

# A Simulation Study on Turnpikes in Stochastic LQ Optimal Control

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**Abstract:** This paper presents a simulation study on turnpike phenomena in stochastic optimal control problems. We employ the framework of Polynomial Chaos Expansions (PCE) to investigate the presence of turnpikes in stochastic LQ problems. Our findings indicate that turnpikes can be observed in the evolution of PCE coefficients as well as in the evolution of statistical moments. Moreover, the turnpike phenomenon can be observed for optimal realization trajectories and with respect to the optimal stationary distribution. Finally, while adding variance penalization to the objective alters the turnpike, it does not destroy the phenomenon.

*Keywords:* Stochastic optimal control, turnpike properties, stochastic uncertainty, polynomial chaos expansion

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## 1. INTRODUCTION

The last decade has seen substantial progress in terms of optimal and predictive control. This includes the analysis of stochastic MPC for set-point stabilization and the understanding of deterministic economic MPC schemes, wherein the objective is more general than a penalization of the distance to a given set-point.

A crucial point in the analysis of economic MPC is the interplay of turnpike and dissipativity properties. The former refers to the phenomenon that for many Optimal Control Problems (OCPs), the optimal solutions for varying horizon and varying initial conditions are structurally similar. More precisely, the turnpike phenomenon refers to the fact that in the middle part of the horizons the optimal solutions stay within a neighborhood of the optimal steady state, see (Dorfman et al., 1958; McKenzie, 1976; Carlson et al., 1991) for classical references and (Trélat and Zuazua, 2015; Damm et al., 2014; Faulwasser and Grüne, 2021) for recent results. It is worth to be noted that the research on turnpike properties of OCPs originated in economics. The analysis of the interplay between turnpike and dissipativity notions of OCPs has been investigated in a number of papers; indeed it can be shown that under mild assumptions the turnpike property is equivalent to a certain strict dissipativity notion (Grüne and Müller, 2016; Faulwasser et al., 2017). Moreover, this close relation can be exploited in the analysis of economic MPC schemes, see (Faulwasser et al., 2018) for a recent overview. However, when it comes to economic MPC under uncertainties much less has been done in terms of analysis—see (Bayer et al., 2016)—despite the fact that in the economics literature a number of investigations on turnpike properties in stochastic problems have been conducted see (Marimon, 1989; Kolokoltsov and Yang, 2012).

Regarding numerical computations with stochastic uncertainties, there has been recent interest into Polynomial Chaos Expansions (PCEs). The core idea of PCE is that a random variable can be modeled as an  $L^2$  function in an appropriate Hilbert space and that in this space a polynomial basis can be used to parametrize the random variable by deterministic coefficients. The idea dates back to Wiener (1938). In recent years, PCE methods have been subject to renewed interest and have been widely investigated for uncertainty quantification (Sullivan, 2015). While in principle the number of terms in the series expansion is infinite, numerical implementation requires truncation. Recently, it was shown that for polynomial explicit mappings the truncation from applying Galerkin projection to the first  $L + 1$  basis functions can be characterized in closed form, which enables to quantify the error and to choose sufficiently many basis functions such that the error vanishes (Mühlpfordt et al., 2018). In systems and control, PCE has been used in a number of papers, e.g., in (Paulson et al., 2014; Mesbah and Streif, 2015; Kim et al., 2013). The PCE approach is also considered for uncertainty quantification in electrical power systems (Mühlpfordt et al., 2017) and gas networks (Gerster et al., 2019). A major advantage of the PCE framework is that it allows the consideration of a large class of non-Gaussian random variables with finite variance.

The present note conducts a simulation study on turnpike properties in stochastic Linear-Quadratic (LQ) OCPs. Specifically, we employ the `PolyChaos.jl` package by Mühlpfordt et al. (2020) to solve example problems. The contribution is to demonstrate that in understanding turnpike properties in stochastic LQ OCPs, one needs to compare the stochastically optimal steady state with the optimal distribution in the middle of the optimization

horizon. Moreover, our numerical experiments show that the deterministic PCE coefficients of the state and input variables also exhibit a turnpike phenomenon. Indeed one can also observe the turnpike phenomenon if the disturbance realization sequence is identical for different initial conditions. Finally, we also show that the phenomenon is robust under consideration of variance penalization in the objective. Our results illustrate the prospect of systematically using stochastic turnpikes in the analysis of stochastic MPC schemes.

The remainder of the paper is structured as follows: Section 2 recalls the turnpike phenomenon via an illustrative example and we introduce the considered problem set-up. Moreover, we recall the basics of PCE and how one can avoid PCE truncation errors in the LQ-setting. Section 3 presents several examples of stochastic LQ problems, including uncertainty in the initial condition and additive stochastic disturbances. We also present an example which goes beyond the usual minimization of expected value objective. Finally, Section 4 provides a concise summary.

*Notation* Deterministic {state, input} variables are written as  $x(k), u(k)$  etc., while their stochastic counterparts are written by  $X(k), U(k)$ . The expected value and variance are denoted as  $\mathbb{E}$  and  $\mathbb{V}$ . The deterministic PCE coefficients of random variables  $X(k), U(k)$  are written as  $x_0, x_1, \dots$ , respectively, as  $u_0, u_1, \dots$ .  $\mathbb{I}_{[k_1, k_2]}$  denotes the set of positive integers  $\{k_1, k_1 + 1, \dots, k_2\}$ .

## 2. PROBLEM SET-UP

To motivate the considered problem setting and the later stochastic examples, we first consider a motivating deterministic example taken from (Grüne, 2013).

*Example 1.* (Motivating example). Consider the following deterministic OCP

$$\begin{aligned} & \min_{u(\cdot), x(\cdot)} \sum_{k=0}^{N-1} u^2(k) \\ & \text{subject to} \\ & x(k+1) = 2x(k) + u(k), x(0) = x_0 \in \mathbb{R} \\ & x(k) \in [-2, 2] \end{aligned}$$

We solve the problem for an increasing sequence of horizons  $N = 3, 6, \dots, 24$  and varying initial conditions. The results are presented in Fig. 1, where the left side shows the state trajectories and the right side depicts corresponding inputs. It can be seen that the optimal solutions all approach a neighborhood of  $(0, 0)$  and depart from it towards the end of the horizon. This phenomenon is known as *turnpike property* (McKenzie, 1976; Carlson et al., 1991).

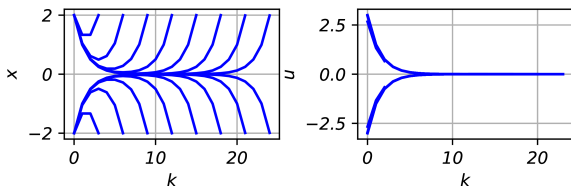


Fig. 1. Solutions of the motivating example.

### 2.1 Problem Statement

Subsequently, we are interested in exploring the turnpike phenomenon in stochastic LQ OCPs. To this end, we consider stochastic LQ OCPs of the following form

$$\min_{U(\cdot), X(\cdot)} \sum_{k=0}^{N-1} \ell(X(k), U(k)) \quad (1a)$$

subject to

$$X(k+1) = AX(k) + BU(k) + EW(k), \quad (1b)$$

$$X(0) = X_0, \quad (1c)$$

$$\mathbb{P}[X(k) \in \mathbb{X}] \leq \varepsilon_x, k \in \mathbb{I}_{[0, N]}, \quad (1d)$$

$$\mathbb{P}[U(k) \in \mathbb{U}] \leq \varepsilon_u, k \in \mathbb{I}_{[0, N-1]}, \quad (1e)$$

whereby at each time step  $k$ ,  $X(k) \in L^2(\Omega, \mathcal{F}_k, \mu; \mathbb{R}^{n_x})$  is an *i.i.d.* random variable on the underlying filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k=0, \dots, N}, \mu)$ , where  $\Omega$  is the set of realizations,  $\mathcal{F}$  is a  $\sigma$ -algebra,  $(\mathcal{F}_k)_{k=0, \dots, N}$  is a filtration, and  $\mu$  is the probability measure.

Note that the  $\sigma$ -algebras  $\mathcal{F}_k$  are related by the time evolution of the information, i.e.

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq \mathcal{F}_N \subseteq \mathcal{F}.$$

We choose  $(\mathcal{F}_k)_{k=0, \dots, N}$  as the smallest filtration such that  $X$  is an adapted process (which results from the evolution of the dynamics (1b)), i.e.

$$\mathcal{F}_k = \sigma(X(0), \dots, X(k)), \quad k = 0, 1, \dots$$

Then, the control at time  $k$  is modeled as a stochastic process which is adapted to the filtration  $(\mathcal{F}_k)_k$ , i.e.  $U(k) \in L^2(\Omega, \mathcal{F}_k, \mu; \mathbb{R}^{n_u})$ .

The concept of stochastic filtrations can be understood as a stochastic causality requirement, i.e. the stochastic input process  $U(k)$  at time step  $k$  may only depend on the realization of the random variables  $X(0), \dots, X(k)$  up to time step  $k$ . Note that the influence of the noise, which is also an adapted stochastic process, is handled implicitly via the state recursion. That is,  $U(k)$  depends on  $X(0), \dots, X(k)$  and thus also on  $W(0), \dots, W(k-1)$ . For details on stochastic filtrations we refer to (Fristedt and Gray, 2013).

The stage cost is given by

$$\ell : L^2(\Omega, \mathcal{F}, \mu; \mathbb{R}^{n_x}) \times L^2(\Omega, \mathcal{F}, \mu; \mathbb{R}^{n_u}) \rightarrow \mathbb{R}.$$

Indeed typical choice for  $\ell$  is a combination of expected value and variance of some underlying deterministic stage cost function. The considered dynamics are a time-discrete stochastic system subject to noise  $W(\cdot)$  modelled as a stochastic process. The constraints are written as chance constraints for states and inputs, whereby the underlying sets  $\mathbb{X}$  and  $\mathbb{U}$  are assumed to be closed. Moreover,  $\varepsilon_x$  and  $\varepsilon_u$  specify the probabilities with which the chance constraints shall be satisfied.

### 2.2 Basics of Polynomial Chaos Expansion

In order to obtain a tractable reformulation of the stochastic LQ OCP (1) we consider the framework of Polynomial Chaos Expansions (PCE). For an in-depth introduction we refer to (Sullivan, 2015). The underlying idea of PCE is that random variables from some  $L^2(\Omega, \mathcal{F}, \mu; \mathbb{R}^m)$  with  $m \in \{n_x, n_u, n_w\}$  can be described using an appropriate

basis. To this end, we consider an orthogonal polynomial basis  $\{\phi_i\}_{i=0}^{\infty}$  which spans  $L^2(\Omega, \mathcal{F}, \mu; \mathbb{R}^m)$ .

*Definition 1.* (Polynomial chaos expansion). The polynomial chaos expansion of a real-valued random variable  $X \in L^2(\Omega, \mathcal{F}, \mu; \mathbb{R})$  is

$$X = \sum_{i=0}^{\infty} x_i \phi_i, \quad \text{with } x_i = \frac{\langle X, \phi_i \rangle}{\langle \phi_i, \phi_i \rangle}, \quad (2)$$

where  $x_i \in \mathbb{R}$  is called the  $i$ th coefficient and

$$\langle \phi_i, \phi_j \rangle \doteq \int_{\Omega} \phi_i(\omega) \phi_j(\omega) d\mu(\omega),$$

see (Sullivan, 2015; Xiu and Karniadakis, 2002).

In order to obtain a computationally tractable formulation, one has to truncate the PCE series after  $L + 1$  terms

$$X \approx \sum_{i=0}^L x_i \phi_i. \quad (3)$$

The choice of the basis polynomials can be inferred via the Wiener-Askey scheme (Sullivan, 2015). For example, in case of a standard Gaussian random variable one would consider Hermite polynomials as they allow modeling the Gaussian with the first two terms of the PCE series. Moreover, in case of explicit polynomial maps—e.g., consider state transitions  $X(k) = A^k X(0)$ —of finite degree, one can quantify the truncation errors arising from considering only the first  $L + 1$  PCE coefficients. One may also infer  $L \in \mathbb{N}$  such that no truncation error arises. We refer to Mühlpfordt et al. (2018) for details.

Finally, we remark that whenever the truncated series representation (3) is exact, the first two moments of a random variable  $X$  can be computed in terms of PCE coefficients as follows

$$\mathbb{E}[X] = x_0 \quad \text{and} \quad \mathbb{V}[X] = \sum_{i=1}^L x_i^2 \langle \phi_i, \phi_i \rangle.$$

### 3. SIMULATION STUDY

We consider numerical examples that show the turnpike phenomena in the stochastic setting: The first example is an extension of the motivating example but with uncertain initial condition and additive Gaussian noise. The second example considers a linearized chemical reactor subject to non-Gaussian noise. The third example extends the second one via variance penalization in the objective.

#### 3.1 Scalar Dynamics with Noise

Consider the stochastic variant of the motivating example

$$X(k+1) = 2X(k) + U(k) + W(k), \quad X(0) = X_0, \quad (4)$$

where  $X_0$  denotes the initial random variable with known probability distribution  $p_{X_0}$ ,  $W(k)$  denotes system noise modeled as a white Gaussian noise such that all  $W(k)$  have an identical known probability distribution  $p_W$ . We arrive at the following stochastic LQ OCP

$$\min_{U(\cdot), X(\cdot)} \sum_{k=0}^{N-1} \mathbb{E}[U^2(k)] \quad (5a)$$

$$\text{s.t. } (4), \quad \forall k \in \mathbb{I}_{[0, N-1]} \quad (5b)$$

$$\mathbb{P}[x_{\min} \leq X(k) \leq x_{\max}] \geq \varepsilon_x, \quad \forall k \in \mathbb{I}_{[0, N]} \quad (5c)$$

$$X_0 \sim p_{X_0}, \quad W(k) \sim p_W, \quad \forall k \in \mathbb{I}_{[0, N-1]} \quad (5d)$$

where  $x_{\min} = -2$  and  $x_{\max} = 2$ . We approximate the chance constraint as

$$x_{\min} \leq \mathbb{E}[X(k)] \pm \lambda(\varepsilon_x) \sqrt{\mathbb{V}[X(k)]} \leq x_{\max} \quad (6)$$

with  $\lambda = \sqrt{(1 + \varepsilon_x)/(1 - \varepsilon_x)}$ , for a derivation see (Farina et al., 2013).

We consider  $X_0$  to follow a uniform distribution with the support  $[x_{0\min}, x_{0\max}] = [0.6, 1.4]$ . The noise at time  $k \in \mathbb{I}_{[0, N-1]}$ ,  $W(k)$ , is a Gaussian distribution with mean  $\mathbb{E}[W(k)] = 0$  and variance  $\mathbb{V}[W(k)] = 0.5^2$ . Additionally,  $\varepsilon_x$  is set to 0.8 and thus we have  $\lambda = 3$ .

Without the noise, considering a first-order PCE ( $L = 1$ ) for  $X(k)$  and  $U(k)$  with identical basis functions, exactness of the PCE representation of  $X(k+1)$  is guaranteed since the system dynamic is linear, see Mühlpfordt et al. (2018). Therefore, including noise, the PCE dimension needed for an exact representation is determined by the horizon  $N$ . More precisely,  $L = N + 1$ , where two PCE terms are induced by the uncertainty of the initial condition and the rest is caused by the noise. The PCE basis and coefficients read

$$X(k) = \sum_{i=0}^1 x_i(k) \phi_i + \sum_{i=0}^{N-1} x_i^w(k) \phi_i^w \quad (7a)$$

$$U(k) = \sum_{i=0}^1 u_i(k) \phi_i + \sum_{i=0}^{N-1} u_i^w(k) \phi_i^w \quad (7b)$$

$$W(k) = \sum_{i=0}^{N-1} w_i^w(k) \phi_i^w \quad (7c)$$

$$X_0 = \sum_{i=0}^1 x_{0i} \phi_i \quad (7d)$$

with

$$\phi_0 = 1, \quad \phi_1 = \xi - 0.5, \quad \phi_i^w = \theta_i, \quad i \in \mathbb{I}_{[0, N-1]} \quad (7e)$$

$$w_k^w(k) = \sqrt{\mathbb{V}[W(k)]}, \quad w_i^w(k) = 0, \quad i \in \mathbb{I}_{[0, N-1]} \setminus \{k\} \quad (7f)$$

where  $\xi$  is a standard uniformly distributed random variable while  $\theta_i$  are independent Gaussian random variables. Additionally,  $x_{00} = (x_{0\min} + x_{0\max})/2$  and  $x_{01} = x_{0\max} - x_{0\min}$ , since  $X_0$  is uniformly distributed.

#### Expressing Stochastic Filtrations in the PCE Framework

Stochastic modeling via adapted filtrations expresses the idea that the noise  $W(k)$  at time  $k$  influences the next state  $X(k+1)$  and input  $U(k+1)$  and the subsequent time instances  $i > k$  but not at  $i \leq k$ . In terms of PCE representation this implies that

$$\begin{bmatrix} Z(0) \\ Z(1) \\ Z(2) \\ \vdots \\ Z(M) \end{bmatrix} = \begin{bmatrix} z_0(0) & z_1(0) \\ z_0(1) & z_1(1) \\ z_0(2) & z_1(2) \\ \vdots & \vdots \\ z_0(M) & z_1(M) \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} + Z^w \begin{bmatrix} \phi_0^w \\ \phi_1^w \\ \phi_2^w \\ \vdots \\ \phi_{M-1}^w \end{bmatrix} \quad (8)$$

with

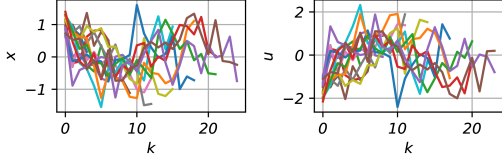


Fig. 2. 16 sample realization trajectories of optimal solutions to OCP (8).

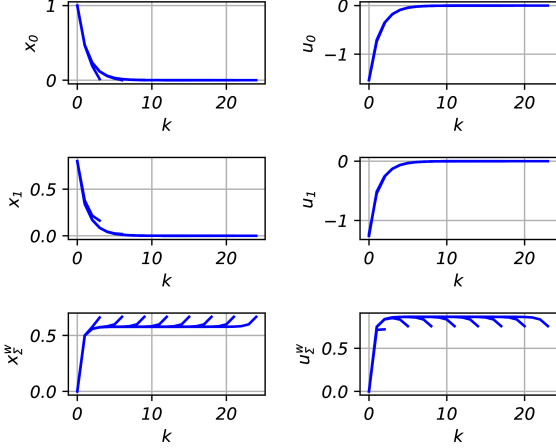


Fig. 3. PCE coefficients solutions to OCP (8).

$$Z^w \doteq \begin{bmatrix} 0 & 0 & \cdots & 0 \\ z_0^w(1) & 0 & \cdots & 0 \\ z_0^w(2) & z_1^w(2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_0^w(M) & z_1^w(M) & \cdots & z_{M-1}^w(M) \end{bmatrix} \in \mathbb{R}^{(M+1) \times M},$$

and where, respectively,  $Z \in \{X, U\}$  and  $\mathbf{z} \in \{x, u\}$  are placeholders and  $M = N$  for  $X$  and  $M = N - 1$  for  $U$ .

### Results

To illustrate turnpike behavior, we solve the stochastic OCP (5) as given above over different optimization horizons  $N = 3, 6, \dots, 24$  using `PolyChaos.jl` (Mühlpfordt et al., 2020). Fig. 2 shows the trajectories of the optimal solutions for a total of 16 realizations of the uncertainties. At first glance, the realizations of the solutions are noisy and appear not to exhibit the turnpike property.

Hence, in order to uncover the turnpike, we consider the trajectories of the PCE coefficients. Note that each PCE basis  $\phi_i^w$  induced by the noise is a standard Gaussian distributed random variable, the sum  $\sum_{i=0}^{k-1} z_i^w(k) \phi_i^w$ ,  $\mathbf{z} \in \{x, u\}$  is equal to a new Gaussian distributed random variable with zero mean and variance. Hence, for the sake of simplified illustration, we consider

$$z_\Sigma^w(k) \phi_\Sigma^w = \sum_{i=0}^{k-1} z_i^w(k) \phi_i^w \quad (9a)$$

with

$$\phi_\Sigma^w = \theta_\Sigma, \quad z_\Sigma^w(k) = \sqrt{\sum_{i=0}^{k-1} (z_i^w(k))^2}. \quad (9b)$$

Here  $\phi_\Sigma$  is a random variable with standard Gaussian distribution (zero mean). Therefore, instead of  $z_i^w(k)$ ,  $i \in$

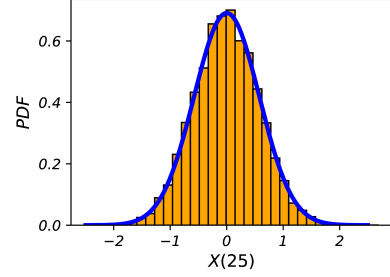


Fig. 4. Comparison between PDF of the optimal steady state and the histogram of state trajectories at  $k = 25$ .

$\mathbb{I}_{[0, N-1]}$ , only one PCE coefficient  $z_\Sigma^w(k)$  suffices to represent the uncertainty caused by noise. Note that this transformation is used only for illustration purposes and not in the underlying computation.

Using the PCE reformulation detailed above, Fig. 3 illustrates that the turnpike phenomenon occurs in terms of PCE coefficients. Actually, the turnpike property of PCE coefficients suggests that the optimal steady-state is a random variable with stationary distribution. We approximate this distribution by solving

$$\min_{\bar{X}, \bar{U}} \mathbb{E}[\bar{U}^2]$$

subject to

$$\bar{X} = 2\bar{X} + \bar{U} + \bar{W}$$

$$x_{min} \leq \mathbb{E}[\bar{X}] \pm \lambda(\varepsilon_x) \sqrt{\mathbb{V}[\bar{X}]} \leq x_{max},$$

where similar to before  $\bar{W}(k) \sim p_W$  is modelled as Gaussian noise and  $\bar{X}$ ,  $\bar{U}$  are expressed via PCE. We obtain the stationary distribution depicted in Fig. 4. This figure also shows the histogram at  $k = 25$  obtained from sampling  $10^4$  realizations of the uncertainty from the optimal PCE solution for  $N = 50$ . As one can see, the behavior in the middle of the horizon corresponds to the solution obtained for the steady-state problem.

The time evolution of the state histograms and the distributions obtained via PCE is shown in Fig. 5 for  $k = 0, 10, 20, 30, 40, 50$ . It is not surprising that the histograms follow the calculated PDF perfectly and the state  $X(k)$  keeps the same distribution in the middle of trajectory. This illustrates the turnpike phenomenon in the distributions.

### 3.2 Stochastic LQ OCP for a CSTR

As a second example, we consider a linearized CSTR reactor which appeared in several papers such as (Zanon and Faulwasser, 2018). The expected value quadratic stage cost and the linear discrete-time system with noise read

$$\ell(X, U) = \mathbb{E}[X_A + 0.1U^2 + 24X_B - 0.5U] \quad (10a)$$

$$f(X, U) = \begin{bmatrix} 0.76 & 0 \\ 0.12 & 0.88 \end{bmatrix} \begin{bmatrix} X_A \\ X_B \end{bmatrix} + \begin{bmatrix} 0.005 \\ -0.005 \end{bmatrix} U + \begin{bmatrix} W_A \\ W_B \end{bmatrix} \quad (10b)$$

where  $W_A, W_B$  are modeled as independent uniformly distributed random variables with support  $[-0.1, 0.1]$ . The initial state  $X(0)$  is a Gaussian distributed random vector with known mean  $(0.5, 0.8)$  and variance  $(0.05^2, 0.08^2)$ .

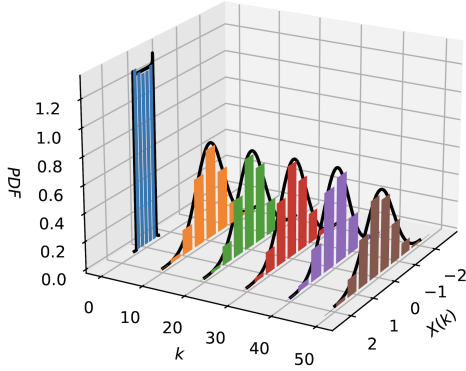


Fig. 5. Comparison of PDF and histogram of  $10^4$  samples for OCP (8) with  $N = 50$ .

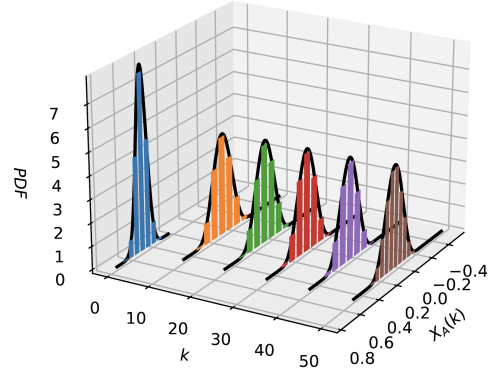


Fig. 8. Comparison of PDF and histogram of  $10^4$  samples for the OCP with (10)  $N = 50$ .

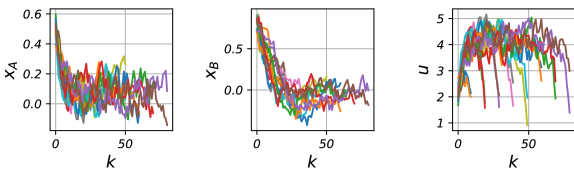


Fig. 6. Sample trajectories for the OCP with (10).

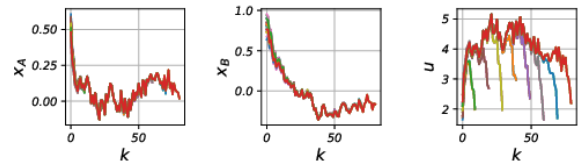


Fig. 9. System response and corresponding input for the OCP with (10): different initial conditions and horizons with one identical noise realization.

- Pick horizons  $N_1, \dots, N_s \leq M$  and corresponding realizations of the initial condition  $X_0(\omega_1), \dots, X_0(\omega_s)$ .
- For all horizons  $N_1, \dots, N_s$  and the initial condition  $X_0(\omega_1), \dots, X_0(\omega_s)$ , simulate the response of the dynamics under the optimal input policy, while the disturbance sequence is fixed to  $\{W(k; \omega_k)\}$  (or a subpart thereof).

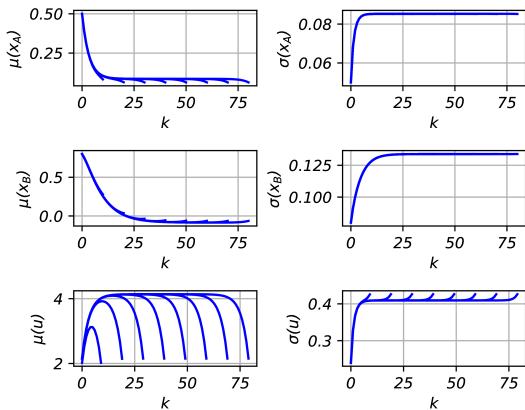


Fig. 7. Mean and variance obtained from the PCE coefficients for the OCP with (10).

We solve the stochastic OCP via PCE over different horizons  $N = 10, 20, \dots, 80$ . We obtain the trajectories of the optimal solutions for a total 16 realization sequences of the uncertainties, see Fig. 6. As the dimension of PCE is quite large, we plot the first two moments instead of PCE coefficients, i.e. mean and the variance of state and input random variables, see Fig. 7. Similar to the previous example, the trajectories of mean and variance exhibit the turnpike property. Similar to before, Fig. 8 depicts the histograms of the state  $X_A$  at  $k = 0, 10, 20, 30, 40, 50$  for the considered realizations and the PDF obtained via PCE, see Fig. 8.

As an additional means of assessing the turnpike phenomenon via simulation, we consider the following numerical setting:

- Compute a random realization of the disturbance  $W(k)$ , for  $k \in \mathbb{I}_{[0, M]}$  denoted as  $\{W(k; \omega_k)\}$

The results of this numerical experiment are depicted in Figure 9. As one can see, all the trajectories approach the same solution after some time. One can understand this solution as a time-varying path of a stationary turnpike solution, whose shape is governed by the considered disturbance sequence. Observe the difference to Figure 6, wherein for each trajectory a different disturbance realization sequence is considered.

### 3.3 Stochastic LQ OCP with Variance Penalization

In order to obtain an optimal but narrow steady-state distribution the penalization of the variance is one option. Hence, we consider the previous example augmented with a penalty of the variance of the state  $X_A$  in the stage cost

$$\ell(X, U) = \mathbb{E}[X_A + 0.1U^2 + 24X_B - 0.5U] + \gamma \mathbb{V}[X_A] \quad (11)$$

Here we choose  $\gamma = 10^4$  and solve the stochastic OCP via PCE over different horizons  $N = 10, 20, \dots, 80$ . The trajectories of the optimal solutions for a total 16 realization sequences of the uncertainties are shown in Fig. 10. Observe that the state  $x_A$  is much less effected by the noise. It can also be seen in Fig. 11 that the variance of state  $X_A$  is smaller than in Example 2, while the variance of state  $X_B$  and the variance of the input  $U$  increase. Fig. 12 also shows that the optimal steady state  $\bar{X}_A$  has a narrower distribution.

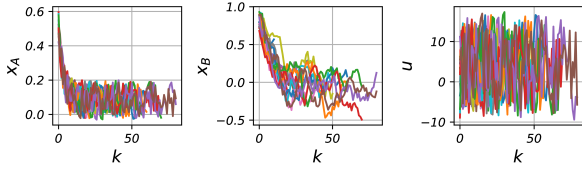


Fig. 10. Sample trajectories for the OCP with (11).

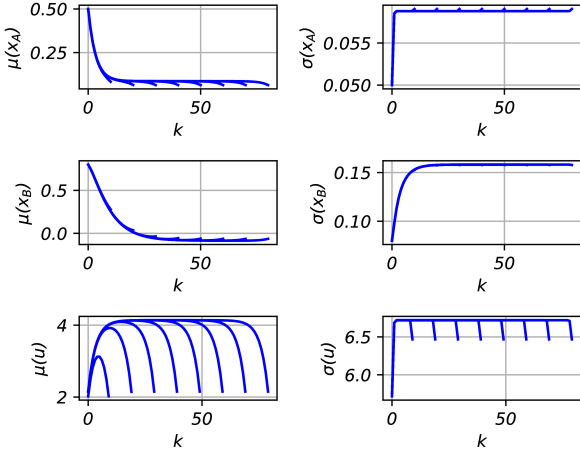


Fig. 11. Mean and variance obtained from the PCE coefficients for the OCP with (11).

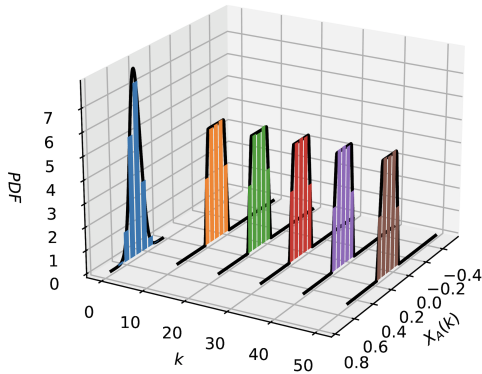


Fig. 12. Comparison of PDF and histogram of  $10^4$  samples for the OCP with (11)  $N = 50$ .

#### 4. SUMMARY

This paper has conducted a simulation study on turnpike properties in stochastic OCPs. It has presented three examples of stochastic LQ OCPs all of which exhibit the turnpike phenomenon. Indeed, the examples illustrate that turnpike phenomena can be observed in different contexts:

- in terms for statistical moments (or PCE coefficients which can be mapped to moments),
- in terms of probability distributions of state and input variables staying close to their optimal stationary distributions, and
- in terms of the realization trajectories staying close to an orbit defined by the noise realization.

Moreover, our simulation study demonstrates that beyond the usual minimization of expected values, the turnpike phenomenon is also present in combination of expected value and min-variance objectives.

While this note did merely present simulation results, there is a clear prospect of extending the established notations of turnpike properties to stochastic OCPs and corresponding stochastic MPC formulations. Yet, at this stage, there is also an evident need for analytic results to understand the turnpike phenomenon in stochastic settings.

#### REFERENCES

- Bayer, F., Lorenzen, M., Müller, M., and Allgöwer, F. (2016). Robust economic model predictive control using stochastic information. *Automatica*, 74, 151–161.
- Carlson, D., Haurie, A., and Leizarowitz, A. (1991). *Infinite Horizon Optimal Control: Deterministic and Stochastic Systems*. Springer Verlag.
- Damm, T., Grüne, L., Stieler, M., and Worthmann, K. (2014). An exponential turnpike theorem for dissipative optimal control problems. *SIAM Journal on Control and Optimization*, 52(3), 1935–1957.
- Dorfman, R., Samuelson, P., and Solow, R. (1958). *Linear Programming and Economic Analysis*. McGraw-Hill, New York.
- Farina, M., Giulioni, L., Magni, L., and Scattolini, R. (2013). A probabilistic approach to model predictive control. In *52nd IEEE Conference on Decision and Control*, 7734–7739. IEEE.
- Faulwasser, T. and Grüne, L. (2021). Turnpike properties in optimal control: An overview of discrete-time and continuous-time results. In E. Zuazua and E. Trelat (eds.), *Handbook of Numerical Analysis*. Elsevier. Arxiv: 2011.13670. In press.
- Faulwasser, T., Grüne, L., and Müller, M. (2018). Economic nonlinear model predictive control: Stability, optimality and performance. *Foundations and Trends in Systems and Control*, 5(1), 1–98. doi:10.1561/26000000014.
- Faulwasser, T., Korda, M., Jones, C., and Bonvin, D. (2017). On turnpike and dissipativity properties of continuous-time optimal control problems. *Automatica*, 81, 297–304. doi:10.1016/j.automatica.2017.03.012.
- Fristedt, B. and Gray, L. (2013). *A modern approach to probability theory*. Springer Science & Business Media.
- Gerster, S., Herty, M., Chertkov, M., Vuffray, M., and Zlotnik, A. (2019). Polynomial chaos approach to describe the propagation of uncertainties through gas networks. In *Progress in Industrial Mathematics at ECMI 2018*, 59–65. Springer.
- Grüne, L. (2013). Economic receding horizon control without terminal constraints. *Automatica*, 49(3), 725–734.
- Grüne, L. and Müller, M. (2016). On the relation between strict dissipativity and turnpike properties. *Sys. Contr. Lett.*, 90, 45 – 53.
- Kim, K., Shen, D., Nagy, Z., and Braatz, R. (2013). Wiener’s polynomial chaos for the analysis and control of nonlinear dynamical systems with probabilistic uncertainties. *IEEE Control Systems*, 33(5), 58–67.
- Kolokoltsov, V. and Yang, W. (2012). Turnpike theorems for Markov games. *Dyn. Games Appl.*, 2(3), 294–312.
- Marimon, R. (1989). Stochastic turnpike property and stationary equilibrium. *J. Econom. Theory*, 47(2), 282–306.
- McKenzie, L. (1976). Turnpike theory. *Econometrica: Journal of the Econometric Society*, 44(5), 841–865.
- Mesbah, A. and Streif, S. (2015). A probabilistic approach to robust optimal experiment design with chance constraints. *IFAC-PapersOnLine*, 48(8), 100–105.
- Mühlpfordt, T., Faulwasser, T., Roald, L., and Hagenmeyer, V. (2017). Solving optimal power flow with non-gaussian uncertainties via polynomial chaos expansion. In *Proc. of 56th IEEE Conference on Decision and Control*, 4490–4496. Melbourne, Australia. doi:10.1109/CDC.2017.8264321.
- Mühlpfordt, T., Findeisen, R., Hagenmeyer, V., and Faulwasser, T. (2018). Comments on quantifying truncation errors for polynomial chaos expansions. *IEEE Control Systems Letters*, 2(1), 169–174. doi: 10.1109/LCSYS.2017.2778138.
- Mühlpfordt, T., Zahn, F., Hagenmeyer, V., and Faulwasser, T. (2020). PolyChaos.jl – a julia package for polynomial chaos in systems and control. In *Proceedings of the 21. IFAC World Congress*.
- Paulson, J., Mesbah, A., Streif, S., Findeisen, R., and Braatz, R. (2014). Fast stochastic model predictive control of high-dimensional systems. In *53rd IEEE Conference on decision and Control*, 2802–2809. IEEE.
- Sullivan, T.J. (2015). *Introduction to uncertainty quantification*, volume 63. Springer.
- Trelat, E. and Zuazua, E. (2015). The turnpike property in finite-dimensional nonlinear optimal control. *Journal of Differential Equations*, 258(1), 81–114.
- Wiener, N. (1938). The homogeneous chaos. *American Journal of Mathematics*, 897–936.
- Xiu, D. and Karniadakis, G.E. (2002). The Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM Journal on Scientific Computing*, 24(2), 619–644.
- Zanon, M. and Faulwasser, T. (2018). Economic MPC without terminal constraints: Gradient-correcting end penalties enforce stability. *Journal of Process Control*, 63, 1–14. doi:10.1016/j.jprocont.2017.12.005.