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Optimal realizable networked controllers for networked systems

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***Abstract*— In this paper, we provide the optimal solution for the H_2 control problem in the case when both plant and controller are interconnected systems interacting over an arbitrary directed causal network. We first analyze the structure of network realizable systems and then characterize the set of all stabilizing controllers that are realizable over the given network using the Youla parametrization. The H_2 control problem is then cast as a convex optimization problem and its solution is shown to provide the optimal distributed controller over the given network. The results of this paper allow one to apply many classical results and approaches of multi-variable robust control to networked systems.**

***Index Terms*—** Distributed control, Network realizability theory, Networked control systems, Optimal H_2 control

I. INTRODUCTION

With increasing number of applications in the field of networked or spatially interconnected systems, there has been a great surge in research towards distributed control problems. The interconnected systems in consideration are a (possibly large) group of individual sub-systems interacting over a communication network. The main objective is to find controllers satisfying the desired performance criteria and implementable over the existing communication network in a distributed fashion. In the recent literature on distributed controller synthesis, the problem has been analyzed for various classes of interconnected systems like spatially invariant systems [1]–[4], systems with triangular and band structures [5], [6], symmetrically interconnected systems [7] and a more general case of systems satisfying quadratic invariance property [8], [9]. Identical sub-systems connected over a graph with diagonalizable "pattern matrix" were considered by [10] and heterogeneous sub-systems connected over an arbitrary undirected graphs were considered by [11].

In [1]–[8] and many references within, the symmetry and invariance properties are well exploited in order to obtain tractable algorithms to solve distributed optimal control problems for large scale distributed systems. The methodologies used in the literature can be classified into two categories: transfer function and state-space approaches. In both these cases, conditions are given for a particular transfer function or a state-space model to be well-posed and be compatible with a given network interconnection [8], [11]. However, in transfer function approaches, it is not clear how to implement a designed controller in terms of local controllers exchanging information over the given network. Only in special cases [12], reconstruction of local controllers is known. In a state-space approach, the conditions for realizability over the given network can be easily specified [13].

In [8], given a system, a characterization of all stabilizing controllers that belong to a constraint set S is provided when the set S is quadratically invariant under the given system. In the case where the system and the controller are restricted to be interconnected systems over a given network, the results cannot be directly extended since [8] does not provide any necessary conditions for the designed controller to be implementable on the given network as a group of n sub-systems interacting over the causal network $\mathcal{G}$. In this paper, we address this issue and provide an alternative solution that ensures the network realizability of the designed optimal controller.

The outline of this paper is as follows: Section II introduces the notation used in the paper to describe systems on networks. In Section III, interconnected systems over causal networks are described using state-space and input-output representations. The main problem under consideration is described in Section IV, followed by the results on network realizability and parametrization of all stabilizing network realizable controllers in Section V.

II. NOTATION

A. Graph model

In this paper, we deal with networks of systems that are best described using a directed pseudograph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ where $\mathcal{V} = \{1, \dots, n\}$ represents the nodes of the graph or the sub-systems in the network, and $\mathcal{A} \subset \mathcal{V}^2$ represents the arc-set or the set of communication links between different sub-systems. We say, arc $(i, j) \in \mathcal{A}$ if there exists a directed link from node i to node j. For ease of notation, we consider $(i,i) \in \mathcal{A}\ \forall i \in \mathcal{V}$. Define directed neighborhoods around each node i, $\mathcal{N}_i^- = \{j | (j,i) \in \mathcal{A}\}$ and $\mathcal{N}_i^+ = \{j | (i,j) \in \mathcal{A}\}$, which are the sets of nodes that have arcs to and from node i. With a slight abuse of notation, we shall refer to the network by $\mathcal{G}$ along with the underlying graph representing the network.

B. General

We refer to a column-vector as *vector*. To make representations compact, we use the notation $[x_i]_{i \in \mathcal{I}}$ for vertical concatenation of vectors or matrices $\{x_i\}_{i \in \mathcal{I}}$, of appropriate dimension, where $\mathcal{I}$ is an index set. Let $[x_{ij}]_{i,j \in \mathcal{I}}$ represent a block-matrix where the $(i, j)^{\text{th}}$ block is a matrix x_{ij}. Also, let $\mathbf{diag}[x_i]_{i \in \mathcal{I}}$ denote the matrix formed by arranging the vectors or matrices $\{x_i\}_{i \in \mathcal{I}}$ in a block-diagonal fashion and the remaining entries being zeros. Sometimes, if the

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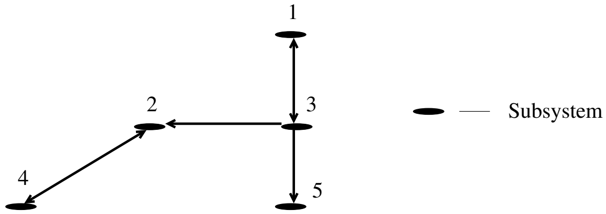

Fig. 1. An example of an interconnected system represented by a directed pseudograph.

index set $\mathcal{I}$ equals $\{1,\ldots,n\}$, then we will not explicitly mention the index set. Given a matrix $A = \left[a_1 \ldots a_n\right] \in \mathbb{C}^{m\times n}$, where $\{a_i\}_i$ denote the columns of A, we associate a vector $\mathbf{vec}(A) = [a_i]_i \in \mathbb{C}^{mn}$ which is a vector formed by vertically concatenating the columns of matrix A. Define $\mathbf{vec}^{-1}(\cdot)$ as the inverse operation of the $\mathbf{vec}(\cdot)$ such that $\mathbf{vec}^{-1}(\mathbf{vec}(A)) = A$.

A block-matrix A is said to be *structured according to* a graph $\mathcal{G} = (\mathcal{V},\mathcal{A})$ if it is of the form $[A_{ij}]_{i,j\in\{1,\ldots,n\}}$ and $A_{ij} = 0$ when $(j,i) \notin \mathcal{A}$. When required, we shall use I for an identity matrix and 0 for a zero matrix of appropriate size. The dimensions of the sub-matrices $\{A_{ij}\}_{i,j}$ are described using two integer-valued vectors as follows. Let $\mathcal{P}_a = (a_1,\ldots,a_n)$ and $\mathcal{P}_b = (b_1,\ldots,b_n)$ be two $n-$tuples with a_i and b_i being integers for all i. Then, matrix A is said to be *partitioned according to* $(\mathcal{P}_a,\mathcal{P}_b)$ if the sub-matrix A_{ij} has dimension $a_i \times b_j\ \forall i, j$. This definition of partitioning is easily extended to the case of vectors too. A vector x is said to be *partitioned according to* $\mathcal{P}_a$ if it can be written as $[x_i]_{i\in\{1,\ldots,n\}}$ where x_i is a real vector of size a_i for all $i \in \{1,\ldots,n\}$.

III. Interconnected systems

A group of sub-systems interacting over a communication network is termed as a *networked* or an *interconnected system* (see Fig. 1). The interconnected system is characterized by 1) topology of the network; 2) local dynamics of the sub-systems; 3) interaction between the sub-subsystems over the communication network.

We consider n sub-systems $\{P_i\}_{i\in\{1,\ldots,n\}}$ interacting over a network represented by a directed graph $\mathcal{G} = (\mathcal{V},\mathcal{A})$ with the sub-systems at its vertices and communication links corresponding to the arcs. Let each sub-system P_i be described by its states $x_i(k)$, local inputs $u_i(k)$, local outputs $y_i(k)$, network inputs $\nu_i(k)$ and network outputs $\eta_i(k)$. In this paper, we consider each sub-system P_i to be a discrete-time causal finite-dimensional linear time-invariant (FDLTI) system with a state-space description

$$\begin{bmatrix} x_i(k+1) \\ y_i(k) \\ \eta_i(k) \end{bmatrix} = \begin{bmatrix} A_i & B_i^u & B_i^\nu \\ C_i^y & D_i^{yu} & D_i^{y\nu} \\ C_i^\eta & D_i^{\eta u} & D_i^{\eta\nu} \end{bmatrix} \begin{bmatrix} x_i(k) \\ u_i(k) \\ \nu_i(k) \end{bmatrix} \tag{1}$$

Note that, $\eta_i(k) = [\eta_{ij}(k)]_{j\in\mathcal{N}_i^+\setminus\{i\}}$ and $\nu_i(k) = [\nu_{ij}(k)]_{j\in\mathcal{N}_i^-\setminus\{i\}}\ \forall\ i$ which correspond to the overall set of messages transmitted and received by P_i, where $\eta_{ij}(k)$ is the message vector passed from plant P_i to P_j at the time instant k and $\nu_{ij}(k)$ is the message passed in the other direction.

A. Causal interconnection

A network is said to be a *causal interconnection* if the nodes or the sub-systems on the network cannot relay the incoming information obtained from the neighboring nodes in the same time instant, i.e. the outgoing information at each time instant can only depend on the local information available at that time instant.

In the case of an interconnected system described by (1), the interconnection is said to be causal if $D_i^{\eta\nu} = 0$ for all i, i.e. P_i can not relay the information about $\nu_i(k)$ to its neighbors in the same time instant k that it receives. In this paper, we assume that the considered network is a causal interconnection and the communication links are noiseless, delay-free and have no bandwidth constraints. Thus $\nu_{ij}(k) = \eta_{ji}(k)$ for all i, j. In more practical scenarios, we have to incorporate the stochastic nature and frequency response of the communication links, which will be addressed in a future work.

Remark 1: In some cases, an additional condition $D_i^{y\nu} = 0$ for all i may appear while modeling systems on networks to ensure that the incoming information from neighboring nodes of P_i cannot be relayed to the controller unit during the same time instant. The procedure developed in this paper can be extended to such cases with slight modifications that incorporate the additional constraint.

B. State-space realizations of interconnected systems

Let the state-space equations corresponding to the dynamics of each sub-system P_i be given by (1) with $D_i^{\eta\nu} = 0$ for all i. From the individual dynamics of P_i and the causal interconnection, the state-space equations for the complete interconnected system P are obtained in the following manner.

$$\begin{aligned} B_i^\nu \nu_i(k) &= B_i^\nu [\eta_{ji}(k)]_{j\in\mathcal{N}_i^-\setminus\{i\}} \\ &= B_i^\nu [C_{ji}^\eta x_j(k) + D_{ji}^{\eta u} u_j(k)]_{j\in\mathcal{N}_i^-\setminus\{i\}} \\ &= \sum_{j\in\mathcal{N}_i^-\setminus\{i\}} (A_{ij}x_j(k) + B_{ij}^u u_j(k)) \end{aligned} \tag{2}$$

for some appropriate values of A_{ij} and B_{ij}^u. Similarly, by expressing $D_i^{y\nu}\nu_i(k)$ in terms of some appropriate matrices C_{ij}^y and D_{ij}^{yu}, (1) can be written as

$$\begin{aligned} x_i(k+1) &= \sum_{j\in\mathcal{N}_i^-} A_{ij}x_j(k) + \sum_{j\in\mathcal{N}_i^-} B_{ij}^u u_j(k) \\ y_i(k) &= \sum_{j\in\mathcal{N}_i^-} C_{ij}^y x_j(k) + \sum_{j\in\mathcal{N}_i^-} D_{ij}^{yu} u_j(k) \end{aligned} \tag{3}$$

where $A_{ii} = A_i$, $B_{ii}^u = B_i^u$, $C_{ii}^y = C_i^y$, $D_{ii}^{yu} = D_i^{yu}$ for all i. By defining A_{ij}, B_{ij}^u, C_{ij}^y and D_{ij}^{yu} to be zero for $(j,i) \notin \mathcal{A}$, the state-space representation of the interconnected system P can be represented as

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_u \\ C_y & D_{yu} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} \tag{4}$$

where $x(k) = [x_i(k)]_i$, $u(k) = [u_i(k)]_i$ and $y(k) = [y_i(k)]_i$. Note that $A := [A_{ij}]_{i,j}$, $B_u := [B^u_{ij}]_{i,j}$, $C_y := [C^y_{ij}]_{i,j}$ and $D_{yu} := [D^{yu}_{ij}]_{i,j}$ are structured according to network $\mathcal{G}$. These denote the *sparsity* constraints on the state-space matrices of the interconnected systems. Also note that A is partitioned according to $(\mathcal{P}_x, \mathcal{P}_x)$, B_u according to $(\mathcal{P}_x, \mathcal{P}_u)$, C_y according to $(\mathcal{P}_y, \mathcal{P}_x)$ and D_{yu} according to $(\mathcal{P}_y, \mathcal{P}_u)$, where $\mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y$ are the partitions of $x(k)$, $u(k)$ and $y(k)$, respectively.

Definition 1: Given the network $\mathcal{G}$ and the partitions $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$, the set of state-space realizations (A, B_u, C_y, D_{yu}) with the state-space matrices structured according to $\mathcal{G}$ is denoted by $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$.

We also define the set $\mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y) = \bigcup_{\mathcal{P}_x \in \mathbb{N}^n} \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$. Note that the state-space realization of any interconnected system built on a causal network interaction $\mathcal{G}$ belongs to the set $\mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Later, we show that given any element of $\mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, one can obtain a corresponding interconnected system with n sub-systems interacting over a given network while maintaining internal stability.

Remark 2: In the case of Remark 1, when $D^{y\nu}_i = 0$ for all i, note that the resulting state-space matrices in (4) will have the following format: A and B_u are structured according to $\mathcal{G}$ while C_y and D_{yu} are block-diagonal. This constraint only changes the structure of the state-space realizations of interconnected systems built on $\mathcal{G}$ but the framework designed in this paper still holds, with minor modifications. More details regarding this case will be provided in a future work.

C. Transfer functions of interconnected systems

In this section, we briefly consider the structure of the transfer functions of interconnected systems over a given network. Applying the ideas presented in [14] to a discrete-time model, one can show that the transfer function of any discrete-time, causal, FDLTI interconnected system over a given causal network interconnection $\mathcal{G}$ can be described in terms of the *delay* and *sparsity* constraints. Note that $\mathcal{R}_p$ denotes the set of real-rational proper transfer function matrices and $\mathcal{RH}_\infty$ denotes the set of real-rational proper stable transfer function matrices.

Let $P(z) = [P_{ij}(z)]_{i,j}$ be a transfer function matrix corresponding to the considered interconnected system over $\mathcal{G}$, where $P_{ij}(z)$ is the transfer function matrix from input vector $u_j(k)$ to output vector $y_i(k)$. Note that $P(z)$ is partitioned according to $(\mathcal{P}_y, \mathcal{P}_u)$.

- *Delay constraints:* If $l(j,i)$ is the length of the shortest path from node j to node i, then the input vector $u_j(k)$ at node j can effect the output vector $y_i(k)$ at node i only after $l(j,i) - 1$ time instants in order to preserve the causality of the interconnection. Thus, $P_{ij}(z)$ can be written as $z^{-l(j,i)+1} H_{ij}(z)$ where $H_{ij}(z) \in \mathcal{R}_p$.
- *Sparsity constraints:* In the case when there exists no directed paths from node j to node i on $\mathcal{G}$, $P_{ij}(z)$ is a zero matrix with appropriate dimensions. This implies that the input at node j can not affect the output at node i. The sparsity condition can be treated as a special case of delay constraint where the delay is infinite.

Definition 2: Given a network $\mathcal{G}$ and the input and output partitions, $\mathcal{P}_u$ and $\mathcal{P}_y$, the set of transfer function matrices $P(z)$ that satisfy the corresponding delay and sparsity constraints is denoted by $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

Note that the transfer function matrix of any interconnected system built on a network $\mathcal{G}$ with appropriate input and output partitions belongs to the set $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. But given an element of $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, it is non-trivial to obtain a corresponding interconnected system with n sub-systems interacting over a given network while maintaining internal stability.

D. Realizable systems over a causal network

A system is said to be *realizable over a causal network* if it can be implemented as n individual sub-systems (with their local states, inputs and outputs corresponding to each node of the network) that pass messages to each other along the directed links while respecting the causal network interconnection and maintaining internal stability.

In the later part of the paper, we can see that this property is essential in designing distributed controllers over causal networks. Also note that the realizability conditions might change according to the properties of the network considered. We analyze the network realizability property of the elements of $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ and $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ in Section V.

IV. Problem description

We consider a group of n sub-systems, $\{P_i\}_i$, interacting over a given causal network interconnection $\mathcal{G}$ where the links are assumed to be noiseless, delay-free and have unlimited bandwidth. Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$. Including the local exogenous inputs $w_i(k)$ and local regulated outputs $z_i(k)$ in (1), the state-space description of the sub-system P_i is written as

$$\begin{bmatrix} x_i(k+1) \\ z_i(k) \\ y_i(k) \\ \eta_i(k) \end{bmatrix} = \begin{bmatrix} A_i & B^w_i & B^u_i & B^\nu_i \\ C^z_i & D^{zw}_i & D^{zu}_i & D^{z\nu}_i \\ C^y_i & D^{yw}_i & D^{yu} & D^{y\nu}_i \\ C^\eta_i & D^{\eta w}_i & D^{\eta u}_i & 0 \end{bmatrix} \begin{bmatrix} x_i(k) \\ w_i(k) \\ u_i(k) \\ \nu_i(k) \end{bmatrix} \tag{5}$$

In this paper, we consider only the class of interconnected plants for which D^{yu}_i, $D^{y\nu}_i$ and $D^{\eta u}_i$ are zero matrices for all i. By interconnecting these sub-systems over the causal network $\mathcal{G}$ and following the equations similar to (2), we can write (5) as follows

$$\begin{bmatrix} x(k+1) \\ z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B_w & B_u \\ C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \\ u(k) \end{bmatrix} \tag{6}$$

where $x(k) = [x_i(k)]_i$ and similarly the other inputs and outputs are made of the local inputs and outputs of all the plants. Note that the resulting state-space matrices A, B_w, C_z and D_{zw} are structured according to $\mathcal{G}$ while B_u, C_y, D_{zu} and D_{yw} are block-diagonal matrices, with appropriate partitions.

One can view the system P as $\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, a mapping from $\begin{bmatrix} w(k) \\ u(k) \end{bmatrix}$ to $\begin{bmatrix} z(k) \\ y(k) \end{bmatrix}$. The actual plant or process is in

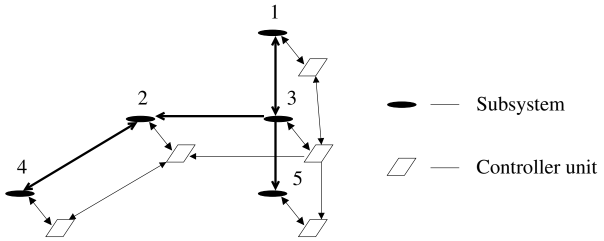

Fig. 2. An example of an interconnected plant and controller pair that are realizable over the same network.

fact P_{22} which is the map from $u(k)$ to $y(k)$. Note that, $P_{22} = (A, B_u, C_y, 0) \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ where $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$ are the partitions of $x(k)$, $u(k)$ and $y(k)$, respectively, with B_u and C_y being block-diagonal. We assume that the plant is stabilizable and detectable over the network $\mathcal{G}$, i.e. there exists a proper controller K (realizable over the network $\mathcal{G}$) that internally stabilizes P through output feedback.

In this paper, we consider interconnected systems with structure described in (6) with block-diagonal B_u and C_y to make the analysis easier (and the closed-loop interconnection well-posed) while conveying the basic methodology followed in the later sections. Analysis for a more general case, when B_u and C_y are not block-diagonal, will be considered in future.

The main objective of this paper is to design optimal controllers that are realizable over the given network $\mathcal{G}$ while minimizing the H_2 norm of the closed-loop mapping from $w(k)$ to $z(k)$. Figure 2 depicts such a plant-controller pair when both are constrained to be interconnected systems over the same network $\mathcal{G}$.

V. Main results

In this section, we first analyze the network realizability properties of the elements of $\mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ and $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Using these results, we describe the set of all stabilizing network realizable controllers corresponding to a given interconnected plant described by (6). This parametrization is then used to solve the distributed H_2 control problem using techniques corresponding to a centralized H_2 control problem.

A. Network realizability of $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ and $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$

Lemma 1: Given a causal network interconnection $\mathcal{G}$ and the partitions $\mathcal{P}_x$, $\mathcal{P}_u$ and $\mathcal{P}_y$, any system $Q \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ is realizable over the network $\mathcal{G}$.

Proof: To prove this statement, we need to find n sub-systems $\{Q_i\}_i$ with state-space representation of the form (1) which result in the given Q by interacting over the network $\mathcal{G}$ while maintaining internal stability.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$. By definition, Q has a state-space realization (A, B_u, C_y, D_{yu}), where A, B_u, C_y and D_{yu} are structured according to $\mathcal{G}$ and partitioned accordingly. So, we have matrices $\{A_{ij}\}_{i,j}, \{B^u_{ij}\}_{i,j}, \{C^y_{ij}\}_{i,j}$ and $\{D^{yu}_{ij}\}_{i,j}$ for all $i, j \in \{1, \ldots, n\}$ such that they are zero matrices when $(j,i) \notin \mathcal{A}$. Let the state, input and output vectors be defined according to the corresponding partitions. Then the dynamics of the system Q can be written as

$$\begin{aligned} x_i(k+1) &= \sum_{j \in \mathcal{N}_i^-} A_{ij} x_j(k) + \sum_{j \in \mathcal{N}_i^-} B^u_{ij} u_j(k) \\ y_i(k) &= \sum_{j \in \mathcal{N}_i^-} C^y_{ij} x_j(k) + \sum_{j \in \mathcal{N}_i^-} D^{yu}_{ij} u_j(k) \end{aligned} \qquad \forall i \in \mathcal{V} \quad (7)$$

We rewrite the above equations as

$$\begin{aligned} x_i(k+1) &= A_{ii} x_i(k) + B^u_{ii} u_i(k) \\ &\quad + \sum_{j \in \mathcal{N}_i^- \setminus \{i\}} A_{ij} x_j(k) + \sum_{j \in \mathcal{N}_i^- \setminus \{i\}} B^u_{ij} u_j(k) \\ &= A_i x_i(k) + B^u_i u_i(k) + B^\nu_i \nu_i(k) \\ y_i(k+1) &= C^y_{ii} x_i(k) + D^{yu}_{ii} u_i(k) \\ &\quad + \sum_{j \in \mathcal{N}_i^- \setminus \{i\}} C^y_{ij} x_j(k) + \sum_{j \in \mathcal{N}_i^- \setminus \{i\}} D^{yu}_{ij} u_j(k) \\ &= C^y_i x_i(k) + D^{yu}_i u_i(k) + D^{y\nu}_i \nu_i(k) \qquad \forall i \in \mathcal{V} \end{aligned}$$

where

$$A_i = A_{ii},\ B^u_i = B_{ii},\ C^y_i = C^y_{ii},\ D^{yu}_i = D^{yu}_{ii},$$
$$B^\nu_i = \mathbf{hor}\left[A_{ij} \,\vdots\, B^u_{ij}\right]_{j \in \mathcal{N}_i^- \setminus \{i\}},\ D^{y\nu}_i = \mathbf{hor}\left[C^y_{ij} \,\vdots\, D^{yu}_{ij}\right]_{j \in \mathcal{N}_i^- \setminus \{i\}},$$
$$\eta_{ji}(k) = \begin{bmatrix} x_j(k) \\ u_j(k) \end{bmatrix} \ \forall j \in \mathcal{N}_i^- \setminus \{i\},$$
$$\nu_i(k) = \left[\eta_{ji}(k)\right]_{j \in \mathcal{N}_i^- \setminus \{i\}}, \quad \forall i \in \mathcal{V}$$

where $\mathbf{hor}(\cdot)$ is used to denote horizontal concatenation of matrices. To ensure $\eta_{ji}(k) = \left[\begin{smallmatrix} x_j(k) \\ u_j(k) \end{smallmatrix}\right]$ for all $i \in \mathcal{V}$ and $j \in \mathcal{N}_i^- \setminus \{i\}$, we choose $C^\eta_{ij} = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $D^{\eta u}_{ij} = \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$ for all $j \in \mathcal{N}_i^+ \setminus \{i\}$, while $C^\eta_i := [C^\eta_{ij}]_{j \in \mathcal{N}_i^+ \setminus \{i\}}$ and $D^{\eta u}_i := [D^{\eta u}_{ij}]_{j \in \mathcal{N}_i^+ \setminus \{i\}}$. This choice of C^η_i and $D^{\eta u}_i$ leads to

$$\left[\eta_{ij}(k)\right]_{j \in \mathcal{N}_i^+ \setminus \{i\}} = \eta_i(k) = \begin{bmatrix} x_i(k) \\ u_i(k) \end{bmatrix}_{j \in \mathcal{N}_i^+ \setminus \{i\}}.$$

By defining Q_i as

$$\begin{bmatrix} x_i(k+1) \\ y_i(k) \\ \eta_i(k) \end{bmatrix} = \begin{bmatrix} A_i & B^u_i & B^\nu_i \\ C^y_i & D^{yu}_i & D^{y\nu}_i \\ C^\eta_i & D^{\eta u}_i & 0 \end{bmatrix} \begin{bmatrix} x_i(k) \\ u_i(k) \\ \nu_i(k) \end{bmatrix} \quad \forall i \in \mathcal{V},$$

we obtain n sub-systems $\{Q_i\}_i$ that interact over the network $\mathcal{G}$ to make the original system Q.

Note that, in defining the n sub-systems, we do not increase the number of states of the system. Thus, no internal instability is introduced. If overall number of states are increased, one needs to show that the additional states do not result in unstable pole-zero cancellations. ■

Since any realizable system on the network $\mathcal{G}$, with input and output partitions as $\mathcal{P}_u$ and $\mathcal{P}_y$, is an element of $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ for some state partition $\mathcal{P}_x$ and any element of $\mathfrak{S}(\mathcal{G}, \mathcal{P}_x, \mathcal{P}_u, \mathcal{P}_y)$ is realizable over $\mathcal{G}$ for any $\mathcal{P}_x$, we can say that the set $\mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ represents the set of all discrete-time, causal FDLTI systems that are realizable over a causal network interconnection $\mathcal{G}$. We denote the set of all stable systems in $\mathfrak{S}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ by $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

Theorem 1: Given a network represented by a directed pseudograph $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ and the input and output partitions, $\mathcal{P}_u$ and $\mathcal{P}_y$, any bounded-input bounded-output (BIBO) stable system $Q(z) \in \mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ is realizable over the network $\mathcal{G}$ for some state partition $\mathcal{P}_x$.

In general, a similar result as Theorem 1 cannot be applied to unstable transfer function matrices in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ while maintaining internal stability. Specific unstable transfer function matrices in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ may be shown to be realizable over $\mathcal{G}$ but not every element of $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ can be shown to be network realizable. Note that the procedure followed in the later part of the paper only requires the network realizability property for stable systems in $\mathfrak{T}(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$.

We denote the set of all stable real-rational proper transfer function matrices with input and output partitions as $\mathcal{P}_u$ and $\mathcal{P}_y$, and satisfying the delay and sparsity constraints imposed by the causal network interconnection $\mathcal{G}$ by $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$. Note that, if $Q(z) = [Q_{ij}(z)]_{i,j} \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$, then $Q_{ij}(z) \in \mathcal{RH}_\infty$ for all i, j. One can show that $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ and $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_u, \mathcal{P}_y)$ are equivalent and represent the same set of all stable interconnected systems realizable over $\mathcal{G}$ given the input and output partitions $\mathcal{P}_u$ and $\mathcal{P}_y$.

B. All stabilizing network realizable controllers

In this section, we characterize the set of all stabilizing controllers realizable over a given network for the given interconnected system by means of the following theorem.

Theorem 2: Given an interconnected system P over a causal network $\mathcal{G}$ with a state-space representation given by (6) with block-diagonal B_u and C_y and corresponding partitions $\mathcal{P}_x$, $\mathcal{P}_w$, $\mathcal{P}_u$, $\mathcal{P}_z$ and $\mathcal{P}_y$. Given that P is stabilizable and detectable and given are matrices F and L structured according to $\mathcal{G}$ such that $A + B_uF$ and $A + LC_y$ are stable. F is partitioned according to $(\mathcal{P}_u, \mathcal{P}_x)$ and L is partitioned according to $(\mathcal{P}_x, \mathcal{P}_y)$. Then the set of all stabilizing (FDLTI and causal) controllers, realizable over $\mathcal{G}$, for the given system P is parametrized by a lower fractional transformation (LFT) as

$$K = \mathrm{lft}(J, Q), \tag{8}$$

where Q is FDLTI, causal, stable and realizable over the network $\mathcal{G}$,

$$J = \left[\begin{array}{c|cc} A + B_uF + LC_y & -L & -B_u \\ \hline F & 0 & -I \\ -C_y & I & 0 \end{array}\right], \tag{9}$$

which is also realizable over the network $\mathcal{G}$.

Note that Theorem 2 requires matrices F and L, structured according to $\mathcal{G}$ and partitioned accordingly, such that $A + B_uF$ and $A + LC_y$ are stable. In this paper, we assume that matrices F and L with the above mentioned properties exist. The conditions for the existence of such matrices along with the procedure for generating the matrices will be addressed in future. However, note that for stable systems, F and L can always be chosen to be zero matrices.

C. Optimal solution for H_2 problem

In this section, we provide an optimal solution for the H_2 control problem under the constraint that the designed controller is realizable over the given network represented by $\mathcal{G}$.

The distributed H_2 control problem can be written as

$$\begin{array}{ll} \min & \|T_{zw}\|_2 \\ \text{subject to} & K \in \mathfrak{S}(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u), \\ & T_{zw} \text{ is internally stable} \end{array} \tag{10}$$

where T_{zw} denotes the closed-loop mapping from $w(k)$ to $z(k)$.

First, note that the set of FDLTI, causal and stable systems realizable over the causal network interconnection $\mathcal{G}$ are given by $\mathfrak{S}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ or $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$. Thus, we can assume that $Q(z) \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ to parametrize all stabilizing controllers realizable over $\mathcal{G}$. The set of all closed-loop transfer matrices from $w(k)$ to $z(k)$ can be obtained using Theorem 2 and the results from [15] as

$$T_{zw} = \mathrm{lft}(T, Q) = \{T_{11} + T_{12}QT_{21} : \ Q \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)\}$$

where T is given by

$$\begin{aligned} T &= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \\ &= \left[\begin{array}{cc|cc} A + B_uF & B_uF & B_w & -B_u \\ 0 & A + LC_y & -B_w - LD_{yw} & 0 \\ \hline C_z + D_{zu}F & D_{zu}F & D_{zw} & -Dzu \\ 0 & -C_y & D_{yw} & 0 \end{array}\right]. \end{aligned}$$

Since the closed-loop transfer matrix is simply an affine function of the controller parameter matrix Q while the delay and sparsity constraints on the transfer function of Q are linear, we can rewrite the distributed H_2 problem in (10) as a convex optimization problem

$$\begin{array}{ll} \min & \|T_{11} + T_{12}QT_{21}\|_2 \\ \text{subject to} & Q \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u), \end{array} \tag{11}$$

Vectorization techniques can be applied to write the optimization problem in (11) as an equivalent unconstrained problem. To represent the vectorization of a transfer function matrix, we make a slight change of notation for representing the matrices. Instead of treating Q_{ij} as a sub-matrix of Q, we consider Q_{ij} to be the element of the matrix Q in the i^{th} row and j^{th} column.

Let $\mathbf{vec}(\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)) = \{\mathbf{vec}(Q) | Q \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)\}$ denote the set of vectorized elements of $\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$. If $\mathcal{P}_u = \{\mathcal{P}_u^1, \ldots, \mathcal{P}_u^n\}$ denotes the output partition, then denote $n_u = \sum_i \mathcal{P}_u^i$ to represent the total number of outputs. Similarly, denote n_y to represent the total number of inputs. It can be seen that $\mathbf{vec}(\mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)) \in \mathcal{RH}_\infty^{n_u n_y \times 1}$ is a sub-space due to the delay and sparsity constraints imposed by the network $\mathcal{G}$. Let a denote the total number of elements of $Q \in \mathfrak{T}^s(\mathcal{G}, \mathcal{P}_y, \mathcal{P}_u)$ that are not constrained to be zero. It can be shown that there exists a matrix $H \in \mathcal{R}_p^{n_u n_y \times a}$ whose columns form an orthonormal basis for $\mathbf{vec}(\mathfrak{T}^s(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u))$. Thus, we know that

$$Q \in \mathfrak{T}^s(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u) \Longleftrightarrow \mathbf{vec}(Q) = Hx \quad \text{for some } x \in \mathcal{RH}_\infty^{a\times 1}.$$

Note that H contains the delay and sparsity constraints imposed by the causal network interconnection $\mathcal{G}$. Using the results of vectorization, we get that

$$\begin{aligned} \|T_{11}+T_{12}QT_{21}\|_2 &= \|\mathbf{vec}(T_{11}+T_{12}QT_{21})\|_2 \\ &= \left\|\mathbf{vec}(T_{11}) + (T_{21}^t \otimes T_{12})\,\mathbf{vec}(Q)\right\|_2 \\ &= \left\|\mathbf{vec}(T_{11}) + (T_{21}^t \otimes T_{12})Hx\right\|_2 \end{aligned}$$

Thus, we can pose the problem (11) as an unconstrained H_2 problem

$$\begin{array}{ll} \min & \left\|\mathbf{vec}(T_{11}) + (T_{21}^t \otimes T_{12})Hx\right\|_2 \\ \text{subject to} & x \in \mathcal{RH}_\infty^{a\times 1} \end{array} \qquad (12)$$

which can be solved using standard techniques. Let $x^\star$ denote the solution of the optimization problem (12). Then the corresponding optimal $Q^\star$ is given by $Q^\star = \mathbf{vec}^{-1}(Hx^\star)$. Since $Q^\star \in \mathfrak{T}^s(\mathcal{G},\mathcal{P}_y,\mathcal{P}_u)$, we can obtain a realization $(A_Q^\star, B_Q^\star, C_Q^\star, D_Q^\star)$ (using Theorem 1) that satisfies the constraints imposed by the causal network interconnection $\mathcal{G}$ and the corresponding controller is given by $K^\star = \mathrm{lft}(J, Q^\star)$, where J is given (9). From Theorem 2, we can see that $K^\star$ thus designed is not only optimal but also realizable over the causal network $\mathcal{G}$. Note that the observer-based parametrization of all stabilizing and network realizable controllers followed in this paper requires the existence of a nominal stabilizing controller that is also realizable over the network. We shall address the issue of existence of such nominal controllers, in more detail, in future.

VI. CONCLUSIONS

In this paper, we discussed the structural constraints imposed on the state-space and input-ouput descriptions of interconnected systems that are built on a causal network. We also discussed about the notion of realizability over causal networks and showed that any state-space representation with state-space matrices satisfying some structural constraints is network realizable. We also showed that any stable transfer function matrix satisfying some delay and sparsity constraints is also realizable over the given causal network. Using these two network realizability results, we parametrized the set of all stabilizing controllers realizable over the given network for a class of interconnected plants. This parametrization allowed us to convert a distributed H_2 control problem into an unconstrained convex optimization problem. Thus, we are able to obtain an optimal stabilizing controller realizable over the given causal network for a given interconnected plant over the same network.

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