12th IFAC International Symposium on
Advanced Control of Chemical Processes
July 14-17, 2024. Toronto, Canada

A Transfer State Estimator for Uncertain Parameters and Noise Statistics ⋆

Shuang Gao * **Xiaoli Luan** * **Biao Huang** ** **Shunyi Zhao** *
Fei Liu *

* *Key Laboratory of Advanced Process Control for Light Industry (Ministry of Education), Jiangnan University, Wuxi 214122, China (e-mail: gaoshuang@stu.jiangnan.edu.cn, xlluan@jiangnan.edu.cn, shunyi.s.y@gmail.com, fliu@jiangnan.edu.cn)*
** *Department of Chemical and Materials Engineering, University of Alberta, Edmonton, AB T6G 1H9, Canada (e-mail: bhuang@ualberta.ca).*

Abstract: This paper proposes a novel approach to tackle uncertainties in model parameters and noise statistics for state estimation. The proposed method leverages transfer learning to combine the strengths of the unbiased finite impulse response (UFIR) filter and the Kalman filter (KF), with UFIR serving as the source domain filter and KF as the target domain filter. To bolster the robustness of state estimation within the target domain, the proposed method transfers the predicted state probability density functions (pdfs) from UFIR and fine-tunes the error covariance of the KF filter to achieve seamless integration. Unlike conventional fusion techniques, this method avoids the need for UFIR's error covariance, thus mitigating its adverse impact on estimation accuracy. We demonstrate the competitiveness of this transfer state estimator in handling parameter uncertainties through moving target tracking, showing superior performance compared to existing fusion methods for state estimation.

Keywords: State estimation, parameter uncertainty, variational inference, transfer learning

1. INTRODUCTION

State estimation serves as a crucial step for the effective implementation of advanced control and monitoring techniques across various engineering applications (Xu and Mannor, 2009; Becis-Aubry et al., 2008). The Kalman filter (KF) stands as a powerful tool for reconstructing unmeasurable variables, providing an optimal estimate for linear systems in the minimum mean square error (MSE) sense (Särkkä and Svensson, 2023). Nonetheless, the optimal KF exhibits limitations in robustness, yielding significant errors when the noise sources are not white Gaussian and the model does not match the process exactly or experiences unforeseen temporary changes. These scenarios are inherent to real-world applications, prompting decades of dedicated efforts aimed at enhancing the KF performance including finite-horizon (Fu et al., 2001), adaptive estimation (Xue et al., 2022), and guaranteed-cost filters (Xie et al., 1994; Yang et al., 2015).

Lately, the transfer learning strategy, originally applied in machine learning to address the challenge of balancing complex modeling and the scarcity of labeled data, has found its way into the realm of state estimation to enhance robustness (Karbalayghareh et al., 2018; Quinn et al., 2016). The fundamental concept behind this strategy is to utilize the probability density function from the source domain to refine the posterior distribution in the target domain, effectively formalizing knowledge transfer. Notably, Papež and Quinn (2020) proposes a data-predictor with a student-t distribution that is transferred to the Bayesian posterior distribution, leading to improved robustness against outliers. To mitigate the adverse effects of unknown statistical properties, a probabilistic state predictor driven by a uniform distribution in the source domain, is integrated into the target filter via a knowledge-constrained transfer mechanism (Pavelková et al., 2022). However, it is essential to acknowledge that these filters exhibit infinite impulse response (IIR) characteristics, which may limit their robustness when faced with model mismatches. This limitation implies that estimation errors may persist and accumulate indefinitely with each subsequent time step.

In contrast to the IIR filter, the finite impulse response (FIR) filter only involves measurements within a predefined horizon and cuts off the propagation of errors beyond the horizon (Shmaliy et al., 2017). Accordingly, this type of filter inherently possesses better robustness against abrupt changes in model parameters. One particularly notable FIR filter is the unbiased FIR (UFIR) filter, introduced by Shmaliy (Shmaliy, 2011), which has fascinated the engineering field due to its simple structure and universal noise assumption. This filter employs an optimal horizon length to minimize the estimation error in the MSE sense. It is worth noting that while the UFIR filter is more robust than the KF filter, it does not claim optimality. Consequently, researchers have delved into fusion methods,

⋆ This work was supported by the National Natural Science Foundation of China (61991402, 61833007, 61991400).

seeking to combine the strengths of both filters to create a more versatile and comprehensive solution.

One prevalent fusion approach is weighting fusion, where a mixing probability is computed using the residual and its covariance from two distinct class filters, as demonstrated in previous works (Cho and Kim, 2008; Zhao et al., 2016; You et al., 2019). While these methods have proven effective, they do come with certain limitations that hinder their broad applicability: 1) The estimation performance of the fused filter represents a compromise between two individual filters. This loose fusion approach makes it challenging to fundamentally eliminate performance degradation resulting from model inaccuracies. 2) The fusion process relies on the second-order moment of the residual, which is not provided by the UFIR filter. When the error covariances are computed inaccurately, the weighted fusion becomes inefficient.

Motivated by these challenges, this paper presents a novel approach that leverages the strengths of the UFIR filter to enhance the robustness of the KF, which can address the above problems. Specifically, the proposed method transfers the predicted state probability density functions (pdfs) from the UFIR filter and fine-tunes the error covariance of the KF filter to achieve seamless integration. To the best of our knowledge, this is the first attempt to employ a transfer strategy for combining and preserving the advantages of the UFIR filter and KF. The key contributions of this work can be summarized as follows: 1) This proposed method overcomes the limitations of loose fusion by transferring the state prediction of the UFIR filter, thereby improving the estimation performance. 2) To improve robustness, the tuning factor is introduced to modify the error covariance of KF, which is obtained through variational Bayesian inference using measurements in the target domain. This enables the estimator to account for the unique characteristics and uncertainties of the target domain, ensuring better adaptation and more reliable state estimation. 3) Only the first-order moment of the UFIR filter is projected into the KF, reducing the adverse effects of unknown noise covariance on the estimation accuracy.

The subsequent sections of this paper are organized as follows: Section 2 lays out the problem formulation. Section 3 presents the proposed transfer state estimator in detail. The properties of the proposed algorithm are extensively explored in Section 4. Verification results are presented in Section 5, and finally, conclusions are drawn in Section 6.

In this paper, the following notations are utilized: $\mathcal{N}(\mathbf{x};\hat{\mathbf{x}},\mathbf{P})$ stands for the Gaussian distribution of $\mathbf{x}$ with mean $\hat{\mathbf{x}}$ and covariance $\mathbf{P}$, $\mathbb{E}_{f(\cdot)}[g(\cdot)]$ or $\langle g(\cdot)\rangle_{f(\cdot)}$ represents the expectation of the probability density function (pdf) $g(\cdot)$ with respect to the pdf $f(\cdot)$, $\mathcal{D}_{\mathrm{KL}}(q(\cdot)\|p(\cdot))$ denotes the KL divergence of pdf $q(\cdot)$ with respect to pdf $p(\cdot)$, $\mathrm{tr}(\cdot)$ is the trace operator, and $\exp(\cdot)$ denotes the exponential function.

2. PRELIMINARIES AND PROBLEM FORMULATION

Let us consider a scenario where a process can be represented using the following state-space model:

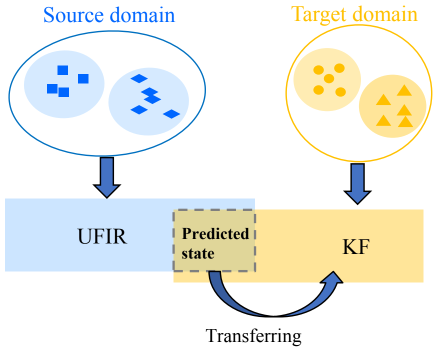

Fig. 1. A schematic description of transfer learning with respect to state estimation.

$$\mathbf{x}_n = \mathbf{F}_n\mathbf{x}_{n-1} + \mathbf{G}_n\mathbf{w}_{n-1}\,, \tag{1}$$

$$\mathbf{y}_n = \mathbf{H}_n\mathbf{x}_n + \mathbf{v}_n\,, \tag{2}$$

where n signifies the time index, $\mathbf{x}_n \in \mathbb{R}^{d_x}$ denotes the state vector, $\mathbb{R}$ is the Euclidean space, $\mathbf{y}_n \in \mathbb{R}^{d_y}$ denotes the measurement vector, and $\mathbf{F}_n$, $\mathbf{G}_n$, and $\mathbf{H}_n$ are system matrices. The process noise $\mathbf{w}_n$ follows a white Gaussian distribution, denoted as $\mathbf{w}_n \sim \mathcal{N}(\mathbf{w}_n;0,\mathbf{Q}_n)$, and the measurement noise $\mathbf{v}_n$ also follows a white Gaussian distribution, written as $\mathbf{v}_n \sim \mathcal{N}(\mathbf{v}_n;0,\mathbf{R}_n)$. To construct a stable filter, it is assumed that the system is both uniformly detectable and stabilizable, as discussed by Anderson and Moore (1981).

While the full-information-based KF remains the optimal choice when the model (1) and (2) are available and reliable, practical scenarios often present challenges where this strategy's performance becomes unacceptable, particularly when unpredictable dynamics come into play. To mitigate the effects of these uncertainties, transfer learning provides a promising avenue by leveraging knowledge from other sources. By tapping into existing knowledge or information from similar domains, we can enhance the performance and robustness of the state estimator when faced with parameter uncertainties.

The problem at hand can be concisely formulated as follows: With the UFIR filter serving as the source domain filter and the KF as the target domain filter, the primary goal is to develop a robust state estimator within the target domain. This estimator can effectively transfer vital model knowledge from the UFIR, ultimately achieving both optimality and robustness. A schematic description of the transfer learning process in the context of state estimation is depicted in Fig. 1.

3. THE PROPOSED APPROACH

In this section, we provide a detailed introduction to a transfer state estimator that leverages transfer learning to enhance the robust performance of the KF by transferring the predicted state probability density functions (pdfs) from the UFIR filter. We will first introduce the source domain filter and then proceed to formulate the target domain filter.

3.1 Source domain filter

The FIR-type filter simultaneously processes N measurements from the horizon $[m = n-N+1, n]$. The extension of (1) and (2) can be achieved following the method described in Shmaliy and Ibarra-Manzano (2012), leading to the following representations:

$$\mathbf{X}_{n,m} = \mathbf{F}_{n,m}\mathbf{x}_{m-1} + \mathbf{G}_{n,m}\mathbf{W}_{n-1,m-1}\,, \quad (3)$$

$$\mathbf{Y}_{n,m} = \mathbf{H}_{n,m}\mathbf{x}_{m-1} + \mathbf{L}_{n,m}\mathbf{W}_{n-1,m-1} + \mathbf{V}_{n,m}\,. \quad (4)$$

Here, the extended vectors are defined as $\mathbf{X}_{n,m} = [\mathbf{x}_n^{\mathrm{T}}, \mathbf{x}_{n-1}^{\mathrm{T}}, \cdots, \mathbf{x}_m^{\mathrm{T}}]^{\mathrm{T}}$, while $\mathbf{Y}_{n,m}$, and $\mathbf{V}_{n,m}$ are the same as $\mathbf{X}_{n,m}$ if we replace $\mathbf{x}_t^{\mathrm{T}}$ with $\mathbf{y}_t^{\mathrm{T}}$, and $\mathbf{v}_t^{\mathrm{T}}$, respectively, $t \in [m,n]$. The extended vector $\mathbf{W}_{n-1,m-1} = [\mathbf{w}_{n-1}^{\mathrm{T}}, \mathbf{w}_{n-2}^{\mathrm{T}}, \cdots, \mathbf{w}_{m-1}^{\mathrm{T}}]^{\mathrm{T}}$. And the extended matrices $\mathbf{F}_{n,m}, \mathbf{G}_{n,m}, \mathbf{L}_{n,m}$ and $\mathbf{H}_{n,m}$ are denoted as

$$\mathbf{F}_{n,m} = [(\mathcal{F}_n^m)^{\mathrm{T}}, (\mathcal{F}_{n-1}^m)^{\mathrm{T}}, \cdots, (\mathcal{F}_{m+1}^m)^{\mathrm{T}}, (\mathcal{F}_m^m)^{\mathrm{T}}]^{\mathrm{T}}, \quad (5)$$

$$\mathbf{G}_{n,m} = \begin{bmatrix} \mathbf{G}_n & \mathcal{F}_n^n\mathbf{G}_{n-1} & \cdots & \mathcal{F}_n^{m+1}\mathbf{G}_m \\ 0 & \mathbf{G}_{n-1} & \cdots & \mathcal{F}_{n-1}^{m+1}\mathbf{G}_m \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{F}_{m+1}^{m+1}\mathbf{G}_m \\ 0 & 0 & \cdots & \mathbf{G}_m \end{bmatrix}, \quad (6)$$

$$\mathbf{H}_{n,m} = \bar{\mathbf{H}}_{n,m}\mathbf{F}_{n,m}, \quad (7)$$

$$\mathbf{L}_{n,m} = \bar{\mathbf{H}}_{n,m}\mathbf{G}_{n,m}, \quad (8)$$

where the auxiliary variables $\bar{\mathbf{H}}_{n,m}$ and $\mathcal{F}_i^j$ are defined by $\bar{\mathbf{H}}_{n,m} = diag\ (\mathbf{H}_n, \mathbf{H}_{n-1}, \cdots, \mathbf{H}_m)$, and $\mathcal{F}_i^j = \mathbf{F}_i\mathbf{F}_{i-1}\cdots\mathbf{F}_j$, respectively.

The FIR filtering estimate at time step n can be computed using discrete convolution, as indicated in Kwon and Han (2005), and is represented as:

$$\hat{\mathbf{x}}_n^S = \mathbf{K}_n^S\mathbf{Y}_{n,m}\,. \quad (9)$$

Here, $\hat{\mathbf{x}}_n^S$ represents the estimate at time step n based on N measurements taken from m to n. To ensure an unbiased estimate, which means that $\mathbb{E}\{\mathbf{x}_n\} = \mathbb{E}\{\hat{\mathbf{x}}_n^S\}$, we can derive the UFIR filter gain as follows:

$$\mathbf{K}_n^S = \mathcal{F}_n^m(\mathbf{H}_{n,m}^{\mathrm{T}}\mathbf{H}_{n,m})^{-1}\mathbf{H}_{n,m}^{\mathrm{T}}. \quad (10)$$

Following Shmaliy (2011), we can derive an alternative iterative expression as follows:

$$\hat{\mathbf{x}}_t^{S-} = \mathbf{F}_t\hat{\mathbf{x}}_{t-1}^S\,, \quad (11)$$

$$\mathbf{M}_t^S = [\mathbf{H}_t^{\mathrm{T}}\mathbf{H}_t + (\mathbf{F}_t\mathbf{M}_{t-1}^S\mathbf{F}_t^{\mathrm{T}})^{-1}]^{-1}\,, \quad (12)$$

$$\mathbf{K}_t^S = \mathbf{M}_t^S\mathbf{H}_t^{\mathrm{T}}\,, \quad (13)$$

$$\hat{\mathbf{x}}_t^S = \hat{\mathbf{x}}_t^{S-} + \mathbf{K}_t^S(\mathbf{y}_t - \mathbf{H}_t\hat{\mathbf{x}}_t^{S-})\,, \quad (14)$$

where $g = m + d_x - 1, t \in [g+1, n]$, d_x is the dimension of states, and

$$\mathbf{M}_g^S = \mathcal{F}_g^m(\mathbf{H}_{g,m}^{\mathrm{T}}\mathbf{H}_{g,m})^{-1}\mathcal{F}_g^{m\,\mathrm{T}}\,, \quad (15)$$

$$\hat{\mathbf{x}}_g^S = \mathcal{F}_g^m(\mathbf{H}_{g,m}^{\mathrm{T}}\mathbf{H}_{g,m})^{-1}\mathbf{H}_{g,m}{}^{\mathrm{T}}\mathbf{Y}_{g,m}\,. \quad (16)$$

It is worth noting that $\hat{\mathbf{x}}_n^S$ is independent of information beyond the horizon and does not depend on the second moment of noise.

3.2 Target domain filter

After formulating the source domain filter, we proceed with the assumption that the predicted state probability density functions (pdfs) in the UFIR, denoted as $p(\mathbf{x}_n|\mathbf{y}_{m:n-1}) = \mathcal{N}(\mathbf{x}_n; \hat{\mathbf{x}}_n^{S-}, \mathbf{P}_n^{S-})$, serve as the transferred knowledge from the source domain to the target domain. It is important to note that the $\mathbf{P}_n^{S-}$ cannot be directly obtained from UFIR. As a result, only the first-order moment of this distribution is transferred, and this transferred knowledge is used to replace the prediction step of the KF in the target domain.

Furthermore, to adapt the source knowledge to the target domain and enhance robustness, we introduce a positive random variable g_n that governs the predicted error covariance $\mathbf{P}_n^-$ of the KF in the target domain, as proposed in Gao et al. (2023). To maintain the positive property of this tunning factor and enable coherent Bayesian derivations, it is assumed that g_n follows an inverse Gamma distribution. Hence, the predicted pdfs in the target domain can be depicted as:

$$p(\mathbf{x}_n|g_n, \mathbf{y}_{m:n-1}) = \mathcal{N}(\mathbf{x}_n; \hat{\mathbf{x}}_n^{S-}, g_n\mathbf{P}_n^-)\,, \quad (17)$$

$$p(g_n|\mathbf{y}_{m:n-1}) = \mathcal{I}(g_n; \alpha_n^-, \beta_n^-)\,. \quad (18)$$

Here, $\hat{\mathbf{x}}_n^{S-}$ represents the transferred predicted state, originating from the source domain filter UFIR. Additionally, $\mathcal{I}(\cdot)$ denotes the inverse Gamma distribution, with $\alpha_n^- > 0$ and $\beta_n^- > 0$ representing the shape and scale parameters, respectively. These parameters can be calculated as $\alpha_n^- = \tau\alpha_{n-1}$ and $\beta_n^- = \tau\beta_{n-1}$, where $0 < \tau \leq 1$ is the heuristic model used.

To determine the tuning parameter g_n in the target domain, we introduce a set that includes the target variables, denoted as $\theta_n = \{\mathbf{x}_n, g_n\}$. We then use proposal distributions $q(\mathbf{x}_n)$ and $q(g_n)$ to approximate the joint posterior distribution $p(\theta_n|\mathbf{y}_{m:n})$. With this approach, we can approximate the posterior probability density function $p(\theta_n|\mathbf{y}_{m:n})$ in the target domain as

$$p(\theta_n|\mathbf{y}_{m:n}) \approx q(\theta_n) = q(\mathbf{x}_n)q(g_n)\,. \quad (19)$$

Specifically, the Kullback-Leibler (KL) divergence (Kulhavỳ, 1990) is employed to evaluate the effectiveness of this approximation. The proposal distributions can be obtained by minimizing the corresponding KL divergence, i.e.,

$$\begin{aligned} &\arg\min_{q(\mathbf{x}_n)q(g_n)} \mathcal{D}_{\mathrm{KL}}(q(\theta_n)\|p(\theta_n|\mathbf{y}_{m:n})) \\ &= \arg\min_{q(\mathbf{x}_n)q(g_n)} \int q(\theta_n)\ln\frac{q(\theta_n)}{p(\theta_n|\mathbf{y}_{m:n})}\mathrm{d}\theta_n\,. \end{aligned} \quad (20)$$

Using the equality that

$$\begin{aligned} \ln p(\mathbf{y}_n|\mathbf{y}_{m:n-1}) = &\,\mathcal{D}_{\mathrm{KL}}(q(\theta_n)\|p(\theta_n|\mathbf{y}_{m:n})) \\ &+\mathcal{L}(\mathbf{y}_n|\mathbf{y}_{m:n-1}), \end{aligned} \quad (21)$$

the minimum of (20) can be obtained as

$$q(\mathbf{x}_n) \propto \exp\Big(\mathbb{E}_{q(g_n)}\big[\ln p(\mathbf{y}_n, \theta_n|\mathbf{y}_{m:n-1})\big]\Big)\,, \quad (22)$$

$$q(g_n) \propto \exp\Big(\mathbb{E}_{q(\mathbf{x}_n)}\big[\ln p(\mathbf{y}_n, \theta_n|\mathbf{y}_{m:n-1})\big]\Big)\,. \quad (23)$$

The updated functions (22) and (23) with respect to target variables strikes a balance between likelihood and prior, ensuring that the posterior approximation captures the information from both the source domain and the target domain.

Utilizing the aforementioned predicted pdfs and using the dependencies between different variables $(\mathbf{x}_n, g_n)$, we can express equation (22) in a more compact manner as:

$$\ln q(\mathbf{x}_n) \propto \mathbb{E}_{q(g_n)}\big[\ln p(\mathbf{x}_n|g_n, \mathbf{y}_{m:n-1})\big] + \ln p(\mathbf{y}_n|\mathbf{x}_n)\,. \tag{24}$$

By setting $p(g_n|\mathbf{y}_{m:n-1})$ as an initializer and referring to the derivations provided in Appendix A, the variational distribution of $\mathbf{x}_n$ at l^{th} iteration is derived as

$$q^{[l]}(\mathbf{x}_n) = \mathcal{N}(\mathbf{x}_n; \hat{\mathbf{x}}_n^{[l]}, \mathbf{P}_n^{[l]})\,, \tag{25}$$

where the mean and covariance are specified by, respectively,

$$\hat{\mathbf{x}}_n^{[l]} = \hat{\mathbf{x}}_n^{S-} + \mathbf{K}_n(\mathbf{y}_n - \mathbf{H}_n\hat{\mathbf{x}}_n^{S-})\,, \tag{26}$$

$$\mathbf{K}_n = \langle g_n\rangle\mathbf{P}_n^{-}\mathbf{H}_n^{\mathrm{T}}(\mathbf{H}_n\langle g_n\rangle\mathbf{P}_n^{-}\mathbf{H}_n^{\mathrm{T}} + \mathbf{R}_n)^{-1}\,, \tag{27}$$

$$\mathbf{P}_n^{[l]} = \langle g_n\rangle\mathbf{P}_n^{-} - \mathbf{K}_n\mathbf{H}_n\langle g_n\rangle\mathbf{P}_n^{-}\,, \tag{28}$$

where $\langle g_n\rangle = \beta_n^{[l-1]}/(\alpha_n^{[l-1]} - 1)$.

Reasoning similarly, employing the fixed variational distribution with respect to $\mathbf{x}_n$, the variational distribution of g_n at the l^{th} iteration can be decomposed as

$$\ln q(g_n) \propto \mathbb{E}_{q(\mathbf{x}_n)}\big[\ln p(\mathbf{x}_n|g_n, \mathbf{y}_{m:n-1})\big] + \ln p(g_n|\mathbf{y}_{m:n-1})\,. \tag{29}$$

According to Appendix A, we can get the updated distribution of g_n at l^{th} iteration as

$$q^{[l]}(g_n) = \mathcal{I}(g_n; \alpha_n^{[l]}, \beta_n^{[l]})\,, \tag{30}$$

where the parameters involved are computed as follows:

$$\alpha_n^{[l]} = \alpha_n^{-} + \frac{d_x}{2}\,, \tag{31}$$

$$\beta_n^{[l]} = \beta_n^{-} + \frac{1}{2}\Big[(\hat{\mathbf{x}}_n^{[l]} - \hat{\mathbf{x}}_n^{S-})^{\mathrm{T}}\mathbf{P}_n^{-}(\hat{\mathbf{x}}_n^{[l]} - \hat{\mathbf{x}}_n^{S-}) + \mathrm{tr}\Big(\mathbf{P}_n^{[l]}(\mathbf{P}_n^{-})^{-1}\Big)\Big]\,. \tag{32}$$

Now, we can summarize the proposed method in Algorithm 1.

4. PERFORMANCE ANALYSIS

Let's turn our attention to line 5 of Algorithm 1, specifically on the variable $\mathbf{P}_n$, which is similar to the covariance update in the standard KF. However, a significant difference is that $\mathbf{P}_n$ is modified by the key quantity $\langle g_n\rangle$. This modification aims to increase the mixed covariance, thereby containing more uncertainties to improve robustness. Notably, the parameter $\langle g_n\rangle$ can be adjusted through α and β. In this regard, we consider two primary scenarios:

- when $\alpha - 1 \to 0$ and $\beta \to \infty$, resulting in $\frac{\beta}{\alpha-1} \to \infty$. In this case, the Kalman gain, denoted as $K \to H^{-1}$ is constant.

Algorithm 1 The Proposed Transfer Filter Algorithm

Inputs: $\hat{\mathbf{x}}_{n-1}$, $\mathbf{P}_{n-1}$, and $\mathbf{y}_n$, α_{n-1}, and β_{n-1}

1: **Run** UFIR to get the transferred predicted state $\hat{\mathbf{x}}_n^{S-}$;
2: **Run** KF prediction steps to get the predicted covariance $\mathbf{P}_n^{-} = \mathbf{F}_n\mathbf{P}_{n-1}\mathbf{F}_n^{\mathrm{T}} + \mathbf{G}_n\mathbf{Q}_n\mathbf{G}_n^{\mathrm{T}}$;
3: $\alpha_n^{-} = \tau\alpha_{n-1}$, $\beta_n^{-} = \tau\beta_{n-1}$, and set them as initializers;
4: **for** $l = 1, 2, 3, \cdots, L$ **do**
5:     **Calculate** $\hat{\mathbf{x}}_n^{[l]}$ and $\mathbf{P}_n^{[l]}$ as (26) and (28);
6:     **Calculate** $\alpha_n^{[l]}$ and $\beta_n^{[l]}$ as (31) and (32);
7: **end for**
8: $\hat{\mathbf{x}}_n = \hat{\mathbf{x}}_n^{[L]}$, $\mathbf{P}_n = \mathbf{P}_n^{[L]}$, $\alpha_n = \alpha_n^{[L]}$, and $\beta_n = \beta_n^{[L]}$;

Outputs: $\hat{\mathbf{x}}_n$, $\mathbf{P}_n$, α_n, and β_n

- when $\alpha - 1 = \beta$, giving $\frac{\beta}{\alpha-1} = 1$. This scenario reverts to the conventional KF.

It is essential to note that the term $\langle g_n\rangle$ converges to optimal values over iterations. Therefore, any adjustments will have a limited impact on the predicted covariance of the KF.

Furthermore, it is important to emphasize that when the time step is less than m, the UFIR does not generate estimates, and consequently, the knowledge transfer does not take place. In cases where the model is perfect, there is no requirement for knowledge transfer either. Both of these scenarios fall into Scenario 2, which essentially reverts to the conventional KF.

5. SIMULATIONS

In this section, we assess the estimation performance of Algorithm 1, denoted as TF, and compare it with the fusion (FU) filter developed in Zhao et al. (2016). We use a scenario in which the robot operates within a 2-D surveillance region, and the dynamic state-space model can be defined as (1) and (2), with $\mathbf{H} = [\mathbf{I}, 0_{2\times2}]$, and

$$\mathbf{F} = \begin{bmatrix} 1 & 0 & \sin(\lambda\epsilon)/\lambda & -(1-\cos(\lambda\epsilon))/\lambda \\ 0 & 1 & -(1-\cos(\lambda\epsilon))/\lambda & \sin(\lambda\epsilon)/\lambda \\ 0 & 0 & \cos(\lambda\epsilon) & -\sin(\lambda\epsilon) \\ 0 & 0 & \sin(\lambda\epsilon) & \cos(\lambda\epsilon) \end{bmatrix}.$$

In this scenario, the values of the parameters are set as follows: $\lambda = 0.05$ and $\epsilon = 0.2$. The process noise characterized by $\mathbf{w}_n \sim \mathcal{N}(\mathbf{w}_n; 0, \mathbf{Q}_n)$, with $\mathbf{Q} = diag(0.01, 0.001, 0.001, 0.01)$. The measurement noise is represented as $\mathbf{v}_n \sim \mathcal{N}(\mathbf{v}_n; 0, \mathbf{R}_n)$, where $\mathbf{R} = [\mathbf{R}_1, \mathbf{R}_2]$ with $\mathbf{R}_1 = [10, 1]^{\mathrm{T}}$ and $\mathbf{R}_1 = [1, 80]^{\mathrm{T}}$. The process runs for 500 time steps, starting from an initial state $\mathbf{x}_0 = [0000]^{\mathrm{T}}$ and $\mathbf{P}_0 = 0.01\mathbf{I}$ with a sampling time of 1 second.

1) Case 1: Effect of Uncertain Noise Statistics: It is a common practical challenge that noise statistics are uncertain and often unavailable to engineers. In this context, we assume that the noise covariances take values $\mathbf{Q} = diag(1, 0.1, 0.1, 1)$ and $\mathbf{R} = [2, 1; 1, 2]$. However, all the algorithms in our study still rely on the aforementioned covariance values. For the simulated data, the optimal horizon for the UFIR filter was determined to be $N_{opt} = 50$, as observed in Fig. 2. In fact, variations in the range of $N = [50, 53]$ within the ellipse have minimal impact on the tracking accuracy. A sample visualization of the tracking results is presented in Fig. 3. Notably, the proposed approach exhibits the closest tracking to the actual

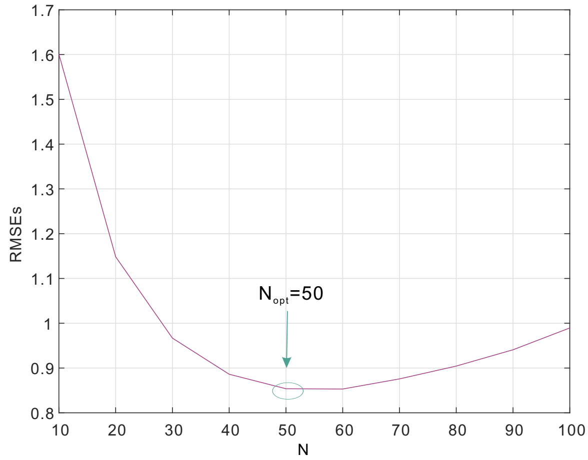

Fig. 2. Effect of estimation horizon N in the UFIR filter

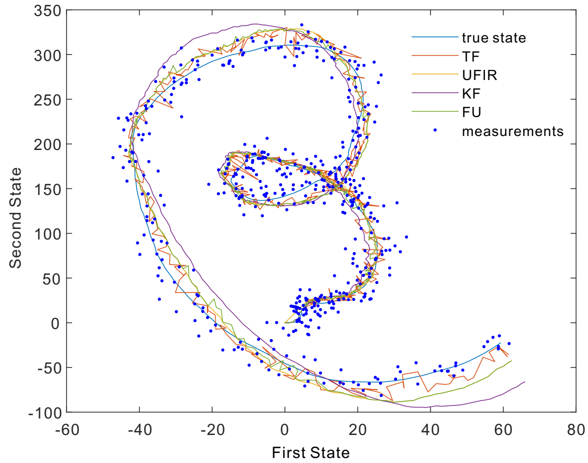

Fig. 3. Example of the tracking results.

trajectory. The average root-mean-square errors (RMSEs) calculated over 150 Monte Carlo (MC) runs for each time step are displayed in Fig. 4. and Fig. 5, corresponding to cases of uncertain process noise and measurement noise, respectively. As observed, the FU filter manages to achieve accuracy close to the most precise subfilter. It essentially represents a compromise between the KF and UFIR. In contrast, the proposed method outperforms both the KF and UFIR, even surpassing the FU filter in terms of accuracy.

2) Case 2: Model Errors: In this case, we evaluate the proposed method under the assumption of model uncertainties that can arise from various factors such as disturbances, short-term environmental changes, and other sources of parameter variability. Specifically, we assume that the model experiences unpredictable changes in the parameter λ, with $\lambda = 0.02$. However, all the algorithms continue to operate using the value of $\lambda = 0.05$ over the course of 500 generated data points. The RMSEs produced by the algorithms and averaged over 50 MC runs are depicted in Fig. 6. Once again, we observe that the proposed method (TF) delivers higher accuracy compared to the best of the two subfilters. It exhibits greater robustness than the FU. This finding aligns with the conclusions drawn in Case 1, emphasizing the robustness and accuracy

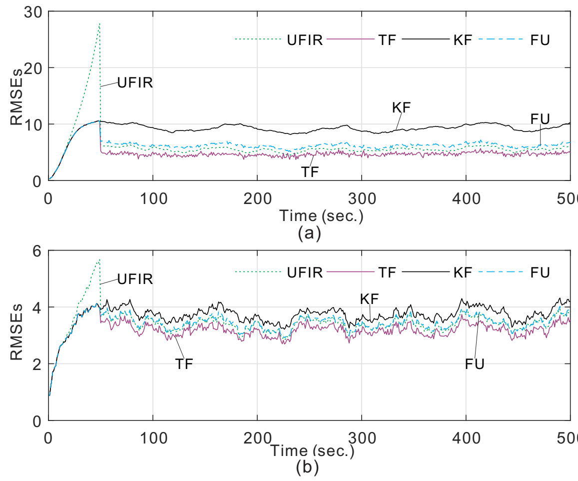

Fig. 4. RMSEs of different algorithms with uncertain process noise: (a) second state and (b) forth state.

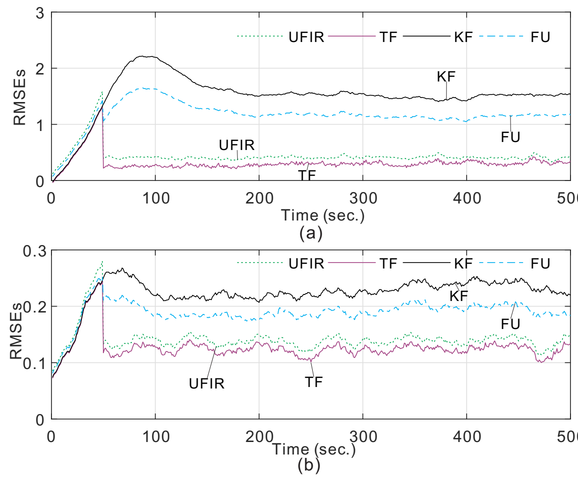

Fig. 5. RMSEs of different algorithms with uncertain measure noise: (a) first state and (b) third state.

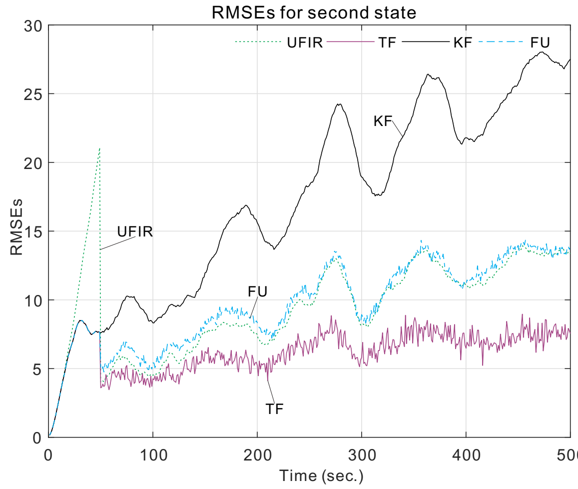

Fig. 6. RMSEs of different algorithms under the presence of model errors.

of the TF method in the presence of model errors and uncertainties.

6. CONCLUSION

This paper proposes a novel method to tackle uncertainties in model parameters for state estimation by utilizing transfer learning. By leveraging the knowledge from the source domain filter UFIR, which exhibits robust properties, the proposed method significantly enhances the robustness of the KF in the target domain. Thus, this work enriches the growing body of literature on applying transfer learning to state estimation under parameter uncertainties. Future research will focus on investigating strategies for effective knowledge transfer and adaptation across a wide array of real-world applications.

REFERENCES

Anderson, B.D. and Moore, J.B. (1981). Detectability and stabilizability of time-varying discrete-time linear systems. *SIAM Journal on Control and Optimization*, 19(1), 20–32.

Becis-Aubry, Y., Boutayeb, M., and Darouach, M. (2008). State estimation in the presence of bounded disturbances. *Automatica*, 44(7), 1867–1873.

Cho, S.Y. and Kim, B.D. (2008). Adaptive iir/fir fusion filter and its application to the ins/gps integrated system. *Automatica*, 44(8), 2040–2047.

Fu, M., De Souza, C.E., and Luo, Z.Q. (2001). Finite-horizon robust kalman filter design. *IEEE Transactions on Signal Processing*, 49(9), 2103–2112.

Gao, S., Zhao, S., Luan, X., and Liu, F. (2023). Adaptive risk-sensitive filter for markovian jump linear systems. *Automatica*, 151, 110897.

Karbalayghareh, A., Qian, X., and Dougherty, E.R. (2018). Optimal bayesian transfer learning. *IEEE Transactions on Signal Processing*, 66(14), 3724–3739.

Kulhavỳ, R. (1990). Recursive nonlinear estimation: A geometric approach. *Automatica*, 26(3), 545–555.

Kwon, W.H. and Han, S.H. (2005). *Receding horizon control: model predictive control for state models*. Springer Science & Business Media.

Papež, M. and Quinn, A. (2020). Bayesian transfer learning between student-t filters. *Signal Processing*, 175, 107624.

Pavelková, L.K., Jirsa, L., and Quinn, A. (2022). Fully probabilistic design for knowledge fusion between bayesian filters under uniform disturbances. *Knowledge-Based Systems*, 238, 107879.

Quinn, A., Kárnỳ, M., and Guy, T.V. (2016). Fully probabilistic design of hierarchical bayesian models. *Information Sciences*, 369, 532–547.

Särkkä, S. and Svensson, L. (2023). *Bayesian filtering and smoothing*, volume 17. Cambridge university press.

Shmaliy, Y.S. (2011). An iterative kalman-like algorithm ignoring noise and initial conditions. *IEEE Transactions on Signal Processing*, 59(6), 2465–2473.

Shmaliy, Y.S. and Ibarra-Manzano, O. (2012). Time-variant linear optimal finite impulse response estimator for discrete state-space models. *International Journal of Adaptive Control and Signal Processing*, 26(2), 95–104.

Shmaliy, Y.S., Zhao, S., and Ahn, C.K. (2017). Unbiased finite impluse response filtering: An iterative alternative to kalman filtering ignoring noise and initial conditions. *IEEE Control Systems Magazine*, 37(5), 70–89.

Xie, L., Soh, Y.C., and De Souza, C.E. (1994). Robust kalman filtering for uncertain discrete-time systems. *IEEE Transactions on automatic control*, 39(6), 1310–1314.

Xu, H. and Mannor, S. (2009). A kalman filter design based on the performance/robustness tradeoff. *IEEE transactions on automatic control*, 54(5), 1171–1175.

Xue, C., Huang, Y., Zhu, F., Zhang, Y., and Chambers, J.A. (2022). An outlier-robust kalman filter with adaptive selection of elliptically contoured distributions. *IEEE Transactions on Signal Processing*, 70, 994–1009.

Yang, C., Yang, Z., and Deng, Z. (2015). Guaranteed cost robust weighted measurement fusion steady-state kalman predictors with uncertain noise variances. *Aerospace Science and Technology*, 46, 459–470.

You, S.H., Ahn, C.K., Shmaliy, Y.S., and Zhao, S. (2019). Fusion kalman and weighted ufir state estimator with improved accuracy. *IEEE Transactions on Industrial Electronics*, 67(12), 10713–10722.

Zhao, S., Shmaliy, Y.S., Shi, P., and Ahn, C.K. (2016). Fusion kalman/ufir filter for state estimation with uncertain parameters and noise statistics. *IEEE transactions on industrial electronics*, 64(4), 3075–3083.

Appendix A. DERIVATIONS IN SECTION 3.2

Considering the property that $\ln p(\mathbf{x}_n|g_n, \mathbf{y}_{m:n-1}) \propto -0.5(\mathbf{x}_n - \hat{\mathbf{x}}_n^-)^{\mathrm{T}}(g_n\mathbf{P}_n^-)^{-1}(\mathbf{x}_n - \hat{\mathbf{x}}_n^-)$ and $\ln p(\mathbf{y}_n|\mathbf{x}_n) \propto -0.5(\mathbf{y}_n - \mathbf{H}_n\mathbf{x}_n)^{\mathrm{T}}(\mathbf{R}_n)^{-1}(\mathbf{y}_n - \mathbf{H}_n\mathbf{x}_n)$, we can rewrite the target marginal distribution with respect to x_n in to a more compact form as

$$\begin{aligned}\ln q(\mathbf{x}_n) \propto &-\frac{1}{2}(\mathbf{x}_n - \hat{\mathbf{x}}_n^-)^{\mathrm{T}}\left(\langle g_n\rangle\mathbf{P}_n^-\right)^{-1}(\mathbf{x}_n - \hat{\mathbf{x}}_n^-) \\ &-\frac{1}{2}(\mathbf{y}_n - \mathbf{H}_n\mathbf{x}_n)^{\mathrm{T}}(\mathbf{R}_n)^{-1}(\mathbf{y}_n - \mathbf{H}_n\mathbf{x}_n)\,,\end{aligned} \tag{A.1}$$

where $\langle g_n\rangle = \beta_n(\alpha_n - 1)^{-1}$. Recovering (A.1) as a Gaussian distribution, we can get (25).

Once $q(\mathbf{x}_n) = q^{[l]}(\mathbf{x}_n)$ is available, the variational distribution $q^{[l]}(g_n)$ becomes

$$\begin{aligned}\ln q(g_n) \propto &-\frac{d_x}{2}\ln g_n|\mathbf{P}_n^-| - \frac{1}{2}\big\{(\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_n^{S-})^{\mathrm{T}} \\ &\times(\mathbf{P}_n^-)^{-1}(\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_n^{S-}) + \mathrm{tr}\left(\mathbf{P}_n(\mathbf{P}_n^-)^{-1}\right)\big\}\frac{1}{g_n} \\ &-(\alpha_n^- + 1)\ln g_n - \beta_n^-/g_n\,.\end{aligned} \tag{A.2}$$

Utilizing the property $\mathcal{I}(g;\alpha,\beta) \propto g^{-(\alpha+1)}\mathrm{e}^{-\beta/g}$, we can rewrite (A.2) equivalently as

$$\begin{aligned}\ln q(g_n) \propto &-(\frac{d_x}{2} + \alpha_n^- + 1)\ln g_n - \frac{1}{g_n}\big\{\beta_n^- \\ &+\frac{1}{2}\big\{(\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_n^{S-})^{\mathrm{T}}(\mathbf{P}_n^-)^{-1}(\hat{\mathbf{x}}_n - \hat{\mathbf{x}}_n^{S-}) \\ &+\frac{1}{2}\mathrm{tr}\left(\mathbf{P}_n(\mathbf{P}_n^-)^{-1}\right)\big\}\,.\end{aligned} \tag{A.3}$$

By employing some approximations, we can recover a inverse Gamma distribution as given in (30), and thus complete the derivations.