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Convergence Analysis and Controller Design for a Team of Mobile Robots Subject to Measurement Error

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Abstract— **This paper deals with the steady-state error analysis in the formation control of wheeled mobile robots with leader follower structure. A nonholonomic model is considered for each robot, and it is assumed that each follower is capable of measuring its relative distance and relative velocity with respect to the leader. However, these measurements are assumed to be subject to error. A control law is proposed, and its convergence properties are investigated. Using the linear matrix inequalities (LMI) approach, the upper bounds of the steady state position and velocity errors due to the measurement error are minimized. Simulations demonstrate the efficacy of the results.**

I. INTRODUCTION

Recently, there has been a vast increase in the use of wheeled mobile robots (WMR) in industry. This type of robot is in particular very suitable for the cases where autonomous motion capabilities are required over reasonably smooth grounds and surfaces. Applications of WMRs ranges from planetary exploration to service for the handicapped to mobile sensor networks. The problems of motion planning and control of WMRs, which pose several theoretical and practical challenges, have been studied extensively in the literature; e.g. see [1], [2], [3].

Tracking control of nonholonomic robots is a challenging area of research. In this type of system, it is sometimes more desirable to design a control law which stabilizes the system around a trajectory, instead of a point. The main objective of the trajectory tracking problem is to design a controller such that the position of the robot asymptotically converges to a prescribed desired path. This problem has been throughly investigated in the literature, and a number of methods are introduced; e.g., nonlinear control, dynamic feedback linearization, and backstepping approach [4], [5], [6].

Coordination of multiple mobile robots, on the other hand, has attracted much interest recently. Exploiting a group of robots instead of a single robot or human for performing a prescribed spatially distributed task has significant advantages in various applications [2]. In particular, formation control problem is one of the most promising research areas in mobile robotics. This problem is concerned with a group of robots moving in formation and performing a single mission in a cooperative fashion. It is desired in this type of problem to control the relative position and orientation of the robots with respect to each other. Applications of formation control of cooperative robots include simultaneous localization and mapping, RoboCup, and the exploration of an unknown environment, to name only a few [7], [8], [9], [10]. The most effective approaches introduced in the literature for formation control of mobile robots are behavior-based, virtual structure, and leader-follower [11].

In a basic leader-follower approach, a particular robot is assigned to be the leader, and other robots are followers [11], [12], [13]. A predefined trajectory is to be tracked by the leader; the followers are then supposed to follow the leader and keep a desired relative pose (distance and orientation) from it. The control of a group of robots is investigated in [13], where the agents are aimed to maintain the desired position in the formation. Vision-based sensors are used in [12] to provide information to the other robots for localization in leader-follower formation. These approaches are effective in the coordination of a team of robots. However, existing coordination strategies often ignore some practical issues such as measurement error, in order to simplify the stability analysis and controller design problem.

In this paper, the problem of controlling relative position of a group of robots with leader-follower formation structure is considered. The control design is carried out for the case of unicycle kinematics, which is the most common among WMRs. The desired relative position trajectory is assumed to be known by the follower robot. It is also assumed that each follower is capable of measuring its relative position and relative velocity with respect to the leader. First, stability of the system is investigated in the case of perfect sensing. A feedback control law is subsequently proposed which guarantees that the error is exponentially convergent to a ball. Controller's gains are adjusted to minimize the radius of steady-state error. The impact of measurement error on the follower is then studied and a control design methodology is introduced to minimize the effect of error. The proposed control design procedures rely on the linear matrix inequalities (LMI).

This paper is organized as follows. The problem is for-

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mulated in Section II, where the main objectives of the work are also presented. In Section III, control strategies are proposed in the absence and presence of measurement error. Section IV presents simulations to support the theoretical results obtained in the paper. Finally, the paper is concluded in Section V.

II. Problem Formulation

Let $z \in \mathbb{R}^n$ denote the set of all n-vectors of generalized coordinates for a wheeled mobile robot. This type of robot is often modeled as a single upright wheel. This model, which is also referred to as the unicycle model, will be used in this paper to describe the behavior of robots. The generalized coordinates for an unicycle are $z = (x, y, \theta)$, where x, y represent the Cartesian coordinates, and θ is the angular orientation with respect to the x-axis in an inertial reference frame. The objective here is to control the relative position of follower robot with respect to a leader robot, which tracks a certain trajectory unknown to the follower. Consider Figure 1, and let the inertial reference frame be centered at the origin O; the dynamic equations for each robot can then be expressed as:

$$\begin{aligned} \dot{x} &= v\cos\theta \\ \dot{y} &= v\sin\theta \\ \dot{\theta} &= \omega \\ \dot{v} &= a \end{aligned} \tag{1}$$

where the acceleration a (which is directly related to force) and ω are treated as the input variables.

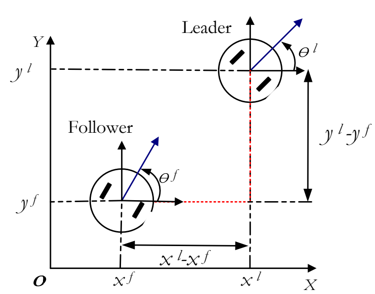

Fig. 1. Relative position of the follower with respect to the leader.

The error vector for the follower is defined as:

$$e^f = \begin{bmatrix} e_p^f \\ e_v^f \end{bmatrix} \tag{2}$$

where e_p^f and e_v^f are position and velocity errors of the follower, respectively, and are defined by:

$$e_p^f := \begin{bmatrix} e_{p_x}^f \\ e_{p_y}^f \end{bmatrix} = \begin{bmatrix} x^f - x^l - d_x^f(t) \\ y^f - y^l - d_y^f(t) \end{bmatrix} \tag{3}$$

and:

$$e_v^f := \begin{bmatrix} e_{v_x}^f \\ e_{v_y}^f \end{bmatrix} = \begin{bmatrix} \dot{x}^f - \dot{x}^l \\ \dot{y}^f - \dot{y}^l \end{bmatrix} \tag{4}$$

where x^l and y^l represent the position of the leader. Also, $d_x^f(t)$ and $d_y^f(t)$ denote the desired relative positions between the follower and leader in the x and y directions, respectively. Ideally, it is aimed to make the position and velocity errors as close as possible to zero. If this error approaches zero as time increases, then the follower robot will be aligned with the leader robot in the steady-state.

Assumption 1: $d_x^f(t)$ and $d_y^f(t)$ are either constant, or are the outputs of autonomous dynamical systems represented by:

$$\begin{aligned} \dot{q}(t) &= \Gamma q(t) \\ d_x^f(t) &= \Pi_1 q(t) \\ d_y^f(t) &= \Pi_2 q(t) \end{aligned}$$

where $q \in \mathbb{R}^\kappa$ and Γ, Π_1, Π_2 are constant matrices of appropriate dimensions. Note that the matrix Γ can have one single eigenvalue at the origin, but all other eigenvalues are located in the open left half-plane. Denote the real part of the rightmost non-zero eigenvalues of Γ with $-\lambda$ (note that $\lambda > 0$).

It is supposed that each follower is equipped with the proper sensors to measure its relative position and velocity (with respect to its forward robot). Thus, the error vector e^f can be used in constructing the control input. Now, using equations (3) and (4), one can write:

$$\dot{e}_p^f = \begin{bmatrix} \dot{e}_{p_x}^f \\ \dot{e}_{p_y}^f \end{bmatrix} = \begin{bmatrix} e_{v_x}^f - \dot{d}_x^f(t) \\ e_{v_y}^f - \dot{d}_y^f(t) \end{bmatrix} \tag{5}$$

and similarly:

$$\dot{e}_v^f = \begin{bmatrix} \dot{e}_{v_x}^f \\ \dot{e}_{v_y}^f \end{bmatrix} = \begin{bmatrix} \dot{v}^f\cos\theta^f - v^f\dot{\theta}^f\sin\theta^f - \ddot{x}^l \\ \dot{v}^f\sin\theta^f + v^f\dot{\theta}^f\cos\theta^f - \ddot{y}^l \end{bmatrix}$$

By rewriting the above equations and using the relations $\dot{\theta}^f = \omega^f$ and $\dot{v}^f = a^f$, it can be shown that:

$$\dot{e}_v^f = \begin{bmatrix} \dot{e}_{v_x}^f \\ \dot{e}_{v_y}^f \end{bmatrix} = \begin{bmatrix} \cos\theta^f & -v^f\sin\theta^f \\ \sin\theta^f & v^f\cos\theta^f \end{bmatrix} \begin{bmatrix} a^f \\ \omega^f \end{bmatrix} - \begin{bmatrix} \ddot{x}^l \\ \ddot{y}^l \end{bmatrix}$$

Define:

$$\begin{bmatrix} u_1^f \\ u_2^f \end{bmatrix} = \begin{bmatrix} \cos\theta^f & -v^f\sin\theta^f \\ \sin\theta^f & v^f\cos\theta^f \end{bmatrix} \begin{bmatrix} a^f \\ \omega^f \end{bmatrix} \tag{6}$$

which yields:

$$\begin{bmatrix} \dot{e}_{v_x}^f \\ \dot{e}_{v_y}^f \end{bmatrix} = \begin{bmatrix} u_1^f \\ u_2^f \end{bmatrix} - \begin{bmatrix} \ddot{x}^l \\ \ddot{y}^l \end{bmatrix} \tag{7}$$

Combining the two equations (5) and (7), the error dynamics can be obtained as:

$$\begin{bmatrix} \dot{e}^f_{p_x} \\ \dot{e}^f_{p_y} \\ \dot{e}^f_{v_x} \\ \dot{e}^f_{v_y} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} e^f_{p_x} \\ e^f_{p_y} \\ e^f_{v_x} \\ e^f_{v_y} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u^f_1 \\ u^f_2 \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} \ddot{x}^l \\ \ddot{y}^l \end{bmatrix} + \underbrace{\begin{bmatrix} -\dot{d}^f_x(t) \\ -\dot{d}^f_y(t) \\ 0 \\ 0 \end{bmatrix}}_{\phi(t)}$$

or equivalently:

$$\dot{e}^f = Ae^f + Bu^f - Bs^l + \phi(t) \tag{8}$$

where u^f denotes the control input, and s^l is defined as:

$$s^l := \begin{bmatrix} \ddot{x}^l \\ \ddot{y}^l \end{bmatrix}$$

Note that Assumption 1 implies there exists $\eta > 0$ such that:

$$\|\phi(t)\| \leq \eta e^{-\lambda(t-t_0)}, \quad t \geq t_0$$

where $\|.\|$ denotes the 2-norm.

Assumption 2: It is assumed that each robot's acceleration is uniformly bounded, i.e., $\|s^l\| \leq \rho$. Note that ρ is assumed to be known *a priori*.

The following control law is proposed for each follower:

$$u^f = K^f e^f \tag{9}$$

where $K^f \in \mathbb{R}^{2\times 4}$ is a constant matrix.

Remark 1: It is to be noted that Assumption 2 always holds in practice, because the actuator motors can only provide finite torques. The values of ρ can be obtained from the robot manufacturer.

Remark 2: To apply u^f to the follower robot, a^f and ω^f should be first obtained from (6). To avoid singularity, it is assumed that $v^f \neq 0$.

It is desired next to find the control gain K^f for the followers such that the steady-state error is sufficiently small in the following two scenarios:

- Perfect sensing
- Noisy measurements

Remark 3: The problem formulation provided in this section can be extended to a platoon of mobile robots, in which all robots except the ones at the head of platoon can potentially be the followers of the robots immediately in front of them (and similaly all robots except the ones at the tail of the platoon can be leaders of some other robots).

III. MAIN RESULTS

The lemma presented below play a key role in developing the main results of the paper.

Lemma 1: Assume that $g(t)$ is an exponentially decaying signal; i.e., there exist positive constants λ and ε such that:

$$\|g(t)\| \leq \varepsilon e^{-\lambda(t-t_0)}, \quad t \geq t_0 \tag{10}$$

Given positive real parameters ξ, σ and $\check{\kappa}$, let the function $V(t)$ satisfy the following inequality:

$$\dot{V}(t) + \xi V(t) - \frac{1}{\sigma}\|g(t)\|^2 \leq \check{\kappa} \tag{11}$$

Then:

$$V(t) \leq e^{-\xi(t-t_0)}V(t_0) + \frac{\varepsilon^2}{\sigma(2\lambda-\xi)}\{e^{-\xi(t-t_0)} - e^{-2\lambda(t-t_0)}\} + \frac{\check{\kappa}}{\xi}[1-e^{-\xi(t-t_0)}] \tag{12}$$

Proof: Multiply (11) by $e^{\xi t}$ and integrate the result from t_0 to t. Then use (10) to obtain:

$$\begin{aligned} \int_{t_0}^{t} \frac{d}{ds}[e^{\xi s}V(s)]ds &\leq \frac{1}{\sigma}\int_{t_0}^{t} e^{\xi s}\varepsilon^2 e^{-2\lambda(s-t_0)}ds + \check{\kappa}\int_{t_0}^{t} e^{\xi s}ds \\ e^{\xi t}V(t) &\leq e^{\xi t_0}V(t_0) \\ &+ \frac{-\varepsilon^2 e^{2\lambda t_0}}{\sigma(2\lambda-\xi)}\{e^{-(2\lambda-\xi)t} - e^{-(2\lambda-\xi)t_0}\} \\ &+ \frac{\check{\kappa}}{\xi}[e^{\xi t} - e^{\xi t_0}] \end{aligned} \tag{13}$$

Multiplying by $e^{-\xi t}$ yields:

$$V(t) \leq e^{-\xi(t-t_0)}V(t_0) + \frac{\varepsilon^2}{\sigma(2\lambda-\xi)}\{e^{-\xi(t-t_0)} - e^{-2\lambda(t-t_0)}\} + \frac{\check{\kappa}}{\xi}[1-e^{-\xi(t-t_0)}]$$

This completes the proof. ■

A. Perfect Sensing

Theorem 1: Consider a group of mobile robots moving in formation with leader-follower structure, where the error dynamics of the followers obey equation (8). Suppose that the conditions of Assumptions 1 and 2 hold. Given $\alpha > 0$, solve the following optimization problem:

Find the maximum of $\delta > 0$ subject to the following LMI:

$$R^f A^T + AR^f + {S^f}^T B^T + BS^f + \alpha R^f + \delta BB^T < 0 \tag{14}$$

For $R^f > 0$, S^f and $\delta > 0$ obtained from the above problem, if the controller (9) with $K^f = S^f {R^f}^{-1}$ is applied to the follower, e^f is exponentially convergent to the ball $\mathscr{B}(r)$ with the rate $\bar{\lambda} = \frac{1}{2}\xi$, where r is given by:

$$r = \sqrt{\frac{\rho^2 \lambda_{max}(R^f)}{\delta\xi}} \tag{15}$$

and:

$$0 < \xi = \min\{\alpha, 2\lambda\} - \varepsilon_0 \tag{16}$$

for a sufficiently small positive value ε_0.

Proof: System (8) under controller (9) can be described by:

$$\dot{e}^f = (A + BK^f)e^f - Bs^l + \phi(t)$$

Consider the Lyapunov function as $V^f = {e^f}^T P^f e^f$. One can write:

$$\begin{aligned}\dot{V}^f + \xi V^f =& 2{e^f}^T P^f (A + BK^f)e^f + \xi {e^f}^T P^f e^f \\ &- 2{e^f}^T P^f Bs^l + 2{e^f}^T P^f \phi(t)\end{aligned}$$

On the other hand, it is known that:

$$\begin{aligned}-2{e^f}^T P^f Bs^l &\le \delta \|{e^f}^T P^f B\|^2 + \frac{\|s^l\|^2}{\delta} \\ 2{e^f}^T P^f \phi(t) &\le \sigma \|e^f\|^2 + \frac{1}{\sigma}\|P^f \phi(t)\|^2 \end{aligned} \tag{17}$$

where δ and σ are arbitrary positive constants. Thus:

$$\begin{aligned}\dot{V}^f + \xi V^f \le& 2{e^f}^T P^f (A + BK^f)e^f + \alpha {e^f}^T P^f e^f + \frac{1}{\sigma}\|P^f \phi(t)\|^2 \\ &- (\alpha - \xi){e^f}^T P^f e^f + \sigma {e^f}^T e^f + \delta {e^f}^T P^f BB^T P^f e^f \\ &+ \frac{\rho^2}{\delta}\end{aligned}$$

Choose:

$$\sigma \le (\alpha - \xi)\lambda_{\min}(P^f) \tag{18}$$

Now, if:

$$(A + BK^f)^T P^f + P^f (A + BK^f) + \alpha P^f + \delta P^f BB^T P^f < 0 \tag{19}$$

then it follows that:

$$\dot{V}^f + \xi V^f - \frac{1}{\sigma}\|P^f \phi(t)\|^2 \le \frac{\rho^2}{\delta} \tag{20}$$

Multiplying (19) by ${P^f}^{-1}$ from left and right yields:

$$\begin{aligned}{P^f}^{-1}A^T + {P^f}^{-1}{K^f}^T B^T + A{P^f}^{-1} + BK^f {P^f}^{-1} + \alpha {P^f}^{-1} \\ + \delta BB^T < 0\end{aligned} \tag{21}$$

Let ${P^f}^{-1} = R^f$ and $K^f {P^f}^{-1} = S^f$; then (21) becomes equivalent to (14). Let also $V(t)$ in (11) be set to ${e^f}^T P^f e^f$, and choose $g(t) = P^f \phi(t)$ and $\check{\kappa} = \rho^2/\delta$. It follows from Lemma 1 and (20) that:

$$\begin{aligned}V^f(t) \le& e^{-\xi(t-t_0)}V^f(t_0) \\ &+ \frac{\varepsilon^2}{\sigma(2\lambda - \xi)}\{1 - e^{-(2\lambda-\xi)(t-t_0)}\}e^{-\xi(t-t_0)} \\ &+ \frac{\rho^2}{\delta\xi}[1 - e^{-\xi(t-t_0)}]\end{aligned} \tag{22}$$

where $\varepsilon = \|P^f\|\eta$. Since $\xi < 2\lambda$, it can be concluded that

$$V^f(t) \le e^{-\xi(t-t_0)}V^f(t_0) + \frac{\|P^f\|^2\eta^2}{\sigma(2\lambda - \xi)}e^{-\xi(t-t_0)} + \frac{\rho^2}{\delta\xi} \tag{23}$$

Furthermore, since $\lambda_{\min}(P^f)||e(t)||^2 \le V^f(t)$, it can be concluded that:

$$||e(t)|| \le e^{-\bar{\lambda}(t-t_0)}\psi + \sqrt{\frac{\rho^2}{\delta\xi\lambda_{min}(P^f)}} \tag{24}$$

where:

$$\psi = \sqrt{\frac{1}{\lambda_{min}(P^f)}\left({e^f}^T(t_0)P^f e^f(t_0) + \frac{\|P^f\|^2\eta^2}{\sigma(2\lambda - \xi)}\right)}$$

and $\bar{\lambda} = \frac{1}{2}\xi$. This completes the proof. ∎

Remark 4: It is to be noted that the design parameter α in (14) can be chosen properly such that the underlying LMI conditions are feasible. A similar comment can be made on the LMI conditions given in the theorem presented in the next subsection.

B. Noisy Measurements

In order to take into account the effect of measurement error on the follower's motion, the control law is written as:

$$u^f = K^f \tilde{e}^f \tag{25}$$

where $\tilde{e}^f = e^f + \delta_e^f$, and the measurement error δ_e^f is assumed to have a known upper bound represented by:

$$\Delta_e^f := \max_{t>t_0} ||\delta_e^f||^2$$

In this subsection, an upper bound on the steady-state error is first obtained. An algorithm is subsequently proposed to find the control gain K^f such that this upper bound is minimized.

Lemma 2: Assume that $g(t)$ is an exponentially decaying signal. Given the positive real parameters ξ and $\check{\kappa}$, let the following inequality hold:

$$\dot{V}(t) + \xi V(t) - b{\delta_e^f}^T Q \delta_e^f - \|g(t)\|^2 \le \check{\kappa} \tag{26}$$

where b is a positive constant and Q is a symmetric positive definite matrix. Then:

$$V(\infty) \le \frac{b}{\xi}\max_{t>t_0}[{\delta_e^f}^T(t)Q\delta_e^f(t)] + \frac{\check{\kappa}}{\xi} \tag{27}$$

Proof: Multiplying (26) by $e^{\xi t}$, it follows that:

$$\frac{d}{dt}[e^{\xi t}V(t)] \le b\delta_e^f(t)^T Q\delta_e^f(t)e^{\xi t} + \|g(t)\|^2 e^{\xi t} + \check{\kappa}e^{\xi t}$$

Integrating both sides of the above relation from t_0 to t, one arrives at:

$$e^{\xi t}V(t) - e^{\xi t_0}V(t_0) \le \int_{t_0}^{t}[b{\delta_e^f}^T(\tau)Q\delta_e^f(\tau) + \|g(\tau)\|^2 + \check{\kappa}]e^{\xi\tau}d\tau$$

or equivalently:

$$\begin{aligned}V(t) \le& \frac{b}{\xi}\max_{t>t_0}[{\delta_e^f}^T(t)Q\delta_e^f(t)(e^{\xi t} - e^{\xi t_0})]e^{-\xi t} + \frac{\check{\kappa}}{\xi}[1 - e^{\xi(t_0-t)}] \\ &+ e^{\xi(t_0-t)}V(t_0) + \int_{t_0}^{t}\|g(\tau)\|^2 e^{-\xi(t-\tau)}d\tau\end{aligned}$$

The proof follows now on noting that the integral

$$\lim_{t\to\infty}\int_{t_0}^{t}\|g(\tau)\|^2 e^{-\xi(t-\tau)}d\tau$$

approaches zero as $t \to \infty$. ∎

Consider the system described in Theorem 1, and define the quadratic Lyapunov function $V^f = (e^f)^T P^f e^f$. Let $R^f = P^{f^{-1}}$, and assume this matrix has a lower bound R_l and an upper bound R_r given below:

$$R_l = diag([\beta_1, \beta_1, \beta_2, \beta_2]) \tag{28a}$$

$$R_r = diag([\gamma_1, \gamma_1, \frac{1}{\varepsilon_0}, \frac{1}{\varepsilon_0}]) \tag{28b}$$

where β_1 and β_2 are variable weighting factors corresponding to the position and velocity errors, respectively. Furthermore, $\gamma_1 > 0$ is a variable weighting factor as well, and ε_0 is a sufficiently small positive number.

Let the design parameters $\alpha > 0$ and $b_f > 0$ be given. For any scalar $0 < b < b_f$, define $\Delta^f_{e_p} := \max_{t>t_0} ||\delta^f_{e_p}||^2$, $\Delta^f_{e_v} := \max_{t>t_0} ||\delta^f_{e_v}||^2$, and solve the following optimization problem:

min $[\Omega_{\beta_1}\beta_1 + \Omega_{\beta_2}\beta_2 + \Omega_{\gamma_1}\gamma_1 + \Omega_\delta \delta]$

subject to following LMIs:

$$\begin{bmatrix} R^f A^T + AR^f + S^{f^T} B^T + BS^f + \alpha R^f + \delta BB^T & BS^f \\ S^{f^T} B^T & -bR^f \end{bmatrix} < 0 \tag{29}$$

and:

$R_l < R^f < R_r$

w.r.t.:

$\beta_1 > 0,\ \beta_2 > 0,\ \gamma_1 > 0,\ \delta > 0$

where Ω_{β_1}, $\Omega_{\beta_2} < 0$, $\Omega_{\gamma_1} > 0$ and $\Omega_\delta < 0$ are weighting coefficients. If the above problem is feasible, then calculate:

$$\bar{e}_p = \sqrt{\frac{b\gamma_1}{\xi\beta_1}\Delta^f_{e_p} + \frac{b\gamma_1}{\xi\beta_2}\Delta^f_{e_v} + \frac{\gamma_1\rho^2}{\delta\xi}}$$

where ξ is obtained from (16). Define:

$$\bar{e}_{p,\min} := \min_{0<b<b_f} \bar{e}_p$$

The next theorem provides an upper bound on the steady-state position error.

Theorem 2: If the controller (25) with $K^f = S^f R^{f^{-1}}$ and $P^f = R^{f^{-1}}$ provided above is applied to the follower, then $\lim_{t\to\infty} ||e^f_p(t)|| < \bar{e}_{p,\min}$.

Proof: The system (8) under controller (25) can be represented in the closed-loop form as:

$$\dot{e}^f = (A + BK^f)e^f + BK^f\delta^f_e - Bs^l + \phi(t) \tag{30}$$

Choose, $Q = R^{f^{-1}}$, $g(t) = \frac{1}{\sqrt{\sigma}} P^f \phi(t)$ and $\check{\kappa} = \frac{\rho^2}{\delta}$, where σ satisfies (18). Substituting $\dot{e}^f$ from equation (30) into (26) and using (17), it can be deduced (similarly to the proof of Theorem 1) that:

$$\begin{aligned} &\dot{V} + \xi V - b\delta^{f^T}_e Q\delta^f_e - \frac{1}{\sigma}\|P^f\phi(t)\|^2 \le e^{f^T} A^T R^{f^{-1}} e^f \\ &+ e^{f^T} R^{f^{-1}} A e^f + e^{f^T} K^{f^T} B^T R^{f^{-1}} e^f + \delta^{f^T}_e K^{f^T} B^T R^{f^{-1}} e^f \\ &+ e^{f^T} R^{f^{-1}} BK^f e^f + e^{f^T} R^{f^{-1}} BK^f \delta^f_e + \alpha e^{f^T} R^{f^{-1}} e^f \\ &- b\delta^{f^T}_e R^{f^{-1}} \delta^f_e + \frac{\rho^2}{\delta} + \delta e^{f^T} P^f BB^T P^f e^f \end{aligned} \tag{31}$$

The right hand expression of (31) can now be rewritten in the following matrix form:

$$\begin{bmatrix} e^{f^T} & \delta^{f^T}_e \end{bmatrix} \begin{bmatrix} \Pi & R^{f^{-1}} BK^f \\ K^{f^T} B^T R^{f^{-1}} & -bR^{f^{-1}} \end{bmatrix} \begin{bmatrix} e^f \\ \delta^f_e \end{bmatrix} \tag{32}$$

where:

$$\begin{aligned} \Pi = &A^T R^{f^{-1}} + R^{f^{-1}} A + K^{f^T} B^T R^{f^{-1}} + R^{f^{-1}} BK^f \\ &+ \alpha R^{f^{-1}} + \delta R^{f^{-1}} BB^T R^{f^{-1}} \end{aligned}$$

On the other hand, it is implied from (29) and $S^f = K^f R^f$ that:

$$\begin{bmatrix} R^f A^T + AR^f + R^f K^{f^T} B^T + BK^f R^f + \alpha R^f + \delta BB^T & BK^f R^f \\ R^f K^{f^T} B^T & -bR^f \end{bmatrix} < 0 \tag{33}$$

By pre and post-multiplying (33) by $\begin{bmatrix} R^{f^{-1}} & 0 \\ 0 & R^{f^{-1}} \end{bmatrix}$ one can obtain:

$$\begin{bmatrix} \Pi & R^{f^{-1}} BK^f \\ K^{f^T} B^T R^{f^{-1}} & -bR^{f^{-1}} \end{bmatrix} < 0 \tag{34}$$

Thus, the expression in (32) is non-positive for every e^f and δ^f_e, and consequently:

$$\dot{V} + \xi V - b\delta^{f^T}_e Q\delta^f_e - \frac{1}{\sigma}\|P^f\phi(t)\|^2 \le 0$$

Therefore, it is concluded from Lemma 2 that:

$$(e^f(\infty))^T R^{f^{-1}} e^f(\infty) \le \frac{b}{\xi}\max_{t>t_0}[(\delta^f_e(t))^T R^{f^{-1}} \delta^f_e(t)] + \frac{\rho^2}{\delta\xi} \tag{35}$$

In order to have an upper bound for the steady-state error, R^f should be limited by two matrices from right and left. Using the left bound as defined in (28a), one arrives at:

$$\begin{aligned} &(\delta^f_e(t))^T R^{f^{-1}} \delta^f_e(t) \le \\ &\begin{bmatrix} \delta^f_{e_p} & \delta^f_{e_v} \end{bmatrix} \begin{bmatrix} \frac{1}{\beta_1} & 0 & 0 & 0 \\ 0 & \frac{1}{\beta_1} & 0 & 0 \\ 0 & 0 & \frac{1}{\beta_2} & 0 \\ 0 & 0 & 0 & \frac{1}{\beta_2} \end{bmatrix} \begin{bmatrix} \delta^f_{e_p} \\ \delta^f_{e_v} \end{bmatrix} \\ &\le \frac{1}{\beta_1}||\delta^f_{e_p}||^2 + \frac{1}{\beta_2}||\delta^f_{e_v}||^2 \end{aligned} \tag{36}$$

Similarly, for the right bound:

$$\frac{1}{\gamma_1}||e^f_p||^2 \le \frac{1}{\gamma_1}||e^f_p||^2 + \varepsilon_0||e^f_v||^2 \le (e^f(t))^T R^{f^{-1}} e^f(t)$$

Thus, it follows from the above relation as well as (35) that:

$$\frac{1}{\gamma_1}||e_p^f(\infty)||^2 \le \frac{b}{\xi}\max_{t>t_0}[(\delta_e^f(t))^T {R^f}^{-1}\delta_e^f(t)] + \frac{\rho^2}{\delta\xi} \quad (37)$$

Substituting the upper limits for the position and velocity measured error, an upper bound on the steady-state position error is obtained from (36) and (37) as follows:

$$||e_p^f(\infty)||^2 \le \frac{b\gamma_1}{\xi\beta_1}\Delta_{e_p}^f + \frac{b\gamma_1}{\xi\beta_2}\Delta_{e_v}^f + \frac{\gamma_1\rho^2}{\delta\xi} \quad (38)$$

The upper bound of the error provided above can be suppressed by proper choices of the parameters β_1, β_2, γ_1 and δ as the free variables (this can be performed by using an appropriate minimization problem). ■

Remark 5: A similar approach can be used to obtain an upper bound on the steady-state velocity error. To this end, R_r needs to be replaced by:

$$R_r = diag([\frac{1}{\varepsilon_0}, \frac{1}{\varepsilon_0}, \gamma_1, \gamma_1])$$

which leads to the following upper bound:

$$||e_v(\infty)|| < \sqrt{\frac{b\gamma_1}{\alpha\beta_1}\Delta_{e_p} + \frac{b\gamma_1}{\alpha\beta_2}\Delta_{e_v} + \frac{\gamma_1\rho^2}{\delta\xi}}$$

Remark 6: It is to be noted that the condition (26) is relaxed when the measurement error δ_e^f is sufficiently larger. Thus, smaller values could be found for the ratios γ_1/β_1 and γ_1/β_2 in this case. This concludes that the upper bound introduced in (38) is not directly proportional to $\Delta_{e_p}^f$ and $\Delta_{e_v}^f$.

IV. Simulation Results

Example 1: Consider two mobile robots, one leader and one follower, and assume the leader moves on a circular trajectory given by:

$$x_r^l = 2\cos(0.025t)$$
$$y_r^l = 2\sin(0.025t)$$

The follower is to follow the leader with the following desired distance:

$$d(t) = \begin{bmatrix} d_x(t) \\ d_y(t) \end{bmatrix} = \begin{bmatrix} 0.4 + 1(1-e^{-t}) \\ 0.5 - 0.1(1-e^{-t}) \end{bmatrix}$$

and control input (9). Assume that the upper bound for acceleration is $\rho = 0.04$ m/sec, and that there is no measurement error. One can use Theorem 1 with $\alpha =$ 0.8 to obtain a controller which satisfies the design specifications. In this case, the gain matrix K in (9) will be:

$$K = \begin{bmatrix} -19.7203 & 0 & -47.1743 & 0 \\ 0 & -19.7203 & 0 & -47.1743 \end{bmatrix}$$

where r in (15) is obtained as 4×10^{-3} m. In Figure 2, the relative position of the follower with respect to the leader along the x-axis is compared with its desired trajectory d_x. A similar comparison is made in the y direction in Figure 3. These figures demonstrate that the desired position tracking is achieved with a small error (1.5×10^{-3} m position error is observed in this case). Figure 4, on the other hand, shows that the velocity regulation error e_v approaches zero (with a good precision) in both x and y directions. Figure 5 depicts the trajectory of the leader and follower moving toward the circular path from their initial positions (2.3,-0.2) and (2.9,0.3), respectively.

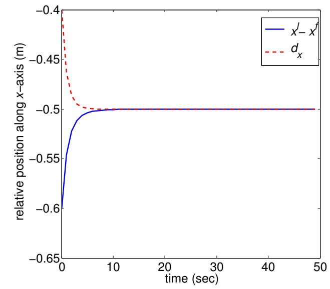

Fig. 2. Relative position of the follower with respect to the leader along the x-axis for the leader-follower circular trajectory tracking of Example 1.

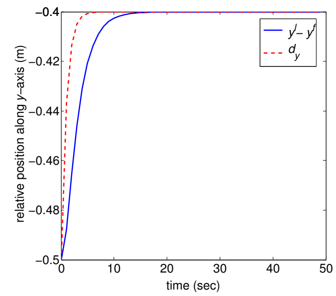

Fig. 3. Relative position of the follower with respect to the leader along the y-axis for the leader-follower circular trajectory tracking of Example 1.

Example 2: Consider a multi-agent system, where 2 followers are to follow a leader in a linear path. Suppose that the leader and followers are initially located on an equilateral triangle with the length of the sides equal to 0.5 m. The final desired formation is another equilateral triangle with the length of the sides equal to 0.4 m, while the leader is tracking a ramp reference signal along both axes, characterized by:

$$x_r^l(t) = y_r^l(t) = 0.04t$$

Let $\alpha = 0.8$, $\Omega_{\beta_1} = -1$, $\Omega_{\beta_2} = -1$, $\Omega_{\gamma_1} = 5$ and $\Omega_\delta = -5$. Assume that the measurement error is modeled by a random process which is uniformly distributed in intervals $(0, 8\times10^{-4})$ and $(0, 10^{-3})$ for position and velocity measurements,

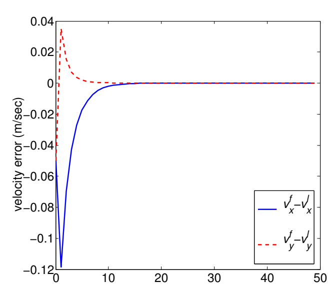

Fig. 4. The velocity error of the follower for the leader-follower circular trajectory tracking of Example 1.

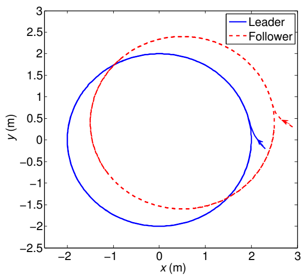

Fig. 5. The leader and follower trajectories in the 2-D plane for the leader-follower circular trajectory tracking of Example 1.

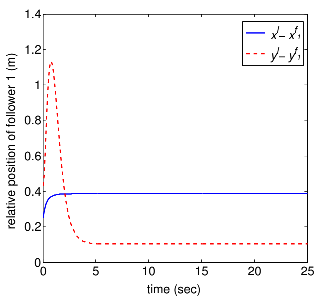

Fig. 6. Relative position of follower 1 with respect to the leader along the x and y axes for the leader-follower trajectory tracking of Example 2.

respectively. Using Theorem 2 with $b = 20$, the gain matrix given below is obtained for both followers:

$$K = \begin{bmatrix} -8.6539 & 0 & -6.1034 & 0 \\ 0 & -8.6539 & 0 & -6.1034 \end{bmatrix}$$

It results from Theorem 2 that an upper bound for the steady-state position error is 8.1×10^{-3}. In fact, the steady-state position error obtained from simulation is approximately equal to 3×10^{-3} which confirms the corresponding theoretical result. Figure 6 depicts the relative position of follower 1 with respect to the leader in both x and y directions. The velocity regulation error of follower 1 along the x and y axes is plotted in Figure 7. This figure shows that the error approximately approaches zero in both directions. The planar motion of the formation is sketched in Figure 8.

V. Conclusions

This paper investigates the problem of controlling relative position of a follower robot with respect to a leader. Convergence analysis for the tracking error is provided for both cases of perfect sensing and noisy measurements. Assuming that a known upper bound exists on the measurement error, a controller is designed using linear matrix inequalities (LMI) to minimize the upper bound on the steady-state error. Two examples of path following are examined by simulation, which confirm the efficacy of the proposed methods.

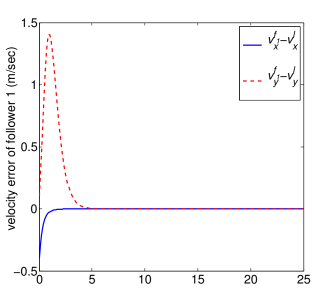

Fig. 7. The velocity error of follower 1 for the leader-follower trajectory tracking of Example 2.

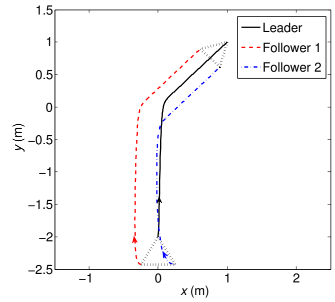

Fig. 8. The planar motion of the formation for the leader-follower trajectory tracking of Example 2.

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