Dual mode finite-time seeking control for a class of unknown dynamical system

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Abstract: In this study, we propose a dual-mode finite-time extremum seeking control system for the solution of real-time optimization problems for a class of unknown nonlinear dynamical systems. The technique provides fast convergence of the closed-loop to the optimum of a measured objective function in the absence of exact knowledge of the process dynamics. The finite-time approach provides a control system suitable for a large class of process control problems where exact models are not available. A simulation study is presented to demonstrate the effectiveness of the technique.

Keywords: Extremum-seeking control, Finite-time stability, Uncertain nonlinear systems, real-time optimization

1. INTRODUCTION

Recent developments in continuous-time optimization have led to the development of finite-time optimization techniques. In [Garg and Panagou, 2018], a model-based gradient descent with finite-time convergence was proposed. A Lyapunov stability approach was used to establish the finite-time stability of the optimum. Model-based continuous-time algorithms have also been proposed to force the finite-time convergence of real-time optimization problems subject to time-varying objective functions [Romero and Benosman, 2020].

In the absence of exact model information, data-driven model free techniques must be used. Extremum seeking control (ESC) provides an effective data-driven control mechanism to solve real-time optimization problems. ESC is a well established real-time optimization technique with a solid theoretical foundation. Its stability properties were fully characterized in Krstic and Wang [2000] and Tan et al. [2006] for the solution of steady-state optimization problems. Building on these results, considerable research has been conducted over the last two decades to address the limitations of the basic ESC methodology.

A number of recent studies have led to the development of extremum seeking control schemes with finite-time and fixed-time with practical convergence properties. In Poveda and Krstić [2020], an ESC design technique was develop to achieve fixed-time practical convergence of a class of static maps. An alternative technique was also proposed in Guay and Benosman [2020] for the same class of systems where finite-time practical convergence was achieved. The main difference between the two techniques is related to the dynamics of the averaged closed-loop system. The averaged system obtained in Guay and Benosman [2020] achieves finite-time stability of the unknown optimum as perturbations vanish. Newton seeking generalizations of these techniques have also been proposed in Guay [2020] and Poveda and Krstic [2020].

One important limitation of ESC is the lack of guaranteed transient performance. When solving steady-state optimization problems, the leading strategy is to operate the real-time optimization near the steady-state to overcome the effect of the unknown process dynamics. Fast ESC techniques (e.g., Moase and Manzie [2012], Scheinker and Krstic [2016]) have been proposed in the literature to overcome the requirement for slow transients. In Guay [2016] and Guay and Atta [2018], a dual-mode ESC approach was proposed to improve the transient performance of ESC systems. In this study, we seek a dual-mode method that provides finite-time convergence of an ESC system to its unknown optimum conditions.

In this study, we propose a dual mode approach that achieves finite-time convergence of an unknown dynamic system to a neighbourhood of the optimum of a measured objective function. We consider a dual-mode ESC formulation that achieves finite-time practical stability of ESC system. Following the methodology proposed in Guay [2016], the proportional feedback component of the dual mode is used to achieve finite-time stability while the integral feedback corrects for the correct value of the input variable at steady-state.

The paper is structured as follows. Some preliminaries are given in Section 2. The problem formulation is given in Section 3. The proposed ESC is presented in Section 4. Section 5 presents a brief simulation study. Conclusions are presented in Section 6.

2. PRELIMINARIES

2.1 Finite-time Stability

In this section, we present the definition of finite-time stability considered in this study (as stated in Hong et al. [2010]). We introduce the following class of finite-dimensional nonlinear systems:

$$\dot{x} = F(x) \tag{1}$$

where $x \in \mathbb{R}^n$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous in x.

The continuity of the right hand side of (1) guarantees existence of at least one solution, possibly non-unique. The set of all solutions of (1) with initial conditions $x(t_0) = x_0$ is denoted by $\phi(t, t_0, x_0)$ for $t \geq t_0$. In the remainder, the set of all solutions of system (1) at time t will be simply denoted by $x(t)$. The equilibrium $x_0 = 0$ is a unique solution of the system in forward time.

Definition 1. The equilibrium $x = 0$ of (1) is said to be finite-time locally stable if it is Lyapunov stable and such that there exists a settling-time function

$$T(x_0) = \inf \left\{ \bar{T} \geq t_0 \;\middle|\; \lim_{t \to \bar{T}} x(t) = 0\, ;\, x(t) \equiv 0,\, \forall t \geq \bar{T} \right\}$$

in a neighbourhood U of $x = 0$. It is globally finite-time stable if $U = \mathbb{R}^n$.

Finite-time stability can be expressed using a special class of $\mathcal{K}$ functions. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a called a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. It is a class $\mathcal{K}_\infty$ function if it is class $\mathcal{K}$ and $\lim_{s\to\infty} \alpha(s) = \infty$.

A continuous function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a generalized class $\mathcal{K}$ function if $\phi(0) = 0$ and

$$\begin{cases} \phi(s_1) > \phi(s_2) & \text{if}\, \phi(s_1) > 0,\, s_1 > s_2 \\ \phi(s_1) = \phi(s_2) & \text{if}\, \phi(s_1) = 0,\, s_1 > s_2. \end{cases} \tag{2}$$

A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a generalized $\mathcal{KL}$ function if, for each fixed $t \geq 0$, the function $\beta(s, t)$ is a generalized $\mathcal{K}$ function and each fixed $s \geq 0$, the function $\beta(s, t)$ is such that $\lim_{t\to T} \beta(s, t) = 0$ for $T \leq \infty$. We can characterize finite-time stability using generalized $\mathcal{K}$ functions as follows:

Definition 2. System (1) is finite-time stable if there exists a generalized $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that every solution $x(t)$ satisfies: $\|x(t)\| \leq \beta(\|x(0)\|, t)$ with $\beta(r, t) \equiv 0$ when $t \geq \bar{T}(r)$ with $\bar{T}(r)$ continuous with respect to r and $\bar{T}(0) = 0$.

Definition 3. Let $V(x)$ be a continuous function. It is called a finite-time Lyapunov function if there exists class $\mathcal{K}_\infty$ functions ϕ_1 and ϕ_2 and a class $\mathcal{K}$ function ϕ_3 such that $\phi_1(\|x\|) \leq V(x) \leq \phi_2(\|x\|)$ and

$$D^+V(x(t)) \leq -\phi_3(\|x\|)$$

where, in addition, ϕ_3 satisfies: $c_1 V(x)^a \leq \phi_3(\|x\|) \leq c_2 V(x)^a$ for some positive constants $a < 1$, $c_1 > 0$ and $c_2 > 0$.

Finally, we will need the following definition of practical finite-time stability.

Definition 4. System (1) is semi-globally practically finite-time stable if there exists a generalized $\mathcal{KL}$ function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a positive constant $\zeta > 0$ such that every solution $x(t)$ starting in $\mathcal{X}$ satisfies:

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \zeta \tag{3}$$

with $\beta(r, t) \equiv 0$ when $t \geq \bar{T}(r)$ with $\bar{T}(r)$ continuous with respect to r and $\bar{T}(0) = 0$.

3. PROBLEM FORMULATION

We consider a class of multivariable unknown nonlinear systems described by the following dynamical system:

$$\dot{x} = f(x) + g(x)u \tag{4a}$$

$$y = h(x) \tag{4b}$$

where $x \in \mathbb{R}^n$ are the state variables, $u \in \mathbb{R}$ is the input variable, and $y \in \mathbb{R}$ is the output variable. It is assumed that the function $h : \mathbb{R}^n \to \mathbb{R}$ is sufficiently smooth. The function h, is assumed to be unknown. It has an unknown minimizer x^* with an optimal value $y^* = h(x^*)$.

The cost function, $h(x)$, meets the following assumption.

Assumption 5. The function $h(x)$ is such that its gradient vanishes only at the minimizer x^*, that is:

$$\left.\frac{\partial h}{\partial x}\right|_{x=x^*} = 0.$$

The Hessian at the minimizer is assumed to be positive and nonzero. In particular, there exists a positive constant α_h such that

$$\frac{\partial^2 h(x)}{\partial x \partial x^\top} \geq \alpha_h I$$

for all $x \in \mathcal{X} \subset \mathbb{R}$.

The objective of this study is to develop an ESC design technique that guarantees finite-time convergence to a neighbourhood of the unknown minimizer, x^*, of the measured function $y = h(x)$.

4. FINITE TIME EXTREMUM SEEKING CONTROLLER DESIGN AND ANALYSIS

4.1 Proposed target average system

The intended target averaged system is given by the control system

$$\begin{aligned} \dot{x} =& f(x) + g(x)u \\ \dot{\hat{u}} =& -\frac{k}{\tau_I}\gamma(L_g h)\frac{(L_g h)^3}{\sqrt{1 + (L_g h)^4}} \\ u =& -k\gamma(L_g h)L_g h + \gamma(L_g h)\frac{(L_g h)^2}{\sqrt{1 + (L_g h)^4}}\hat{u} \end{aligned} \tag{5}$$

where k, τ_I are controller gains to be assigned.

As in Andrey [2012], Lopez-Ramirez et al. [2018], the function $\gamma(v)$ is given by:

$$\gamma(v) = \frac{c_1}{\|v\|^{\alpha_1}} + \frac{c_2}{\|v\|^{\alpha_2}}$$

where $\alpha_1 = \dfrac{q_1 - 2}{q_1 - 1}$ and $\alpha_2 = \dfrac{q_2 - 2}{q_2 - 1}$ for $q_1 \in (2, \infty)$ and $q_2 \in (1, 2)$. In this study, it will be assumed that $c_2 = 0$. Furthermore, we will make the following simplifying assumption.

Assumption 6. The function $|L_g h|$ is assumed to be such that there exists β_1 and β_2:

$$\beta_1(h - h^*) \leq |L_g h|^2 \leq \beta_2(h - h^*).$$

Assumption 7. For the system (4a)-(4b), there exists a $k^* \geq 0$ such for all $k > k^*$:

$$L_f h - kL_g h^2 + \rho(L_g h)L_g h u^* \leq 0 \tag{6}$$

for $x \in D \subset \mathbb{R}^n$ where $\rho : \mathbb{R} \to \mathbb{R}$ is a bounded positive definite continuous function such that $\rho(0) = 0$.

The function $\gamma(v)v$ is not locally Lipschitz continuous at $v = 0$ but it is continuous everywhere. Finite-time stability analysis of the optimal equilibrium ($x = x^*$, $u = u^*$) of the target system (5) is stated in the following lemma.

Lemma 1. Consider the nonlinear system (5). Let Assumptions 5, 6 and 7 be satisfied. Then the optimum $x = x^*$ is a finite-time stable equilibrium of (5).

Proof: Let $\tilde{u} = \hat{u} - u^*$ and pose the following Lyapunov function candidate:3

$$V_1 = h(x) - h(x^*) + \frac{\tau_I}{2k}\tilde{u}^2.$$

Its derivative along the trajectories of (5) yields:

$$\dot{V}_1 = L_f h - k\gamma(L_g h)L_g h^2 + \frac{(L_g h)^3}{\sqrt{1+(L_g h)^4}}\gamma(L_g h)\hat{u} - \tilde{u}\frac{(L_g h)^3}{\sqrt{1+(L_g h)^4}}\gamma(L_g h).$$

Upon substitution of $\hat{u} = \tilde{u} + u^*$. This can be written as:

$$\dot{V}_1 = L_f h + \frac{L_g h^3}{\sqrt{1+(L_g h)^4}}\gamma(L_g h)u^* - k\gamma(L_g h)L_g h^2.$$

By Assumption 7, it follows that:

$$\dot{V}_1 \leq -k\gamma(L_g h)L_g h^2.$$

It follows from Lyapunov stability theory that all trajectories of the system are bounded. That is, there exists a constant c_u such that $\|\tilde{u}\| \leq c_u$.

Next, let $V_2 = h(x) - h(x^*)$. Then it follows by definition that:

$$\dot{V}_2 = \dot{V}_1 + \tilde{u}\gamma(L_g h)\frac{L_g h^3}{\sqrt{1+(L_g h)^4}}.$$

or,

$$\dot{V}_2 \leq -k\gamma(L_g h)L_g h^2 + c_u\gamma(L_g h)\frac{|L_g h|^3}{\sqrt{1+(L_g h)^4}}.$$

Upon substitution of the $\gamma(L_g h)$, we obtain:

$$\dot{V}_2 \leq -kc_1|L_g h|^{2-\alpha_1} + c_u c_1\frac{|L_g h|^{3-\alpha_1}}{\sqrt{1+(L_g h)^4}}.$$

Using Assumption 6, we obtain:

$$\dot{V}_2 \leq -kc_1\beta_1^{2-\alpha_1}V_2^{1-\frac{\alpha_1}{2}} + c_u c_1\beta_3^{3-\alpha_1}\frac{V_2^{\frac{3}{2}-\frac{\alpha_1}{2}}}{\sqrt{1+\beta_1^4 V_2^2}}.$$

If one defines the continuous positive function:

$$\gamma_1(V_2) = c_u c_1\beta_3^{3-\alpha_1}\frac{V_2^{\frac{3}{2}-\frac{\alpha_1}{2}}}{\sqrt{1+\beta_1^4 V_2^2}}$$

Then it is easy to see that,

$$\lim_{V_2\to 0}\frac{\gamma_1(V_2)}{V_2^{1-\frac{\alpha_1}{2}}} = 0.$$

It follows that, for any c_u, one can find a constant $r(c_u) > 0$ such that:

$$\gamma_1(V_2) \leq \frac{k}{2}c_1\beta_1^{2-\alpha_1}V_2^{1-\frac{\alpha_1}{2}}$$

for all $V_2 \leq r(c_u)$. Thus if $V_2(0) < r$ then

$$\dot{V}_2 \leq -\frac{k}{2}c_1\beta_1^{2-\alpha_1}V_2^{1-\frac{\alpha_1}{2}}. \quad (7)$$

Hence, it follows that $V_2(t) \leq r$ for $t \geq 0$. Furthermore, the system will converge to $x = x^*$ in finite-time.

On the other hand, if $V_2(0) > r$ then it is claimed that the system reaches $V_2(t) \leq r$ in finite-time.

By definition, it follows that $V_1 \geq V_2 > r$.

$$V_1(0) \geq V_1(\tau) + \int_0^\tau \frac{k}{2}c_1\beta_1^{2-\alpha_1}V_2(\sigma)^{1-\frac{\alpha_1}{2}}\,d\sigma.$$

By assumption, we get:

$$V_1(0) \geq \tau\frac{k}{2}c_1\beta_1^{2-\alpha_1}r^{1-\frac{\alpha_1}{2}}.$$

This leads to a contradiction when

$$t \geq \frac{V_1(0)}{\frac{k}{2}c_1\beta_1^{2-\alpha_1}r^{1-\frac{\alpha_1}{2}}}.$$

Thus the system enters the set $V_2 \leq r$ in finite-time. By the finite-time stability of the system in this set, it follows the system converges to x^* in finite-time.

This completes the proof. ■

4.2 Proposed Dual-mode Finite-time ESC

The proposed ESC system is given by:

$$\begin{aligned}
\dot{x} &= f(x) + g(x)u\\
\dot{\xi} &= -\omega_l(\xi+\delta_1)\gamma(\xi+\delta_1)\\
\dot{\hat{u}} &= \frac{k}{\tau_I}\gamma(\xi)\frac{\xi^2}{\sqrt{1+\xi^4}}\delta_1(t,x)\\
u &= -k\xi\gamma(\xi) + \gamma(\xi)\frac{\xi^2}{\sqrt{1+\xi^4}}\hat{u} + a\sin(\omega t)
\end{aligned} \quad (8)$$

where $\delta_1 = \frac{2\omega}{a}\cos(\omega t)h(x)$, ω_l, a and ω are positive constants.

An average of this system is given by:

$$\begin{aligned}
\dot{x}^a &= f(x^a) + g(x^a)u^a\\
\dot{\xi}^a &= -\frac{\omega_l}{T}\int_0^T \frac{\xi^a+\delta_1(\sigma,x^a)}{|\xi^a+\delta_1(\sigma,x^a)|^{\alpha_1}}d\sigma\\
\dot{\hat{u}}^a &= -\frac{k}{\tau_I}\frac{\xi^{a2}}{|\xi^a|^{\alpha_1}\sqrt{1+\xi^{a4}}}L_g h^a\\
u^a &= -k\frac{\xi^a}{|\xi|^{\alpha_1}} + \frac{\xi^{a2}}{|\xi^a|^{\alpha_1}\sqrt{1+\xi^{a4}}}\hat{u}^a.
\end{aligned} \quad (9)$$

Next we note that one can show using standard arguments that:

$$\frac{1}{T}\int_0^T(\xi^a+\delta_1(x^a,t))dt \approx (\xi^a - L_g h(x^a)) + \mathcal{O}\left(\frac{1}{\omega}\right).$$

Similarly, it is also easy to compute that:

$$\frac{1}{T}\int_0^T|\xi^a+\delta_1(x^a,t)|dt \approx |\xi^a - L_g h(x^a)| + \mathcal{O}\left(\frac{1}{\omega}\right).$$

We can now proceed with the stability analysis of the averaged system.

Lemma 2. Let Assumptions 5, 6 and 7 be satisfied. Then the optimum $x^a = x^*$, $\xi^a = 0$ is a finite-time stable equilibrium of the averaged system (9).

Proof: Pose the Lyapunov function: $V_3 = \frac{1}{2}(\xi^a)^2$.

$$\dot{V}_3 = -\omega_l(\xi^a)\frac{1}{T}\int_0^T\frac{\xi^a+\delta_1(\sigma,x^a)}{|\xi^a+\delta_1(\sigma,x^a)|^{\alpha_1}}d\sigma.$$

This can be rewritten as follows:

$$\dot{V}_3 = -\omega_l(\xi^a - L_g h^a)\frac{1}{T}\int_0^T \frac{\xi^a + \delta_1(\sigma, x^a)}{|\xi^a + \delta_1(\sigma, x^a)|^{\alpha_1}}d\sigma - \omega_l L_g h^a \frac{1}{T}\int_0^T \frac{\xi^a + \delta_1(\sigma, x^a)}{|\xi^a + \delta_1(\sigma, x^a)|^{\alpha_1}}d\sigma \quad (10)$$

Using the approach outlined in Guay and Benosman [2020], it can be shown that:

$$\dot{V}_3 \leq -\frac{\omega_l\|\xi^a - L_g h^a\|^2}{T\|\xi^a - L_g h^a\|^{\alpha_1}} - \frac{\omega_l L_g h^a}{T}\int_0^T \frac{\xi^a + \delta_1(\sigma, x^a)}{|\xi^a + \delta_1(\sigma, x^a)|^{\alpha_1}}d\sigma.$$

The last inequality can be further written as:

$$\dot{V}_3 \leq -\frac{\omega_l\|\xi^a - L_g h^a\|^2}{T\|\xi^a - L_g h^a\|^{\alpha_1}} + \frac{\omega_l |L_g h^a|}{T}\int_0^T \frac{|\xi^a + \delta_1(\sigma, x^a)|}{|\xi^a + \delta_1(\sigma, x^a)|^{\alpha_1}}d\sigma.$$

As a result, we obtain:

$$\dot{V}_3 \leq -\frac{\omega_l\|\xi^a - L_g h^a\|^2}{T\|\xi^a - L_g h^a\|^{\alpha_1}} + \frac{\omega_l |L_g h^a|}{T}\int_0^T |\xi^a + \delta_1(\sigma, x^a)|^{1-\alpha_1}d\sigma$$

or,

$$\dot{V}_3 \leq -\frac{\omega_l\|\xi^a - L_g h^a\|^2}{T\|\xi^a - L_g h^a\|^{\alpha_1}} + \frac{\omega_l |L_g h^a|}{T}\left|\int_0^T |\xi^a + \delta_1(\sigma, x^a)|d\sigma.\right|^{1-\alpha_1}$$

Finally, we show that:

$$\dot{V}_3 \leq -\frac{\omega_l}{T}|\xi^a - L_g h^a|^{2-\alpha_1} + \omega_l |L_g h^a||\xi^a - L_g h^a|^{1-\alpha_1}.$$

Next we pose the Lyapunov function candidate: $V_4 = V_3 + h(x^a) - h(x^*) + \frac{\tau_I}{2k}\tilde{u}^{a^2}$.

Upon differentiation with respect to t, we obtain:

$$\dot{V}_4 = \dot{V}_3 + L_f h^a + L_g h^a\Bigg(-k\gamma(\xi^a)\xi^a + \gamma(\xi^a)\frac{\xi^{a^2}}{\sqrt{1+\xi^{a^4}}}\hat{u}^a\Bigg) + \tilde{u}^a\dot{\hat{u}}^a$$

Upon substitution of $\dot{\hat{u}}^a$ and $\tilde{u}^a = \hat{u}^a - u^*$, the last inequality is written as:

$$\dot{V}_4 = \dot{V}_3 + L_f h^a - kL_g h^a\gamma(\xi^a)\xi^a + L_g h^a\gamma(\xi^a)\frac{\xi^{a^2}}{\sqrt{1+\xi^{a^4}}}u^*$$

Next we substitute for $\dot{V}_3$ to get:

$$\dot{V}_4 \leq -\frac{\omega_l}{T}|\xi^a - L_g h^a|^{2-\alpha_1} + \omega_l|L_g h^a||\xi^a - L_g h^a|^{1-\alpha_1} + L_f h^a - kL_g h^a\gamma(\xi^a)\xi^a + L_g h^a\gamma(\xi^a)\frac{\xi^{a^2}}{\sqrt{1+\xi^{a^4}}}u^*.$$

By Assumption 7, one can write:

$$\dot{V}_4 \leq -\frac{\omega_l}{T}|\xi^a - L_g h^a|^{2-\alpha_1} + \omega_l|L_g h^a||\xi^a - L_g h^a|^{1-\alpha_1} - kL_g h^a\gamma(\xi^a)\xi^a.$$

Upon rearrangement, the last inequality as follows:

$$\dot{V}_4 \leq -\frac{\omega_l}{T}|\xi^a - L_g h^a|^{2-\alpha_1} + \omega_l|L_g h^a||\xi^a - L_g h^a|^{1-\alpha_1} - k(L_g h^a - \xi^a)\gamma(\xi^a)\xi^a - k|\xi^a|^2\gamma(\xi^a),$$

leading to the following inequality:

$$\dot{V}_4 \leq -\frac{\omega_l}{T}|\xi^a - L_g h^a|^{2-\alpha_1} + \omega_l|L_g h^a||\xi^a - L_g h^a|^{1-\alpha_1} + k|\xi^a - L_g h^a|\gamma(\xi^a)|\xi^a| - k|\xi^a|^2\gamma(\xi^a).$$

Substituting for $\gamma(\xi^a)$:

$$\dot{V}_4 \leq -\frac{\omega_l}{T}|\xi^a - L_g h^a|^{2-\alpha_1} + \omega_l|L_g h^a||\xi^a - L_g h^a|^{1-\alpha_1} + k|\xi^a - L_g h^a||\xi^a|^{1-\alpha_1} - k|\xi^a|^{2-\alpha_1}.$$

By the triangle inequality, we can upper bound $|L_g h^a| \leq |\xi^a| + |\xi^a - L_g h^a|$. This yields:

$$\dot{V}_4 \leq -\frac{\omega_l}{T}|\xi^a - L_g h^a|^{2-\alpha_1} + \omega_l|\xi^a||\xi^a - L_g h^a|^{1-\alpha_1} + \omega_l|\xi^a - L_g h^a|^{2-\alpha_1} + k|\xi^a - L_g h^a||\xi^a|^{1-\alpha_1} - k|\xi^a|^{2-\alpha_1}.$$

We can then apply Young's inequality to obtain the following expression:

$$\dot{V}_4 \leq -\left(k - \frac{\omega_l}{\gamma_1}\frac{1}{2-\alpha_1} - \frac{k}{\gamma_2}\frac{1-\alpha_1}{2-\alpha_1}\right)|\xi^a|^{2-\alpha_1} - \left(\frac{\omega_l}{T} - \omega_l\gamma_1\frac{1-\alpha_1}{2-\alpha_1} - \omega_l - k\gamma_2\frac{1}{2-\alpha_1}\right)|\xi^a - L_g h^a|^{2-\alpha_1}$$

for positive constants γ_1 and γ_2. It is then easy to find a T^* such that for each $T < T^*$ and a k^* with $k > k^*$, such that:

$$\dot{V}_4 \leq -\gamma_3|\xi^a - L_g h^a|^{2-\alpha_1} - \gamma_4|\xi^a|^{2-\alpha_1}$$

for positive constants γ_3 and γ_4. As a result, we obtain:

$$\dot{V}_4 \leq -\gamma_3|L_g h^a|^{2-\alpha_1} - \gamma_5|\xi^a|^{2-\alpha_1}$$

for some $\gamma_5 > 0$. This gives:

$$\dot{V}_4 \leq -\gamma_3\beta_1^{1-\alpha_1/2}|h(x) - h(x^*)|^{1-\frac{\alpha_1}{2}} - \gamma_5 2^{1-\frac{\alpha_1}{2}}V_3^{1-\frac{\alpha_1}{2}}.$$

It follows that all signals of the averaged dynamics are bounded. As in the proof of Lemma 1, we define the function $V_5 = V_3 + h(x) - h(x^*)$. This function is such that:

$$\dot{V}_5 \leq -\gamma_3\beta_1^{1-\alpha_1/2}|h(x) - h(x^*)|^{1-\alpha_1/2} - \gamma_5 2^{1-\alpha_1/2}V_3^{1-\alpha_1/2} + |\tilde{u}||L_g h^a|\frac{\xi^{a2}}{\sqrt{1+\xi^{a4}}}.$$

By definition of the function V_5, this yields:

$$\dot{V}_5 \leq -\gamma_6 2^{1-\alpha_1/2}V_5^{1-\alpha_1/2} + |\tilde{u}||L_g h^a|\frac{\xi^{a2}\gamma(\xi^a)}{\sqrt{1+\xi^{a4}}}.$$

Since all signals of the system are bounded, there exists a constant $c_u > 0$ such that $\|\tilde{u}^a\| \leq c_u$. The last inequality can be written as:

$$\dot{V}_5 \leq -\gamma_6 2^{1-\alpha_1/2}V_5^{1-\alpha_1/2} + c_u\beta_2^{1/2}(h - h^*)^{1/2}\frac{|\xi^a|^{2-\alpha_1}}{\sqrt{1+\xi^{a4}}},$$

or

$$\dot{V}_5 \leq -\gamma_6 2^{1-\alpha_1/2}V_5^{1-\alpha_1/2} + c_u\beta_2^{1/2}\frac{2^{1-\alpha_1/2}V_5^{3/2-\alpha_1/2}}{\sqrt{1+\xi^{a4}}}.$$

Following the development in the proof of Lemma 1, there exists a constant $r(c_u) > 0$ such that

$$c_u\beta_2^{1/2}\frac{2^{1-\alpha_1/2}V_5^{3/2-\alpha_1/2}}{\sqrt{1+\xi^{a^4}}} \le \frac{\gamma_6}{2}2^{1-\alpha_1/2}V_5^{1-\alpha_1/2}$$

whenever $V_5 < r(c_u)$. It follows that if $V_5(0) = 0$ then,

$$\dot{V}_5 \le -\frac{\gamma_6}{2}2^{1-\alpha_1/2}V_5^{1-\alpha_1/2}.$$

As a result, we conclude that the equilibrium $\xi_a = 0$ and $x^a = x^*$ is finite-time stable. If $V_5(0) > r(c_u)$, we can proceed as in the proof of Lemma 1 and show that the system reaches the set $V_5 < r(c_u)$ in finite-time. This completes the proof. ■

The main result of the previous analysis is that the averaged system (9) are closely related to the stability properties of the target system (5). It thus remains to prove that the trajectories of the closed-loop system (8) remain close to the trajectories of the target system. In this study, we follow the analysis presented in Guay and Benosman [2020] where we applied a classical averaging theorem [Krasnosel'skii and Krein, 1955]. This theorem can be used to show the closeness of solution of the nominal system and the averaged system over a compact set $D \subset \mathbb{R}^2$ as $a \to 0$. A generalization, with proof, for averaging of differential inclusions was proposed in Plotnikova [2005].

The theorem can be stated as follows.

Theorem 1. Krasnosel'skii and Krein [1955] Consider the nonlinear system $\dot{X} = f(t, X, \epsilon)$ where,

(1) the map $f(t, X, \epsilon)$ is continuous in t and X on $\mathbb{R}_{\ge 0} \times \mathbb{R}^n$,
(2) there exists a positive constant $L > 0$ and a compact set $D \subset \mathbb{R}^n$ such that $\|f(t, X, \epsilon)\| \le L$ for $t \in \mathbb{R}_{\ge 0}$, $X \in D$ and $\epsilon \in (0, \epsilon^*]$,
(3) the averaged system

$$\dot{X}^a = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t, X^a, 0)dt$$

exists with solutions defined on the set D with $X^a(0) = X(0)$.

Then, for each $\epsilon \le \epsilon^*$, there exists constants δ and $\overline{T}$ such that:

$$\|X(t) - X^a(t)\| \le \delta$$

for $t \in [0, \overline{T}]$.

We can now state the final result of this study.

Theorem 2. Consider the dual model finite-time ESC system (8). Let Assumptions 5, 6 and 7 be satisfied. Then there exists an ω^* such that for all $\omega > \omega^* > 0$, the optimum $x = x^*$, $\xi = 0$ is a semi-globally practically finite-time stable equilibrium of system (8).

Proof: The proof proceeds in two steps. In the first step, we consider the application of Theorem 1. For the analysis of the proposed finite-time ESC, the Krasnosel'skii-Krein theorem can be applied as follows.

Consider the state, $X = [x, \hat{u}, \xi]^\top$, and the corresponding averaged variables $X^a = [x^a, \hat{u}^a, \xi^a]^\top$. By the analysis provided above, the averaged system has a finite-time stable equilibrium at the origin $X^* = [x^*, u^*, 0]^\top$. Furthermore, the solutions of the system (9) exist and can be contained in a compact set $D \in \mathbb{R}^{n+2}$ containing X^*. Consider the nonlinear system (8). By the smoothness of the cost function $h(x)$ and the periodicity of the dither signal, it follows that the right hand side of the system can be bounded a the compact set $D \in \mathbb{R}^{n+2}$ uniformly in t. The continuity and the boundedness of the right hand side of (8) over a compact set D guarantees existence of solution for the averaged system. As a result, one can invoke the Krasnosel'skii-Krein theorem with $\epsilon = \omega^{-1}$ to guarantee that for any $\omega > \omega^*$ there exist constants $\overline{T}$ and δ such that:

$$\|X(t) - X^a(t)\| \le \delta \tag{11}$$

for $t \in [0, \overline{T}]$.

In the second step, we exploit the finite-time stability of the averaged system and the averaging result established in the first step to establish the finite-time practical semi-global stability of the ESC system.

Using the finite-time stability property of the averaged system and the averaging result for small amplitude signals, one can apply the approach in the proof of Theorem 1 in Teel et al. [2003] to show that there exist a generalized class $\mathcal{K}_\infty$ function, β_X and a constant, c_X, such that:

$$\|X(t)\| \le \beta_X(\|X(t_0)\|, t) + c_X$$

for $X(t_0) \in D$.

By inequality (11), we have that:

$$\|X^a(t)\| \le \|X(t)\| + \delta$$

for $t \in [0, \bar{T}]$.

Assume that there exists a constant r such that the set $\{\|X(t_0)\| \le r\} \subset D$.

First, we pick the constant $\delta_0 > 0$ such that:

$$\sup_{\gamma_0\in[0,r],\, t=[0,\infty)} (\beta_X(\gamma_0 + \delta, t) - \beta_X(\gamma_0, t)) + \delta \le \frac{\delta_0}{2}.$$

There always exists a $\bar{T}$ such that:

$$\beta_X(\|X^a(0)\|, \tau) \le \beta(r, \tau) \le \frac{\delta_0}{2}$$

for $\tau \in [\bar{T}, \infty)$.

For any $t \in [0, 2\bar{T}]$, it follows that:

$$\begin{aligned}\|X(t)\| \le &\beta_X(\|X^a(0)\|, t) + \delta \le \beta_X(\|X(0)| + \delta, t) + \delta\\ \le &\beta_X(\|X(0)|, t) + \frac{\delta_0}{2}.\end{aligned}$$

Thus it follows for $t \in [\bar{T}, 2\bar{T}]$, we conclude that:

$$\|X(t)\| \le \delta_0.$$

Repeating the argument, it follows that:

$$\|X(t)\| \le \delta_0$$

for $t \in [T, \infty)$.

As result, we obtain:

$$\|X(t)\| \le \beta_X(\|X(0)\|, t) + \delta_0.$$

As a result, system (8) has a semi-globally practically finite-time stable equilibrium at the optimal conditions $x = x^*$ and $\xi = 0$.

This completes the proof. ■

5. SIMULATION STUDY

We first consider the following system:

$$\dot{x}_1 = -x_1^3 + x_2,$$
$$\dot{x}_2 = -x_2 + x_1 + u$$
$$y = 1 + 2(x_2 - 1)^2.$$

The objective is to drive the system to the unknown optimum $x_2^* = 1$, $x_1^* = 1$, $y^* = 1$ and $u^* = 0$. The dual mode finite-time extremum seeking controller is implemented with the following tuning parameters: $a = 5$, $\omega = 100$, $\omega_l = 100$, $\alpha_1 = 1/2$, $k = 1$, and $\tau_I = 0.5$.

Figures 1 and 2 show the resulting trajectories of the state variable x_2, the input variable $\hat{u}$ and the cost function y for varying initial conditions. In Figure 1, we consider five different initial conditions for $x_2(0)$(= -10, -5, 0, 5 and 10) with $\hat{u}(0) = 0$. The trajectories corresponding to five different initial conditions in $\hat{u}$(= -10, -5, 0, 5 and 10) with $x_2(0) = 0$ are shown in Figure 2. The trajectories are shown to reach the optimum at the same time for all initial conditions. The results demonstrate that the ESC system achieves finite-time practical convergence. It is important to note that the proposed finite-time ESC system cannot guarantee to convergence of $\hat{u}$ to u^*.

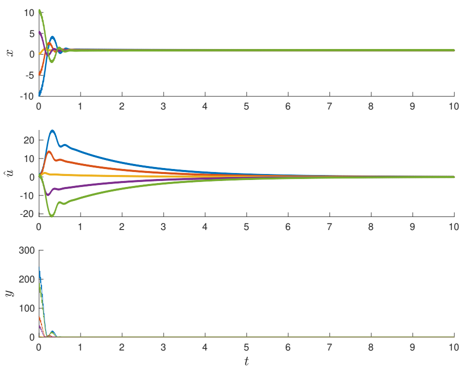

Fig. 1. Performance of the finite-time seeking system with varying initial conditions for $x_2(0)$.

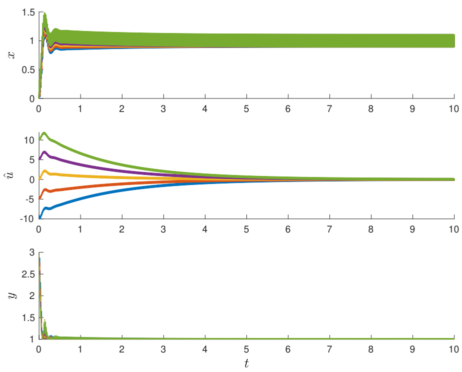

Fig. 2. Performance of the finite-time seeking system with varying initial conditions for $\hat{u}(0)$.

6. CONCLUSION

We proposed a dual-mode finite-time extremum seeking control system for a class of unknown nonlinear dynamical systems. It is shown that practical finite-time stability of the unknown optimum of the measured cost function is achieved. Future work will focus on the study of systems with arbitrary relative degree for the design of data-driven controllers.

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