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Optimal Control of Polynomial Systems with Performance Bounds: A Convex Optimization Approach

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Abstract— **This paper is concerned with an approach for a nonlinear optimal control of polynomial systems. The Hamilton-Jacobi-Bellman (HJB) equation is relaxed into HJB inequalities. Both an upper bound and a lower bound on the cost function, as well as a suboptimal controller, can be computed from solutions of the resulting inequalities. Solving the HJB inequalities can be cast as state-dependent matrix inequalities (SDMIs), whose derivation is based on representation of the given polynomial system in a linear-like form. The resulting SDMI for the upper-bound computation is nonconvex in the decision variables, and hence an iterative procedure is proposed to deal with the nonconvexity. On the other hand, the resulting SDMI for the lower-bound computation can be written as a state-dependent linear matrix inequality (SDLMI), which is a convex optimization problem solvable by existing numerical tools. A numerical example is provided to illustrate the proposed approach.**

I. Introduction

Nonlinear optimal control is known to be a difficult and challenge problem in control theory. Computation of the optimal performance can be formulated in terms of the Hamilton-Jacobi-Bellman (HJB) equation [1]. In the case of linear systems, the HJB equation is reduced to the Riccati equation, which can be efficiently solved. On the other hand, for a general nonlinear system, the resulting HJB equation is difficult to solve. Due to its importance, however, several approaches has been proposed to find approximation schemes to the HJB equation, e.g., a power series expansion approach [10] and the Galerkin spectral approach [2]. An alternative approach based on relaxation of the HJB equation with its counterpart inequalities has been considered by several researchers. Various approaches have been proposed to solve such HJB inequalities in order to obtain upper bounds or lower bounds on the optimal performance. An approach based on gridding on the given state-space region was proposed in [7]. In [19], the author proposed an algorithm to iteratively improve bounds on the optimal performance.

Recently, an approach based on robust-linear-matrix-inequality (robust-LMI) [3], [12], [17] formulation with the sum-of-squares (SOS) relaxation [15], [17] was considered in [16]. In particular, joint search of Lyapunov functions and controller variables satisfying the HJB inequalities can be formulated as state-dependent linear matrix inequalities (SDLMIs), which are special cases of robust LMIs. This formulation is based on representing nonlinear systems in a state-dependent linear-like form. This approach is quite promising since an SDLMI is a convex constraint and can be relaxed into a standard LMI via the SOS technique. Furthermore, upper bounds on the optimal performance, as well as stabilizing controllers can be efficiently obtained when the systems are of low orders. However, the SDLMI formulation is conservative in the sense that such linear-like representation is not unique and the success of the synthesis depends on the chosen representation. Ichihara [6] addressed this conservatism issue by introducing polynomial annihilators, which can represent the non-uniqueness of the linear-like representation, in the design problem.

In this paper, we present a novel approach to a nonlinear optimal control problem for polynomial systems operating in a compact domain. Similarly to [6], [16], Our approach relies on state-dependent matrix inequalities (SDMIs) and the SOS relaxation. The differences between the current approach and the approaches of [6], [16] are threefold. Firstly, both upper bounds and lower bounds on the optimal performance are available in the current approach, unlike [6], [16] which consider only upper bounds on the optimal performance. More precisely, computation of the lower bounds is formulated as a convex SDLMI, which can be efficiently solved. Such formulation for the lower bounds has not been proposed in the literature. Contrary to the lower-bound case, however, computation of the upper bounds yields a nonconvex SDMI in a Lyapunov matrix and a polynomial annihilator. An iterative procedure will be proposed to separately solve the decision variables and convert the nonconvex constraint to an SDLMI. Secondly, we suggest a new parameterization of the polynomial annihilator, which can reduces number of decision variables and equality constraints in the SDLMI conditions. Hierarchical improvement of both the upper bounds and the lower bounds can be done via increasing the degrees of the polynomial annihilators. Once an upper bound and a lower bound is computed, if the gap between the two bounds is small, the suboptimal performance is concluded to be nearly optimal. Finally, the proposed formulation can be applied to polynomial systems whose operating domains are more general than those considered in [6].

The notation used in this paper is rather standard. The symbols O and I denote the zero matrix and the identity matrix of proper dimensions respectively. For a real symmetric matrix A, the inequality $A \succeq O$ means that A is positive semidefinite. Similarly, $A \succ O$ indicates that A is positive definite. Finally, the symbol $\mathrm{He}(B)$ stands for $B + B^{\mathrm{T}}$ for a square matrix B.

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II. PRELIMINALIES

A. Sum-of-squares technique

In this section, we will summarize about the sum-of-squares (SOS) technique, which plays an important role for solving optimization problems with polynomial objectives and constraints. Recent developments of the sum of squares technique and its applications in various types of control problems can be found in [13], [17], [8].

The computational method used in this report relies on the sum-of-squares decomposition of multivariate polynomials. A multivariate polynomial $f(x)$ ($x \in \mathbb{R}^n$) is *a sum of squares* if there exist polynomials $f_1(x), \dots, f_m(x)$ such that $f(x) = \sum_{i=1}^m f_i^2(x)$. This can be shown equivalent to the existence of a special quadratic form stated in the following proposition.

Proposition 1: Let $f(x)$ be a polynomial in $x \in \mathbb{R}^n$ of degree $2d$. In addition, let $z(x)$ be a column vector whose elements are all monomials in x with degree no greater than d. Then $f(x)$ is a sum of squares iff there exists a positive semidefinite W such that

$$f(x) = z(x)^{\mathrm{T}} W z(x)$$

Proof: See [13]. ∎

The result of Proposition 1 tells us that finding a sum of squares decomposition for $f(x)$ can be done by solving an LMI. Even though the sum of squares condition is not necessary for nonnegativity, numerical experiments seem to indicate that the gap between sum of squares and nonnegativity is small [14].

The idea of the sum of squares can also be extended to the polynomial-matrix case. We say that a symmetric matrix $F(x)$ of dimension $m \times m$ is a (matrix) SOS iff it can be expressed as $F(x) = T(x)^T T(x)$ [8], [17] where $T(x)$ is a polynomial matrix in x. Note that $T(x)$ is not necessarily be a square matrix. Very similar to the scalar polynomial case, every SOS polynomial matrix is globally positive semidefinite. Moreover, a polynomial matrix $F(x)$ of degree $2d$ is SOS iff there exists a positive semidefinite matrix W such that $F(x) = [z_{[d]} \otimes I_m]^T W [z_{[d]} \otimes I_m]$ (see [8] for a proof) where $z_{[d]}$ is also the vector containing all monomials of degree d or less and $A \otimes B$ denotes the Kronecker product between A and B. Similarly to Proposition 1, we can find W by solving an LMI.

B. State-dependent LMIs and sum-of-squares relaxations

The methodology in Section II-A is used to solve state-dependent LMIs (SDLMIs) which are formulated from the state-feedback optimal control problem considered in the succeeding sections. We firstly consider a parameter-dependent LMI, which is a semi-infinite convex optimization problem of the form

$$\left.\begin{array}{ll} \text{min.} & c^T y \\ \text{s.t.} & F_0(x) + \sum_{i=1}^m y_i F_i(x) \succeq O, \quad \forall x \in \mathcal{X}, \end{array}\right\} \quad (1)$$

where $F_i(x)$'s are symmetric matrix functions of the parameter $x \in \mathbb{R}^n$ and $\mathcal{X} \subseteq \mathbb{R}^n$ is a given set. If x is a state variable of a system, the parameter-dependent LMI (1) is called a state-dependent LMI (SDLMI). Solving the above optimization problem leads to solving an infinite set of LMIs and hence computationally difficult. However, when the $F_i(x)$'s are symmetric *polynomial matrices* in x, and the set $\mathcal{X}$ is described by

$$\mathcal{X} = \{x \in \mathbb{R}^n \mid g_j(x) \geq 0,\ j = 1, \dots, k\}, \quad (2)$$

with polynomials $g_i(x)$'s, the sum of squares decomposition can provide a computational relaxation for (1). The idea is to consider the optimization problem

$$\left.\begin{array}{ll} \text{min.} & c^T y \\ \text{s.t.} & F_0(x) + \sum_{i=1}^m y_i F_i(x) = S_0(x) + \sum_{j=1}^k g_j(x) S_j(x), \end{array}\right\} \quad (3)$$

where $S_i(x)$'s are SOS polynomial matrices with appropriate degrees. Note here that the set $\mathbb{R}^n$ can be represented in the form (2) with $g_j(x) \equiv 0,\ j = 1, \dots, k$. It is clear that any solution to the sum of squares optimization problem (3) is also a solution to the SDLMI (1). However, (3) is easier to solve than (1) since searching for y and $S_i(x)$'s satisfying the constraint in (3) can be performed by solving LMIs (see [17] for details). The mentioned idea can be easily extended to SDLMIs with multiple constraints. Transformation from the SOS problem to an LMI problem can be performed by the software SOSTOOLS [15] or YALMIP [9]. If $\mathcal{X}$ satisfies some technical assumptions, the optimal value of the relaxed problem (3) tends to the global optimal value of the original SDLMI when we let the degrees of $S_i(x)$'s grow [17].

III. OPTIMAL CONTROL PROBLEM

We start with a problem description. Consider a nonlinear dynamical system

$$\dot{x}(t) = f(x(t)) + B(x(t))u,$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the input. The vector $f(x)$ and the matrix $B(x)$ are assumed to be polynomials in x. We consider regulation at the origin, thus we require that $f(0) = 0$.

For the above system, we consider a state-feedback controller $u = u(x)$, and assume that the closed-loop system is operated on a compact domain $\mathcal{X} \subset \mathbb{R}^n$ containing the origin. Throughout this paper, We assume that

$$\mathcal{X} = \{x \mid a_k^{\mathrm{T}} z(x) \leq 1, \ \ k = 1, \dots, l\} \quad (4)$$

where $z(x)$ is a vector of dimension N that contains x and monomials of higher degrees in x, and $a_k \in \mathbb{R}^N,\ k = 1, \dots, l$ are given constant vectors. Note that a polytope, an ellipsoid, a generalized ellipsoid in [6], and some classes of semi-algebraic sets can be represented in the form of (4) with appropriate $z(x)$ and a_k's. For example, the set $\mathcal{X} = \{x \in \mathbb{R}^2 \mid x_1 - x_2^2 \leq 1,\ x_1^2 + x_2^2 \leq 4\}$ can be represented in the form (4) with $z(x) = \begin{bmatrix} x_1 & x_2 & x_1^2 & x_2^2 \end{bmatrix}^{\mathrm{T}}$, $a_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^{\mathrm{T}}$, and $a_2 = \begin{bmatrix} 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix}^{\mathrm{T}}$.

As a measure of system performance under the control input u we will consider the cost function

$$J(x_0,u) = \int_0^\infty (z(x)^{\mathrm{T}} Q z(x) + u^{\mathrm{T}} R u) dt, \tag{5}$$

with given matrices $Q \succeq O$ and $R \succ O$. Here, the initial condition is $x(0) = x_0$. Our objective is to compute the value of

$$J^*(x_0) = \min_u J(x_0,u), \quad \forall x_0 \in \mathcal{X}, \tag{6}$$

and the controller u that minimizes the cost function.

Existence of an upper bound and a lower bound on $J^*(x_0)$ is guaranteed by existence of solutions of the HJB inequalities as stated in the following propositions.

Proposition 2 (Upper bound): Let $V : \mathcal{X} \to \mathbb{R}$ be a continuously differentiable function satisfying $V(x) > 0$ ($\forall x \in \mathcal{X}\backslash\{0\}$), $V(0) = 0$, and

$$\min_u \left(\frac{\partial V(x)}{\partial x}^{\mathrm{T}} (f(x) + B(x)u) + z(x)^{\mathrm{T}} Q z(x) + u^{\mathrm{T}} R u \right) < 0, \quad \forall x \in \mathcal{X}\backslash\{0\}. \tag{7}$$

If there exists a number $\rho > 0$ such that the set

$$\overline{\mathcal{X}} = \{x \in \mathbb{R}^n \mid V(x) \le \rho\}$$

is included in $\mathcal{X}$, then $V(x_0) \ge J^*(x_0)$ for all initial conditions $x_0 \in \overline{\mathcal{X}} \subset \mathcal{X}$, and $\hat{u}(x) = -\frac{1}{2} R^{-1} B^{\mathrm{T}}(x) \frac{\partial V(x)}{\partial x}$ attaining the minimum of the left-hand side of (7) for each $x \in \mathcal{X}$ is a stabilizing controller for the system. ■

Proposition 3 (Lower bound): Let $W : \mathcal{X} \to \mathbb{R}$ be a continuously differentiable function satisfying $W(x) > 0$ ($\forall x \in \mathcal{X}\backslash\{0\}$), $W(0) = 0$, and

$$\min_u \left(\frac{\partial W(x)}{\partial x}^{\mathrm{T}} (f(x) + B(x)u) + z(x)^{\mathrm{T}} Q z(x) + u^{\mathrm{T}} R u \right) > 0, \quad \forall x \in \mathcal{X}\backslash\{0\}. \tag{8}$$

Then $W(x_0) \le J^*(x_0)$ for all initial conditions x_0 such that $x(t) \in \mathcal{X}$ for all t, and $\tilde{u}(x) = -\frac{1}{2} R^{-1} B^{\mathrm{T}}(x) \frac{\partial W(x)}{\partial x}$ attains the minimum of the left-hand side of (8) for each $x \in \mathcal{X}$. ■

The proofs of Propositions 2 and 3 are omitted due to space limitation. However, they can be done along the same line as those of Theorems 1 and 2 in [7], respectively.

Searching for functions $V(x)$ and $W(x)$ satisfying (7) and (8) is a difficult task. However, restriction of the search of $V(x)$ and $W(x)$ to polynomials leads to more numerically tractable conditions. Computation of such polynomial solutions is discussed in the next section.

IV. Main Results

An approach to compute polynomial-solution candidates of (7) and (8) is discussed in this section.

We firstly consider solution candidates of (7) in the form of

$$V(x) = z(x)^{\mathrm{T}} Y^{-1} z(x) \tag{9}$$

with $Y \succ O$, where $z(x)$ is the monomial vector defined in Section III. It is obvious that the solution $V(x)$ of the form (9) satisfies $V(0) = 0$. In order to search for a solution in the form of (9), we write the given polynomial system in the following state-dependent linear-like representation:

$$\dot{x} = A(x)z(x) + B(x)u, \tag{10}$$

where $A(x)$ is a polynomial matrix in x. It is notable that the representation $f(x) = A(x)z(x)$ is not unique for each $z(x)$.

Before proceeding, we define additionally $L(x)$ to be a $N \times n$ polynomial matrix whose (i,j)-element is given by

$$L_{ij}(x) = \frac{\partial z_i}{\partial x_j}(x), \quad i = 1,2,\ldots,N, \ \ j = 1,2,\ldots,n.$$

The gradient of $V(x)$ can be obtained by the formula $\frac{\partial V}{\partial x}(x) = 2L^{\mathrm{T}}(x) Y^{-1} z(x)$.

An upper bound on $J^*(x_0)$, as well as a suboptimal controller can be computed via the following theorem.

Theorem 1: For the system $\dot{x} = A(x)z(x) + B(x)u$ with the domain X, suppose that there exist a matrix Y, a polynomial matrix $N(x)$ of dimension $N \times N$, and a scalar $\beta > 0$ such that

$$Y \succ O, \tag{11}$$

$$\begin{bmatrix} \mathcal{M}(x) + N(x)Y + YN(x)^{\mathrm{T}} & -YQ^{\frac{1}{2}} \\ -Q^{\frac{1}{2}}Y & -I \end{bmatrix} \prec O, \quad \forall x \in \mathcal{X}, \tag{12}$$

$$N(x)z(x) = 0, \quad \forall x \in \mathbb{R}^n, \tag{13}$$

$$\begin{bmatrix} \beta & -a_k^{\mathrm{T}} Y \\ -Y a_k & Y \end{bmatrix} \succeq O, \quad k = 1,\ldots,l, \tag{14}$$

where $\mathcal{M}(x) := L(x)A(x)Y + YA^{\mathrm{T}}(x)L^{\mathrm{T}}(x) - L(x)B(x)R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)$. Then for any initial conditions x_0 in the invariant set

$$\overline{\mathcal{X}} = \left\{ x \in \mathbb{R}^n \mid z(x)^{\mathrm{T}} Y^{-1} z(x) \le \frac{1}{\beta} \right\},$$

it holds that $J^*(x_0) \le z(x_0)^{\mathrm{T}} Y^{-1} z(x_0)$, and a stabilizing controller is given by $u = -R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)Y^{-1}z(x)$. ■

Proof: Positivity of $V(x) = z(x)^{\mathrm{T}} Y^{-1} z(x)$ over $\mathcal{X}\backslash\{0\}$ is guaranteed by (11). Based on Schur complements [5], inequality (12) is equivalent to

$$\begin{aligned} &L(x)A(x)Y + YA^{\mathrm{T}}(x)L^{\mathrm{T}}(x) + N(x)Y + YN(x)^{\mathrm{T}} \\ &-L(x)B(x)R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x) + YQY \preceq O. \end{aligned}$$

Pre- and post-multiplication of $z(x)^{\mathrm{T}} Y^{-1}$ and $Y^{-1} z(x)$, respectively, to the above inequality yields

$$\begin{aligned} &z(x)^{\mathrm{T}}(Y^{-1}L(x)A(x) + A^{\mathrm{T}}(x)L^{\mathrm{T}}(x)Y^{-1} \\ &-Y^{-1}L(x)B(x)R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)Y^{-1} + Q)z(x) \le 0 \end{aligned} \tag{15}$$

due to (13). Inequality (15) is nothing but a special case of (7) with the substitution $V(x) = z(x)^{\mathrm{T}} Y^{-1} z(x)$, $f(x) = A(x)z(x)$, and $\hat{u} = -\frac{1}{2} R^{-1} B^{\mathrm{T}}(x) \frac{\partial V(x)}{\partial x}$. Let $\rho := 1/\beta$, inequality (14) is equivalent to

$$\begin{bmatrix} 1 & -a_k^{\mathrm{T}} \\ -a_k & \frac{Y^{-1}}{\rho} \end{bmatrix} \succeq O, \quad k = 1,\ldots,l.$$

Pre- and post-multiplication of $\begin{bmatrix}1 & z(x)^{\mathrm{T}}\end{bmatrix}$ and $\begin{bmatrix}1 \\ z(x)\end{bmatrix}$, respectively, to the above inequality yields

$$(1 - a_k^{\mathrm{T}} z(x)) + 0.5(\frac{z(x)^{\mathrm{T}} Y^{-1} z(x)}{\rho} - 1) \geq 0, \quad k = 1, \ldots, l$$

which implies that

$$\overline{\mathcal{X}} = \left\{x \in \mathbb{R}^n \mid z(x)^{\mathrm{T}} Y^{-1} z(x) \leq \rho\right\} \subset \mathcal{X}.$$

■

The constraint (12) is nonconvex in the decision variables Y and $N(x)$ due to the term $N(x)Y + YN(x)^{\mathrm{T}}$. If $N(x)$ is fixed a priori, however, the constraints (11), (12), and (14) are affine in Y. Indeed, they are SDLMIs in the elements of Y, and can be solved using the method described in Section II-B.

Remark: It is notable that the inequality (15) also implies the existence of an annihilator $N(x)$ satisfying (12) and (13). This fact can be proved using Finsler's lemma [5]. Details of the proof is reported in [6]. Moreover, the existence of a *polynomial* annihilator $N(x)$ is also guaranteed when the domain $\mathcal{X}$ is compact [4]. When the solution $V(x)$ is fixed a priori, i.e., the matrix Y in (9) is fixed a priori, therefore, the gap between (11)-(14) and the HJB inequality (7) can be made arbitrarily small by increasing the degree of $N(x)$.

Let us now discuss how to reduce the number of scalar decision variables by considering the structure of $N(x)$. It is known from the constraint $N(x)z(x) = 0$ that the rank of $N(x)$ is at most $N - 1$. Based on the maximum rank decomposition, for each polynomial annihilator $N_b(x)$ of dimension $(N-1) \times N$ with $\mathrm{rank}(N_b(x)) = N - 1$, there exists a polynomial $M(x)$ of dimension $N \times (N-1)$ such that $N(x) = M(x)N_b(x)$. Note that in case of $z(x) = x$, the explicit form of $N_b(x)$ is given as

$$N_b(x) = \begin{bmatrix} -x_2 & x_1 & 0 & \cdots & 0 \\ 0 & -x_3 & x_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & -x_n & x_{n-1} \end{bmatrix}.$$

Construction of $N_b(x)$ for a general monomial vector $z(x)$ can be founded in [11]. With the new parameterization of the polynomial annihilator, Theorem 1 is modified as follows.

Theorem 2: For the system $\dot{x} = A(x)z(x) + B(x)u$ with the domain X and for a given $N_b(x)$, suppose that there exist a matrix Y, a polynomial matrix $M(x)$ of dimension $N \times (N-1)$, and a scalar $\beta > 0$ such that

$$Y \succ O, \tag{16}$$

$$\begin{bmatrix} \mathcal{M}(x) + \mathrm{He}(M(x)N_b(x)Y) & -YQ^{\frac{1}{2}} \\ -Q^{\frac{1}{2}}Y & -I \end{bmatrix} \prec O, \quad \forall x \in \mathcal{X}, \tag{17}$$

$$\begin{bmatrix} \beta & -a_k^{\mathrm{T}} Y \\ -Y a_k & Y \end{bmatrix} \succeq O, \quad k = 1, \ldots, l, \tag{18}$$

where $\mathcal{M}(x) := L(x)A(x)Y + YA^{\mathrm{T}}(x)L^{\mathrm{T}}(x) - L(x)B(x)R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)$. Then for any initial conditions x_0 in the invariant set

$$\overline{\mathcal{X}} = \left\{x \in \mathbb{R}^n \mid z(x)^{\mathrm{T}} Y^{-1} z(x) \leq \frac{1}{\beta}\right\},$$

it holds that $J^*(x_0) \leq z(x_0)^{\mathrm{T}} Y^{-1} z(x_0)$, and a stabilizing controller is given by $u = -R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)Y^{-1}z(x)$. ■

Since the polynomial matrix $M(x)$ has less number of elements than that of $N(x)$, the modified constraints in Theorem 2 has less number of scalar variables than those in Theorem 1. However, we still have a nonconvex constraint in the variables Y and $M(x)$ due to the term $M(x)N_b(x)Y + YN_b(x)^{\mathrm{T}}M(x)^{\mathrm{T}}$ in (17). Therefore, an iterative algorithm is provided to solve the problem, that is, the polynomial matrix $M^{(i)}(x)$ and the Lyapunov matrix $Y^{(i)}$ at the i^{th} iteration are solved alternatively. The iteration procedure is as follows:

Iterative Algorithm:

(I) By fixing the variable $M(x)$ to $M^{(i)}(x)$, we solve the constraints (16)–(18) in the variable Y and β. In this step, we obtain the matrix $Y^{(i)}$, the scalar $\beta^{(i)}$, and the upper bound $z(x_0)^{\mathrm{T}}(Y^{(i)})^{-1}z(x_0)$.

(II) For fixed $Y^{(i)}$, we solve a matrix $\tilde{M}(x)$ and scalars $\alpha > 0$, $\beta > 0$ satisfying

$$\begin{bmatrix} \tilde{\mathcal{M}}(x) + \mathrm{He}(\tilde{M}(x)N_b(x)Y^{(i)}) & -\alpha Y^{(i)}Q^{\frac{1}{2}} \\ -\alpha Q^{\frac{1}{2}}Y^{(i)} & -I \end{bmatrix} \prec O, \quad \forall x \in \mathcal{X}, \tag{19}$$

$$\begin{bmatrix} \beta & -\alpha a_k^{\mathrm{T}} Y^{(i)} \\ -\alpha Y^{(i)} a_k & \alpha Y^{(i)} \end{bmatrix} \succeq O, \quad k = 1, \ldots, l, \tag{20}$$

where $\tilde{\mathcal{M}}(x) := L(x)A(x)(\alpha Y^{(i)}) + (\alpha Y^{(i)})A^{\mathrm{T}}(x)L^{\mathrm{T}}(x) - L(x)B(x)R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)$. In this step, we fix the degree of $\tilde{M}(x)$ to d_M, and thus the constraints (19)–(20) are just SDLMIs in the scalars α, β, and the coefficients of $\tilde{M}(x)$.

(III) With $M^{(i+1)}(x) = \tilde{M}(x)/\alpha$, we solve $Y^{(i+1)}$, $\beta^{(i+1)}$ satisfying (16)–(18) and obtain the new upper bound $z(x_0)^{\mathrm{T}}(Y^{(i+1)})^{-1}z(x_0)$.

Once $M(x)$ is known in Step (I) or (III), we may optimize the upper bound by introducing a new variable T such that $T \succeq Y^{-1}$. Here, a good upper bound can be obtained by minimize $\mathrm{Trace}(T)$ subject to (16)–(18) and $\begin{bmatrix} T & I \\ I & Y \end{bmatrix} \succeq O$. The last inequality is equivalent to $T \succeq Y^{-1}$ by Schur complements.

If the problem in Step (II) is feasible, then the constraints (16)–(18) are also feasible with $M(x) = \tilde{M}(x)/\alpha$ and $Y = \alpha Y^{(i)}$. As a result, $V(x) = z(x)^{\mathrm{T}}(\alpha Y^{(i)})^{-1}z(x)$ is a valid Lyapunov function of the closed-loop system with the controller $u = -R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)(\alpha Y^{(i)})^{-1}z(x)$. Moreover, we have $J^*(x_0) \leq z(x_0)^{\mathrm{T}}(\alpha Y^{(i)})^{-1}z(x_0)$, and maximization of α in Step (II) may improve the upper bound.

The above algorithm will be proceeded until the value $|z(x_0)^{\mathrm{T}}(Y^{(i+1)})^{-1}z(x_0) - z(x_0)^{\mathrm{T}}(Y^{(i)})^{-1}z(x_0)|$ is less than some threshold. It is notable that increasing the degree

d_M in Step (II) may decrease the upper bound as shown in the numerical example in Section V.

The rest of this section is devoted to the lower bound computation. Here, we then consider solution candidates of (8) in the form of $W(x) = z(x)^{\mathrm{T}}Xz(x), \quad X \succ O$. A lower bound on $J^*(x_0)$ can be computed via the following theorem.

Theorem 3: For the system $\dot{x} = A(x)z(x) + B(x)u$ with the domain X and for a given $N_b(x)$ suppose that there exists a matrix X and a polynomial matrix $E(x)$ such that

$$X \succ O, \qquad (21)$$

$$\begin{bmatrix} \mathcal{N}(x) & XL(x)B(x)R^{-\frac{1}{2}} \\ R^{-\frac{1}{2}}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)X & I \end{bmatrix} \succ O, \quad \forall x \in \mathcal{X}, \qquad (22)$$

where $\mathcal{N}(x) := XL(x)A(x) + A^{\mathrm{T}}(x)L^{\mathrm{T}}(x)X + Q + \mathrm{He}(E(x)N_b(x))$. Then $J^*(x_0) \geq z(x_0)^{\mathrm{T}}Xz(x_0)$ for any x_0 such that $x(t) \in \mathcal{X}$ for all t. ■

Proof: The proof can be done similarly to that of Theorem 1. Inequality (21) guarantees positivity of $W(x)$ over $\mathcal{X}\backslash\{0\}$, while inequality (22) is equivalent to

$$XL(x)A(x) + A^{\mathrm{T}}(x)L^{\mathrm{T}}(x)X + Q + N_b(x)^{\mathrm{T}}E^{\mathrm{T}}(x) + E(x)N_b(x) - XL(x)B(x)R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)X \succeq O,$$

based on Schur complements. Since $N_b(x)z(x) = 0$, the above inequality implies

$$z(x)^{\mathrm{T}}(XL(x)A(x) + A^{\mathrm{T}}(x)L^{\mathrm{T}}(x)X + Q - XL(x)B(x)R^{-1}B^{\mathrm{T}}(x)L^{\mathrm{T}}(x)X)z(x) \geq 0, \qquad (23)$$

which is a special case of (8) with the substitution $W(x) = z(x)^{\mathrm{T}}Xz(x)$, $f(x) = A(x)z(x)$, and $\tilde{u} = -\frac{1}{2}R^{-1}B^{\mathrm{T}}(x)\frac{\partial W(x)}{\partial x}$. ■

If we fix the degree of the polynomial annihilator $E(x)$, the constraints (21)-(22) are SDLMI constraints in X and the coefficients of $E(x)$. Therefore, this problem can be solved by the method in Section II-B with the help of softwares SOSTOOLS or YALMIP. Using the same heuristic as in the upper bound computation, a good lower bound can be computed by maximizing $\mathrm{Trace}(X)$ subject to (21)-(22).

By similar arguments to the upper bound computation, the inequality (23) also implies the existence of a polynomial matrix $E(x)$ satisfying (22). This fact can be proved using Finsler's lemma [5] and exploiting the compactness of the domain $\mathcal{X}$. In the current problem setting, hence, the gap between (21)-(22) and the HJB inequality (8) can be made arbitrarily small by increasing the degree of $E(x)$. As opposed to the upper bound computation, however, the upper bound computation yields the convex constraints in the decision variables, and hence no iterative algorithm is required for computing a lower bound. Moreover, the best value of $\mathrm{Trace}(X)$ with respect to each choice of the solution candidate $W(x) = z(x)^{\mathrm{T}}Xz(x)$ is always achieved, in an asymptotic sense, due to the convexity of the problem.

V. Numerical Example

In this section, we provide an example to illustrate the underlined ideas in Section IV. This example is executed on Matlab 7.5, by using YALMIP as a solver for SOS-based problems.

Example We consider the system $\dot{x} = f(x) + B(x)u$ with

$$f(x) = \begin{bmatrix} -x_1 + x_1^2 + \frac{1}{4}x_2 - \frac{3}{2}x_1^3 - x_1^2x_2 - \frac{3}{4}x_1x_2^2 - \frac{1}{2}x_2^3 \\ 0 \end{bmatrix},$$

and $B(x) = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\mathrm{T}}$. The quadratic cost (5) with $Q = \mathrm{diag}(1,1)$ and $R = 1$ is considered, where the domain $\mathcal{X}$ is described by

$$\mathcal{X} = \{x \in \mathbb{R}^2 \mid -3 \leq x_1 \leq 3, -3 \leq x_2 \leq 3\}.$$

The initial point is $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\mathrm{T}}$.

The system is represented as the linear-like form

$$\dot{x} = A(x)x + B(x)u,$$

with the following system matrix $A(x)$:

$$A(x) = \begin{bmatrix} -1 + x_1 - \frac{3}{2}x_1^2 - \frac{3}{4}x_2^2 & \frac{1}{4} - x_1^2 - \frac{1}{2}x_2^2 \\ 0 & 0 \end{bmatrix}.$$

We firstly compute upper bounds on the optimal performance index (6) by solving the constraints in Theorem 2 with the iterative algorithm provided in Section IV.

To start the algorithm, we fix $M^{(0)}(x) = 0$. By minimizing the objective function $\mathrm{Trace}(T)$, we obtain an upper bound $x_0^{\mathrm{T}}(Y^{(0)})^{-1}x_0 = 2.6862$, with $\mathrm{Trace}(T) = 2.2634$. We reduce the upper bound by iteratively solve Y and $M(x)$ of a fixed degree. The algorithm is performed until the decrease of the upper bound is less than some threshold. Table I summarizes the final results with respect to $M(x)$ of various degrees.

As is seen from the table, the upper bound is significantly improved when increasing the degree of $M(x)$ up to 1. However, further increase of the degree seems to not improve the upper bound. The invariant sets, as well as, the trajectories of the closed-loop systems with respect to the cases of $\mathrm{deg}M(x) = 0$ and $\mathrm{deg}M(x) = 1$ are shown in Fig. 1.

For the computation of a lower bound, we repeatedly increase the degree of $E(x)$ in Theorem 2 and solve the corresponding robust SDP until the improvement of the optimal value of $\mathrm{Trace}(X)$ stops. In this example, the best value of $\mathrm{Trace}(X)$ as well as the best lower bound, are achieved with $E(x)$ of degree 1, and further increase of the degree of $E(x)$ does not improve the results. The improvement of $\mathrm{Trace}(X)$ and the lower bound with respect to the degree increase of $E(x)$ are summarized in Table II.

We observe from the numerical results that there is still a small gap between the best values of both bounds. This may be due to several reasons shown below:

(1) In this example, only quadratic solution candidates to the HJB inequalities are considered for the computation of both the upper bounds and the lower bounds. This conservatism may be improved by considering solution candidates of higher degrees.

(2) The iterative algorithm does not guarantee that the

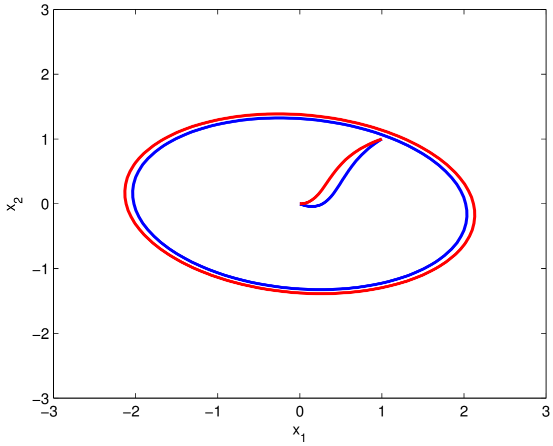

Fig. 1. The invariant sets with the closed-loop trajectories. In the case of $\deg M(x) = 0$ (blue line), and in the case of $\deg M(x) = 1$ (red line).

computed upper bound will converge to its best value after long iteration. In fact, it only guarantees that the upper bound obtained the present step is not worse than that obtained from the previous step.

TABLE I
UPPER BOUNDS ON THE PERFORMANCE INDEX

Degree of $M(x)$	0	1	2
Trace(T)	1.7132	1.1463	1.1463
Upper bound $x_0^{\mathrm{T}} Y^{-1} x_0$	1.9346	1.2113	1.2112

TABLE II
VARIATION OF LOWER BOUNDS WITH THE DEGREES OF $E(x)$

Degree of $E(x)$	0	1	2
Trace(X)	1.0235	1.0447	1.0447
Lower bound $x_0^{\mathrm{T}} X x_0$	0.9011	0.9997	0.9997

VI. CONCLUSION

The nonlinear optimal control of polynomial systems has been addressed in this paper. Finding polynomial solutions of the HJB inequalities, in order to computed bounds on the optimal cost, was formulated into SDMIs in terms of constant matrices and polynomial annihilator matrices. For the upper bound computation, the iterative algorithm is provided to solved the resulting nonconvex SDMI by separating the joint search of decision variables. The problem is therefore converted into an SDLMI, which can be efficiently solved based on the notion of SOS matrices. On the other hand, the resulting SDMI for the lower bound computation is just a convex SDLMI, and thus can be directly solved without any iteration. Increase of the degrees of the polynomial annihilators leads to improvement of both the upper bound and the lower bounds. Possible extensions of the current approach include controller design of nonlinear systems affected by bounded disturbances and/or parametric uncertainties.

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