, Idt), \tag{23c}$$

$$u(t) = u_k, \quad t_k \leq t < t_{k+1}, \, k \in \mathcal{N}, \tag{23d}$$

$$d\boldsymbol{x}(t) = (A_c \boldsymbol{x}(t) + B_c u(t - \tau)) dt + G_c d\boldsymbol{\omega}(t), \tag{23e}$$

$$\boldsymbol{z}(t) = C_c \boldsymbol{x}(t) + D_c u(t - \tau), \tag{23f}$$

$$\bar{z}(t) = \bar{z}_k, \quad t_k \leq t < t_{k+1}, \, k \in \mathcal{N}, \tag{23g}$$

$$\tilde{\boldsymbol{z}}(t) = \boldsymbol{z}(t) - \bar{z}(t). \tag{23h}$$

The corresponding discrete-time stochastic LQ-OCP is

$$\min_{x,u} \quad \psi = E\left\{ \phi = \sum_{k \in \mathcal{N}} l_k(\boldsymbol{x}_k, u_k) + l_{s,k}(\boldsymbol{x}_k, u_k) \right\} \tag{24a}$$

$$s.t. \quad \boldsymbol{x}_0 \sim N(\hat{x}_0, P_0), \tag{24b}$$

$$\boldsymbol{w}_k \sim N_{iid}(0, R_{ww}), \qquad k \in \mathcal{N}, \tag{24c}$$

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + \boldsymbol{w}_k, \qquad k \in \mathcal{N}, \tag{24d}$$

where the variables $\boldsymbol{x}$, $\boldsymbol{w}$ and system matrices A, B, C, D correspond to $\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{w}}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ described in (20).

The stage cost function $l_k(\boldsymbol{x}_k, u_k)$ is

$$l_k(\boldsymbol{x}_k, u_k) = \frac{1}{2} \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix}' Q \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} + q_k' \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} + \rho_k, \tag{25}$$

and $l_{s,k}(\boldsymbol{x}_k, u_k)$ is

$$l_{s,k}(\boldsymbol{x}_k, u_k) = \int_{t_k}^{t_{k+1}} \frac{1}{2} \boldsymbol{w}(t)' Q_{c,ww} \boldsymbol{w}(t) + \boldsymbol{q}_{s,k}' \boldsymbol{w}(t) dt, \tag{26}$$

where Q, q_k, and ρ_k are identical to the deterministic case described in (16) and (17). The state $\boldsymbol{w}(t)$ and system matrices $Q_{c,ww}$ and $\boldsymbol{q}_{s,k}$ of $l_{s,k}(\boldsymbol{x}_k, u_k)$ are

$$\boldsymbol{w}(t) = \int_0^t A(s) G_c d\boldsymbol{\omega}(s), \qquad Q_{c,ww} = C_c' Q_c C_c, \tag{27a}$$

$$\tilde{\boldsymbol{z}}_k = \Gamma(t) \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} - \bar{z}_k, \qquad \boldsymbol{q}_{s,k} = C_c' Q_c \tilde{\boldsymbol{z}}_k. \tag{27b}$$

Based on the previous work by Åström (1970), we can rewrite (24) as

$$\min_{x,u} \psi = \sum_{k \in \mathcal{N}} l_k(x_k, u_k) + \boldsymbol{\rho}_{s,k} \tag{28a}$$

$$s.t. \quad x_0 = \hat{x}_0, \tag{28b}$$

$$x_{k+1} = Ax_k + Bu_k, \quad k \in \mathcal{N}, \tag{28c}$$

Algorithm 1 ODE method for LQ Discretization

Input: $(A_c, B_c, G_c, C_c, D_c, Q_c, T_s, N)$
Output: (A, B, Q, M, R_{ww})
  Set initial states
  $(k = 0,\ A_k = I,\ A_{v,k} = I,\ B_{1,k} = 0,\ B_{2,k} = 0,\ Q_k = 0,\ M_k = 0,\ R_{ww,k} = 0)$
  Compute the step size $h = \frac{T_s}{N}$
  Use (33) to compute (Λ_i, $\Lambda_{v,i}$, $\Theta_{1,i}$, $\Theta_{2,i}$)
  Use (34) to compute (Λ, Λ_v, Θ_1, Θ_2)
  while $k < N$ **do**
    Use (32) to update (A_k, $B_{o,k}$, Q_k, M_k, $R_{ww,k}$)
    Set $k \leftarrow k + 1$
  end while
  Get system matrices $(A(T_s) = A_k, B(T_s) = B_k, Q(T_s) = Q_k, M(T_s) = M_k, R_{ww}(T_s) = R_{ww,k})$

where $\boldsymbol{\rho}_{s,k}$ is

$$\boldsymbol{\rho}_{s,k} = \frac{1}{2}\left[\operatorname{tr}\left(Q\bar{P}_k\right) + \int_{t_k}^{t_{k+1}} \operatorname{tr}\left(Q_{c,ww}P_w\right) dt\right], \quad (29a)$$

and

$$\begin{bmatrix}\boldsymbol{x}_k \\ u_k\end{bmatrix} \sim N(m_k, \bar{P}_k),\ m_k = \begin{bmatrix}x_k \\ u_k\end{bmatrix},\ \bar{P}_k = \begin{bmatrix}P_k & 0 \\ 0 & 0\end{bmatrix}, \quad (29b)$$

$$P_{k+1} = AP_kA' + R_{ww}, \qquad P_w = \operatorname{Cov}\left(\boldsymbol{w}(t)\right). \quad (29c)$$

Proposition 4. The system of differential equations

$$\begin{aligned}
\dot{A}(t) &= A_cA(t), & A(0) &= I, & (30a)\\
\dot{A}_v(t) &= VA_cA_v(t), & A_v(0) &= I, & (30b)\\
\dot{B}_1(t) &= A(t)B_{1c}, & B_1(0) &= 0, & (30c)\\
\dot{B}_2(t) &= A_v(t)\bar{B}_{2c}, & B_2(0) &= 0, & (30d)\\
\dot{Q}(t) &= \Gamma(t)'Q_c\Gamma(t), & Q(0) &= 0, & (30e)\\
\dot{M}(t) &= -\Gamma(t)'Q_c, & M(0) &= 0, & (30f)\\
\dot{R}_{ww}(t) &= \Phi(t)\Phi(t)', & R_{ww}(0) &= 0, & (30g)
\end{aligned}$$

where

$$\bar{B}_{2c} = V(B_{2c} - B_{1c}), \qquad \Phi(t) = A(t)G_c, \quad (31a)$$

$$\Gamma(t) = [C_c\ D_o]\begin{bmatrix}A(t) & B_o(t) \\ 0 & I\end{bmatrix}, \quad (31b)$$

may be used to compute ($A = A(T_s)$, $B_o = B_1(T_s) + B_2(T_s)$, $Q = Q(T_s)$, $M = M(T_s)$, $R_{ww} = R_{ww}(T_s)$) for the discretization of stochastic LQ-OCPs with time delays.

3. NUMERICAL DISCRETIZATION METHODS

In this section, we introduce numerical methods for solving proposed differential equations systems.

3.1 Ordinary differential equation methods

Consider an s-stage fixed-time-step ODE method with $N \in \mathbb{Z}^+$ integration steps and the time step $h = \frac{T_s}{N}$. Define $a_{i,j}$ and b_i for $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, s$ are the Butcher tableau parameters of the ODE method, and we can compute (A, B_o, Q, M, R_{ww}) as

$$A_{k+1} = \Lambda A_k, \qquad B_{1,k+1} = B_{1,k} + \Theta_1 A_k \bar{B}_{1c}, \quad (32a)$$

$$A_{v,k+1} = \Lambda_v A_{v,k}, \qquad B_{2,k+1} = B_{2,k} + \Theta_2 A_{v,k}\bar{B}_{2c}, \quad (32b)$$

$$B_{o,k} = B_{1,k} + B_{2,k},\ \Gamma_{k+1} = \begin{bmatrix}A_{k+1} & B_{o,k+1} \\ 0 & I\end{bmatrix}, \quad (32c)$$

$$M_{k+1} = M_k + h\sum_{i=1}^{s} b_i\Gamma'_{k,i}\bar{M}_c, \quad (32d)$$

$$Q_{k+1} = Q_k + h\sum_{i=1}^{s} b_i\Gamma'_{k,i}\bar{Q}_c\Gamma_{k,i}, \quad (32e)$$

$$R_{ww,k+1} = R_{ww,k} + h\sum_{i=1}^{s} b_iA_{k,i}\bar{R}_{ww,c}A'_{k,i}. \quad (32f)$$

$\bar{B}_{1c} = hB_{1c}$, $\bar{B}_{2c} = hV\left(B_{2c} - B_{1c}\right)$, $\bar{M}_c = -\left[C_c\ D_o\right]' Q_c$, $\bar{Q}_c = -M_c\left[C_c\ D_o\right]$ and $\bar{R}_{ww,c} = G_cG_c'$ are constant. Note that $A_{k,i}$, $A_{v,k,i}$, $B_{1,k,i}$, $B_{2,k,i}$ and $\Gamma_{k,i}$ for $i = 1, 2, \ldots, s$ are stage variables of (A, A_v, B_1, B_2, Γ), we then have

$$A_{k,i} = A_k + h\sum_{j=1}^{s} a_{i,j}\dot{A}_{k,j} = \Lambda_iA_k, \quad (33a)$$

$$A_{v,k,i} = A_{v,k} + h\sum_{j=1}^{s} a_{i,j}\dot{A}_{v,k,j} = \Lambda_{v,i}A_{v,k}, \quad (33b)$$

$$B_{1,k,i} = B_{1,k} + h\sum_{j=1}^{s} a_{i,j}\dot{B}_{1,k,j} = B_{1,k} + \Theta_{1,i}A_k\bar{B}_{1c}, \quad (33c)$$

$$B_{2,k,i} = B_{2,k} + h\sum_{j=1}^{s} a_{i,j}\dot{B}_{2,k,j} = B_{2,k} + \Theta_{2,i}A_{v,k}\bar{B}_{2c}, \quad (33d)$$

$$B_{o,k,i} = B_{1,k,i} + B_{2,k,i}, \quad \Gamma_{k,i} = \begin{bmatrix}A_{k,i} & B_{o,k,i} \\ 0 & I\end{bmatrix}, \quad (33e)$$

and we can compute coefficients Λ, Λ_v, Θ_1, and Θ_2 as

$$\Lambda = I + h\sum_{i=1}^{s} b_iA_c\Lambda_i, \qquad \Theta_1 = \sum_{i=1}^{s} b_i\Lambda_i, \quad (34a)$$

$$\Lambda_v = I + h\sum_{i=1}^{s} b_iVA_c\Lambda_{v,i}, \quad \Theta_2 = \sum_{i=1}^{s} b_i\Lambda_{v,i}. \quad (34b)$$

where the stage variable coefficients Λ_i, $\Lambda_{v,i}$, $\Theta_{1,i}$ and $\Theta_{2,i}$ are functions of Butcher tableau's parameters a and b.

In particular, Γ can be decomposed into the linear combination of A, A_v, B_1 and B_2, such that

$$\begin{aligned}
\Gamma(t) &= \begin{bmatrix}A(t) & B_o(t) \\ 0 & I\end{bmatrix} \\
&= \overbrace{\begin{bmatrix}A(t) & B_1(t) \\ 0 & I\end{bmatrix}}^{H_1(t)} + \overbrace{\begin{bmatrix}A_v(t) & B_2(t) \\ 0 & I\end{bmatrix}}^{H_2(t)} - \overbrace{\begin{bmatrix}A_v(t) & 0 \\ 0 & I\end{bmatrix}}^{H_3(t)} \\
&= E_1H(t)E_2,
\end{aligned} \quad (35a)$$

and

$$H_{1c} = \begin{bmatrix}A_c & B_{1c} \\ 0 & 0\end{bmatrix}, \qquad H_{2c} = \begin{bmatrix}\bar{A}_c & \bar{B}_{2c} \\ 0 & 0\end{bmatrix}, \quad (35b)$$

$$H_{3c} = \begin{bmatrix}A_{v,c} & 0 \\ 0 & 0\end{bmatrix},\ E_1 = [I, I, -I],\ E_2 = [I; I; I], \quad (35c)$$

where $H(t) = \operatorname{diag}(H_1(t), H_2(t), H_3(t)) = e^{H_ct}$ for $H_c = \operatorname{diag}(H_{1c}, H_{2c}, H_{3c})$ and $H_k(t) = e^{H_{kc}t}$ for $k = 1, 2, 3$.

Consequently, we compute $A(T_s) = A_N$, $B_o(T_s) = B_{o,N}$, $M(T_s) = M_N$, $Q(T_s) = Q_N$ and $R_{ww}(T_s) = R_{ww,N}$ with constant coefficients Λ, Λ_v, Θ_1 and Θ_2 using fixed-time-step ODE methods. Algorithm 1 describes the fixed-time-step ODE method for the LQ discretization with time delays.

3.2 Matrix exponential method

Based on formulas introduced by Van Loan (1978) and Moler and Van Loan (1978, 2003), we can discretize the LQ-OCP with time delays by solving the following matrix exponential problems

$$\begin{bmatrix}\Phi_{1,11} & \Phi_{1,12}\\ 0 & \Phi_{1,22}\end{bmatrix} = \exp\left(\begin{bmatrix}-H_c' & E_1'\bar{Q}_c E_1\\ 0 & H_c\end{bmatrix} t\right), \quad (36a)$$

$$\begin{bmatrix}I & \Phi_{2,12}\\ 0 & \Phi_{2,22}\end{bmatrix} = \exp\left(\begin{bmatrix}0 & I\\ 0 & H_c'\end{bmatrix} t\right), \quad (36b)$$

$$\begin{bmatrix}\Phi_{3,11} & \Phi_{3,12}\\ 0 & \Phi_{3,22}\end{bmatrix} = \exp\left(\begin{bmatrix}-A_c & \bar{R}_{ww,c}\\ 0 & A_c'\end{bmatrix} t\right), \quad (36c)$$

The elements of matrix exponential problems (36) are

$$\Phi_{1,22} = H(t) = \mathrm{diag}\left([H_1(t)\ H_2(t)\ H_3(t)]\right), \quad (37a)$$

$$\Phi_{1,12} = H(-t)' \int_0^t H(\tau)' E_1' \bar{Q}_c E_1 H(\tau) d\tau, \quad (37b)$$

$$\Phi_{2,12} = \int_0^t H(\tau)' d\tau, \quad (37c)$$

$$\Phi_{3,22} = A(t), \quad (37d)$$

$$\Phi_{3,12} = A(-t) \int_0^t A(\tau) \bar{R}_{ww,c} A(\tau)' d\tau. \quad (37e)$$

where H_c, $\bar{Q}_c$, $\bar{M}_c$, $\bar{R}_{ww,c}$ are introduced in (32) and (35).

Consequently, set $t = T_s$, we can compute (A, B_o, Q, M, R_{ww}) as

$$A(T_s) = \Phi_{1,22}(1:n_x, 1:n_x), \quad (38a)$$

$$B_o(T_s) = \Phi_{1,22}(1:n_x, n_x+1:\mathrm{end}), \quad (38b)$$

$$\Gamma(T_s) = E_1 \Phi_{1,22} E_2, \quad (38c)$$

$$Q(T_s) = E_2' \Phi_{1,22}' \Phi_{1,12} E_2, \quad (38d)$$

$$M(T_s) = E_2' \Phi_{2,12} \bar{M}_c, \quad (38e)$$

$$R_{ww}(T_s) = \Phi_{3,22}' \Phi_{3,12}. \quad (38f)$$

3.3 Step-doubling method

Consider the matrix $H(t) = e^{H_c t}$, and we can express it in the form of the differential equation as

$$\dot{H}(t) = H_c H(t), \qquad H(0) = I_h, \quad (39)$$

and its ODE expressions are

$$H_{k+1} = \Omega H_k, \quad \Gamma_k = \begin{bmatrix}A_k & B_{o,k}\\ 0 & I\end{bmatrix} = E_1 H_k E_2, \quad (40a)$$

$$H_{k,i} = \Omega_i H_k, \quad \Gamma_{k,i} = \begin{bmatrix}A_{k,i} & B_{o,k,i}\\ 0 & I\end{bmatrix} = E_1 H_{k,i} E_2, \quad (40b)$$

where I_h is an identity matrix that has the same dimension as H_c. $H_{k,i}$ for $i = 1, 2, \dots, s$ indicate the stage variables of $H(t)$ and their coefficients Ω_i are functions of Butcher tableau's parameters $a_{i,j}$ and b_i, such that $\Omega = I + h\sum_{i=1}^{s} b_i H_c \Omega_i$.

Consider the ODE expressions (32) and (40), the matrices

Table 1. The step-doubling functions

	Numerical expression	Step-doubling function
$\tilde{A}(N)$	Λ^N	$\tilde{A}(\frac{N}{2})\tilde{A}(\frac{N}{2})$
$\tilde{B}(N)$	$\sum_{i=0}^{N-1} \bar{\Lambda}^i$	$\tilde{B}(\frac{N}{2})\left(I + \tilde{A}(\frac{N}{2})\right)$
$\tilde{H}(N)$	Ω^N	$\tilde{H}(\frac{N}{2})\tilde{H}(\frac{N}{2})$
$\tilde{M}(N)$	$\sum_{i=0}^{N-1} \Omega^i$	$\tilde{M}(\frac{N}{2})\left(I + \tilde{H}(\frac{N}{2})\right)$
$\tilde{Q}(N)$	$\sum_{i=0}^{N-1} \left(\Omega^i\right)' \tilde{Q}_c \left(\Omega^i\right)$	$\tilde{Q}(\frac{N}{2}) + \tilde{H}(\frac{N}{2})'\tilde{Q}(\frac{N}{2})\tilde{H}(\frac{N}{2})$
$\tilde{R}(N)$	$\sum_{i=0}^{N-1} \left(\Lambda^i\right) \bar{R}_{ww,c} \left(\Lambda^i\right)'$	$\tilde{R}(\frac{N}{2}) + \tilde{A}(\frac{N}{2})\tilde{R}(\frac{N}{2})\tilde{A}(\frac{N}{2})'$

Algorithm 2 Step-doubling meth. for LQ Discretization

Input: $(A_c, B_c, G_c, C_c, D_c, Q_c, T_s, j)$
Output: (A, B_o, Q, M, R_{ww})
Compute the integration step $N = 2^j$
Compute the step size $h = \frac{T_s}{N}$
Use (33) and (40) to compute (Λ_i, $\Lambda_{v,i}$, Ω_i, $\Theta_{1,i}$, $\Theta_{2,i}$)
Use (34) and (40) to compute (Λ, Λ_v, Ω, Θ_1, Θ_2)
Use (43) to compute ($\bar{\Lambda}$, Θ_o, $\bar{B}_{oc}$, $\tilde{M}_c$, $\tilde{Q}_c$)
Set initial states ($i = 1$, $\tilde{A}(i) = \bar{\Lambda}$, $\tilde{B}(i) = I$,
$\tilde{H}(i) = \Omega$, $\tilde{Q}(i) = \tilde{Q}_c$, $\tilde{M}(i) = I$, $\tilde{R}(i) = \bar{R}_{ww,c}$)
while $i \le j$ **do**
  Use step-doubling functions (44) to update
    ($\tilde{A}(i)$, $\tilde{A}_v(i)$, $\tilde{B}(i)$, $\tilde{H}(i)$, $\tilde{Q}(i)$, $\tilde{M}(i)$, $\tilde{R}(i)$)
  Set $i \leftarrow i + 1$
end while
Use (42) to compute (A, B_o, Q, M, R_{ww})

$$\tilde{A}(N) = \bar{\Lambda}^N, \qquad \tilde{A}(1) = \bar{\Lambda}, \quad (41a)$$

$$\tilde{B}_o(N) = \sum_{k=0}^{N-1} \bar{\Lambda}^k, \qquad \tilde{B}(1) = I, \quad (41b)$$

$$\tilde{H}(N) = \Omega^N, \qquad \tilde{H}(1) = \Omega, \quad (41c)$$

$$\tilde{M}(N) = \sum_{k=0}^{N-1} \left(\Omega^k\right)', \qquad \tilde{M}(1) = I_h, \quad (41d)$$

$$\tilde{Q}(N) = \sum_{k=0}^{N-1} \left(\Omega^k\right)' \tilde{Q}_c \left(\Omega^k\right), \quad \tilde{Q}(1) = \tilde{Q}_c, \quad (41e)$$

$$\tilde{R}(N) = \sum_{k=0}^{N-1} \left(\Lambda^k\right) \bar{R}_{ww,c} \left(\Lambda^k\right)', \tilde{R}(1) = \bar{R}_{ww,c}, \quad (41f)$$

that can be used for computing

$$A(T_s) = \tilde{A}(N)(1:n_x, 1:n_x), \ H(T_s) = \tilde{H}(N), \quad (42a)$$

$$B_o(T_s) = \Theta_o \tilde{B}_o(N) \bar{B}_{oc}, \quad M(T_s) = E_2' \tilde{M}(N)' \tilde{M}_c, \quad (42b)$$

$$Q = E_2' \tilde{Q}(N) E_2, \qquad R_{ww} = h\sum_{i=1}^{s} b_i \Lambda_i \tilde{R}(N) \Lambda_i', \quad (42c)$$

where

$$\bar{\Lambda} = \mathrm{diag}(\Lambda, \Lambda_v), \ \Theta_o = [\Theta_1\ \Theta_2], \ \bar{B}_{oc} = [\bar{B}_{1c}; \bar{B}_{2c}], \quad (43a)$$

$$\tilde{M}_c = h\sum_{i=1}^{s} b_i \Omega_i' E_1' \bar{M}_c, \ \tilde{Q}_c = h\sum_{i=1}^{s} b_i \Omega_i' E_1' \bar{Q}_c E_1 \Omega_i. \quad (43b)$$

Al-Mohy and Higham (2010, 2011); Higham (2005) described the squaring and scaling algorithm for solving the matrix exponential problem. We can apply the idea of repeated squaring for computing matrices introduced in (41), and it leads to the step-doubling method.

Let $f(n)$ for $f \in \left[\tilde{A}, \tilde{B}_o, \tilde{H}, \tilde{M}, \tilde{Q}, \tilde{R}\right]$ represents functions described in (41), and the integration step $N = 2^j$ for $j \in \mathbb{Z}^+$. The step-doubling expression of $f(n)$ can be written as

$$f(1) \to f(2) \to f(4) \to \cdots \to f(\frac{N}{2}) \to f(N), \qquad (44a)$$

and

$$f(n) = F(f(\frac{n}{2})), \qquad n \in \left[2, 4, \ldots, \frac{N}{2}, N\right]. \qquad (44b)$$

$F(x)$ is the step-doubling function for computing $f(n)$, and it takes the $\frac{n}{2}^{\text{th}}$ step's result to compute the double step's result $f(n)$. Consequently, we only take j steps to get the same results as the ODE method with N integration steps. Table 1 describes step-doubling functions, and Algorithm 2 describes the step-doubling method for solving proposed differential equations systems.

4. NUMERICAL EXPERIMENTS

This section presents numerical experiments for comparing proposed numerical discretization methods and testing the CT-MPC.

The numerical experiment considers the cement mill system introduced by Olesen et al. (2013) and Prasath et al. (2010), and it can be described as

$$\boldsymbol{Y}(s) = G_u(s)U(s) + G_d(s)(D(s) + \boldsymbol{W}(s)) + \boldsymbol{V}(s), \quad (45)$$

with the transfer functions

$$G_u(s) = \begin{bmatrix} \frac{12.8e^{-s}}{16.7s+1} & \frac{-18.9e^{-3s}}{21.0s+1} \\ \frac{6.6e^{-7s}}{10.9s+1} & \frac{-19.4e^{-3s}}{14.4s+1} \end{bmatrix}, G_d(s) = \begin{bmatrix} \frac{-1.0e^{-3s}}{(32s+1)(21s+1)} \\ \frac{60}{(30s+1)(20s+1)} \end{bmatrix}, \quad (46)$$

where the inputs u_1 = feed flow rate [TPH] and u_2 = separator speed [%] and the outputs z_1 = elevator load [kW] and z_2 = fineness [cm^2/g]. The disturbance D is the clinker hardness [HGI]. $\boldsymbol{W}$ and $\boldsymbol{V}$ are the process and measurement noise.

The cement mill system (45) is converted into a discrete-time state space model with $T_s = 2$ [min],

$$\boldsymbol{x}_{k+1} = A\boldsymbol{x}_k + Bu_k + Ed_k + G\boldsymbol{w}_k, \qquad (47a)$$
$$\boldsymbol{y}_k = C\boldsymbol{x}_k + Du_k + \boldsymbol{v}_k. \qquad (47b)$$

We use the discrete-time state space model as the simulator in the numerical test. The covariance matrices for the process noise $\boldsymbol{w}_k$ and the measurement noise $\boldsymbol{v}_k$ are selected as $R_{ww} = 1.0$ and $R_{vv} = \text{diag}(0.1, 50)$. The simulation time is $T_{sim} = 12$ [h] and the system steady states are $u_s = [128; 60]$ and $z_s = [25; 3100]$. The disturbance $d_k = 20$ for $t \in \{3, 9\}$ [h] and $d_k = 0$ for the rest time.

4.1 Discretization of CT-MPC

The control model of the CT-MPC is $\boldsymbol{Z}(s) = G(s)U(s) + H(s)\boldsymbol{W}(s)$. Here the deterministic model $G(s)$ is identical to $G_u(s)$ and $H(s)$ is the stochastic part of the control model and

$$H(s) = \text{diag}\left(\left[\frac{1}{s}\frac{1}{10s+1}; \frac{1}{s}\frac{1}{10s+1}\right]\right). \qquad (48)$$

The transfer function models are converted into a continuous-time state space model. We use the continuous-time LQ-OCP introduced in (23) as the objective function of the CT-MPC. The weight matrix $Q_c = \text{diag}(1.0, 1.0)$ and the control and prediction horizons are $N = 200$ [min].

Table 2. CPU time and error of the scenario using classic RK4 with $N = 2^{14}$

		Matrix Exp.	ODE	Step-doubling
$e(A)$	[-]	-	$1.03 \cdot 10^{-12}$	$1.03 \cdot 10^{-12}$
$e(B_o)$	[-]	-	$2.31 \cdot 10^{-12}$	$2.31 \cdot 10^{-12}$
$e(R_{ww})$	[-]	-	$3.43 \cdot 10^{-12}$	$3.43 \cdot 10^{-12}$
$e(M)$	[-]	-	$4.76 \cdot 10^{-7}$	$4.76 \cdot 10^{-7}$
$e(Q)$	[-]	-	$5.51 \cdot 10^{-7}$	$5.51 \cdot 10^{-7}$
CPU Time	[s]	0.29	2.94	0.03

We discretize the continuous-time LQ-OCP using the proposed three numerical methods. Table 2 illustrates the CPU time and error of the fixed-time-step ODE and step-doubling methods using the classic RK4 with $N = 2^{14}$. We consider the results from the matrix exponential method as the true solution and the error is computed as $e(i) = \|i(T_s) - i(N)\|_\infty$ for $i \in [A, B_o, R_{ww}, M, Q]$. From Table 2, we notice that the step-doubling method is the fastest among all three methods, spending only 3 [ms] while keeping the same accuracy as the fixed-time-step ODE method.

4.2 Closed-loop simulation

Consequently, we obtain the discrete-time equivalent (28). Define the input sequence $u = [u_0; u_1; \ldots, u_{N-1}]$ with $u_k = I_k u$, and we have

$$b_k = A^k x_0, \quad \Gamma_k = \sum_{i=0}^{k} A^{k-1-i} B I_i, \quad x_k = b_k + \Gamma_k u. \quad (49)$$

Replacing x_k and u_k with the above expressions, we then get the following quadratic program (QP)

$$\min_{\{u_k\}_{k=0}^{N-1}} \quad \phi = \frac{1}{2}u'Hu + g'u \qquad (50a)$$
$$s.t. \quad u_{min} \le u_k \le u_{max}, \quad k \in \mathcal{N}, \qquad (50b)$$
$$\Delta u_{min} \le \Delta u_k \le \Delta u_{max}, \quad k \in \mathcal{N}, \qquad (50c)$$

with the quadratic and linear terms matrices

$$H = \sum_{k \in \mathcal{N}}^{N-1} \begin{bmatrix} \Gamma_k \\ I_k \end{bmatrix}' Q \begin{bmatrix} \Gamma_k \\ I_k \end{bmatrix}, g = \sum_{k=0}^{N-1} \begin{bmatrix} \Gamma_k \\ I_k \end{bmatrix}' \left(Q \begin{bmatrix} b_k \\ 0 \end{bmatrix} + q_k \right), \qquad (51)$$

where $q_k = M\bar{z}_k$. $u_{min} = -20$, $u_{max} = 20$ and $\Delta u_{min} = -2.0$, $\Delta u_{max} = 2.0$ are input box and input rate-of-movement (ROM) constraints.

We implement the CT-MPC along with a linear Kalman filter on the simulated cement mill system. The covariance matrix of the measurement noise is $R_{vv} = [0.1; 50]$ and the process noise covariance R_{ww} is obtained from the differential equation $R_{ww}(T_s)$. The cross-covariance matrix is assumed to be $S = 0$. Fig. 1 illustrates the closed-loop simulation result of the cement mill system. Initially, the CT-MPC takes a while to bring two outputs to the reference (indicated by the blue lines). Then, there are overshoots on the outputs at $t = 3$ [h] caused by the disturbance d_k. The controller captures the unknown disturbance and rejects it after a few iterations. The references have a step change at $t = 6$ [h], and the system outputs are controlled to follow the new reference points. We withdraw the disturbance

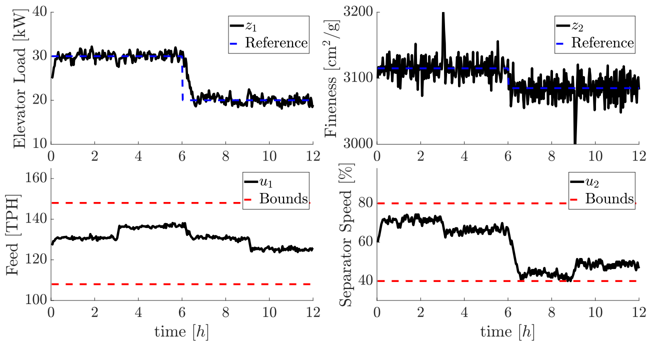

Fig. 1. The closed-loop simulation of a simulated cement mill controlled by a LQ discretization-based MPC.

at $t = 9$ [h]. Thus, the overshoots appear again, and the controller successfully handles it.

5. CONCLUSIONS

This paper introduced the discretization of continuous-time LQ-OCPs with time delays. We expressed the discrete weight matrices as the systems of differential equations, leading to the discrete equivalent of the continuous-time LQ-OCPs with time delays. Three numerical methods are described for solving proposed differential equation systems. We tested the CT-MPC with proposed numerical methods in the numerical experiment on a simulated cement mill system. The step-doubling is the fastest among all three methods and keeps the same accuracy level as the fixed-time-step ODE method. The closed-loop simulation results indicate that the proposed CT-MPCs can stabilize and control the simulated cement mill system.

REFERENCES

Al-Mohy, A.H. and Higham, N.J. (2010). A new scaling and squaring algorithm for the matrix exponential. *SIAM Journal on Matrix Analysis and Applications*, 31(3), 970–989.

Al-Mohy, A.H. and Higham, N.J. (2011). Computing the action of the matrix exponential, with an application to exponential integrators. *SIAM Journal on Scientific Computing*, 33(2), 488–511.

Åström, K.J. (1970). *Introduction to stochastic control theory*. Academic Press New York.

Åström, K.J. and Wittenmark, B. (2011). *Computer-Controlled Systems: Theory and Design, Third Edition*. Dover Books on Electrical Engineering. Dover Publications.

Capolei, A., Foss, B., and Jørgensen, J.B. (2015). Profit and risk measures in oil production optimization. *IFAC-PapersOnLine*, 48(6), 214–220. 2nd IFAC Workshop on Automatic Control in Offshore Oil and Gas Production OOGP 2015.

Franklin, G.F., Powell, J.D., and Workman, M.L. (1990). *Digital Control of Dynamic Systems*. Electrical and Computer Engineering; Control Engineering. Addison-Wesley, Reading Massachusetts, second edition.

Frison, G. and Jørgensen, J.B. (2013). Efficient implementation of the riccati recursion for solving linear-quadratic control problems. In *2013 IEEE International Conference on Control Applications (CCA)*, 1117–1122.

Gu, K., Kharitonov, V.L., and Chen, J. (2003). *Stability of Time-Delay Systems*. Birkhäuser.

Hendricks, E., Sørensen, P.H., and Jannerup, O. (2008). *Linear Systems Control: Deterministic and Stochastic Methods*. Springer, Berlin, German.

Higham, N.J. (2005). The scaling and squaring method for the matrix exponential revisited. *SIAM Journal on Matrix Analysis and Applications*, 26(4), 1179–1193.

Jørgensen, J.B., Frison, G., Gade-Nielsen, N.F., and Damman, B. (2012). Numerical methods for solution of the extended linear quadratic control problem. *IFAC Proceedings Volumes*, 45(17), 187–193. 4th IFAC Conference on Nonlinear Model Predictive Control.

Jørgensen, J.B. and Jørgensen, S.B. (2007a). Comparison of Prediction-Error-Modelling Criteria. In *American Control Conference 2007*, 140–146.

Jørgensen, J.B. and Jørgensen, S.B. (2007b). Continuous-Discrete Time Prediction-Error Identification Relevant for Linear Model Predictive Control. In *European Control Conference 2007*, 4752–4758.

Jørgensen, J.B. and Jørgensen, S.B. (2007c). MPC-Relevant Prediction-Error Identification. In *American Control Conference 2007*, 128–133.

Kassas, Z.M. and Dunia, R. (2006). Discretization of mimo systems with nonuniform input and output fractional time delays. In *Proceedings of the 45th IEEE Conference on Decision and Control*, 2541–2546. IEEE.

Lee, E. (1995). Time-delay systems: Stability and performance criteria with applications: By j. e. marshall, h. górecki, k. walton and a. korytowski. ellis horwood, chichester (1992). *Automatica*, 31(5), 793–795.

Moler, C. and Van Loan, C. (1978). Nineteen dubious ways to compute the exponential of a matrix. *SIAM Rev.*, 20(4), 801–836.

Moler, C. and Van Loan, C. (2003). Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. *SIAM Review*, 45(1), 3–49.

Olesen, D.H., Huusom, J.K., and Jørgensen, J.B. (2013). A tuning procedure for arx-based mpc of multivariate processes. In *2013 American Control Conference*, 1721–1726.

Otto, A. and Radons, G. (2017). *Time Delay Systems: Theory, Numerics, Applications, and Experiments*.

Pannocchia, G., Rawlings, J.B., Mayne, D.Q., and Mancuso, G.M. (2015). Whither discrete time model predictive control? *IEEE Transactions on Automatic Control*, 60(1), 246–252.

Pannocchia, G., Rawlings, J.B., Mayne, D.Q., and Marquardt, W. (2010). On computing solutions to the continuous time constrained linear quadratic regulator. *IEEE Transactions on Automatic Control*, 55(9), 2192–2198.

Prasath, G., Recke, B., Chidambaram, M., and Jørgensen, J.B. (2010). Application of soft constrained mpc to a cement mill circuit. *IFAC Proceedings Volumes*, 43(5), 302–307. 9th IFAC Symposium on Dynamics and Control of Process Systems.

Rockafellar, R. and Uryasev, S. (2001). Conditional value-at-risk: Optimization approach. *Stochastic optimization: algorithms and applications*, 411–435.

Van Loan, C. (1978). Computing integrals involving the matrix exponential. *IEEE Transactions on Automatic Control*, 23(3), 395–404.