2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Rational Relations for Modelling and Analyzing LTI Systems

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***Abstract*— If M is an R-module over an abelian ring R, then the set of all total submodules of M^2 is a seminearring $(T,+,\cdot)$, where $(+)$ is relation addition, and $(\cdot)$ is composition. If B is a Bezout domain of linear surjections on M, we construct a subseminearring Q of T consisting of so-called *rational relations* on M. An example is the set Q of single-input single-output relations defined by linear time-invariant (LTI) differential equations. A subseminearring of this Q is the field F of transfer functions, which approximate such relations as operators by neglecting their free response. Since rational relations include the free response, we propose using them instead of transfer functions to model and analyze LTI systems. Connections to results in behavioral systems theory are described.**

I. Background

In [4], a linear time-invariant (LTI) dynamic system is modelled as a *behavior*, defined as the kernel of a matrix of polynomial differential operators. For a single-input single-output (SISO) system, a behavior takes the form

$$r = \{(u,y) \in C_\infty^2 : ua(D) = yb(D)\}. \tag{1}$$

where u is the input, y is the output, $a \in \mathbb{R}[x]$ and $b \in \mathbb{R}[x]$ are real polynomials with $b \neq 0$, and $D : C_\infty \to C_\infty$ is the differential operator.[1]

In this paper, we view (1) as a binary relation on C_∞. Since the operators $a(D)$ and $b(D)$ in (1) are also binary relations, we may write $r = a(D)b(D)^{-1}$, i.e. the composition of $a(D)$ with the converse of $b(D)$. If $\deg(b) > 0$, the relation $r = a(D)b(D)^{-1}$ is nondeterministic: for every input $u \in C_\infty$, there are (infinitely) many outputs $y \in C_\infty$ satisfying (1). Naturally, we call r a *rational relation*.

A related notion is introduced in [6], where the polynomials a and b in (1) are replaced by rational functions. This introduces a redundancy in (1) for controller parameterization. Every rational function $G \in \mathbb{R}(x)$ defines a unique *rational representation* $G(D)$. Given $G = ab^{-1}$, with a and b coprime, $G(D)$ is defined as r in (1) above (see pp. 228-229 of [6]). In contrast, a rational relation $r = a(D)b(D)^{-1}$ does not require a and b to be coprime. Thus, the set of all rational relations is larger than the set of all rational representations (the set of all rational functions). Rational relations also should not be confused with fractional representations [1]. The former relate signals, while the latter relate transfer functions (e.g. proper stable transfer functions).

It is shown in [4] that addition (a parallel connection) or composition (a series connection) of two rational relations always yields a rational relation, and rules for adding and composing them are given there.[2] If Q is the set of all binary relations in the form of (1), then it follows that $(Q,+,\cdot)$ is a closed algebraic system, where $+$ denotes addition and $\cdot$ denotes composition.

But what type of algebraic system is $(Q,+,\cdot)$? Unlike rational functions, which form a field, we will show that $(Q,+,\cdot)$ is not even a ring or a semiring or a near-ring, but rather a (noncommutative) seminearring, i.e. a ring lacking additive inverses and one distributive law. However, it does have some field-like properties: negation yields a unique inverse[3] for the additive semigroup $(Q,+)$, and converse yields an inverse[3] for the multiplicative semigroup $(Q,\cdot)$. We will also show that the set of all total linear relations on C_∞ is also a seminearring, i.e. a superseminearring of $(Q,+,\cdot)$.

Our results are developed in an abstract algebraic framework and then applied to LTI systems as a special case. In Section IV, given an R-module M and a Bezout domain B of linear surjections on M, we construct a seminearring $(Q,+,\cdot)$ of rational relations on M. In the case of LTI systems, M is the $\mathbb{R}$-module of smooth functions and $B = \mathbb{R}[D]$ is the set of polynomial differential operators.

We also utilize results and notions that apply to binary relations generally, such as the converse of a relation, which generalizes the inverse of an operator. For example, in Section VII, the relation from the reference to the error of a closed-loop system is found to be $(1+CP)^{-1}$, where C is the control relation and P is the plant relation. This expression is valid for nonlinear multiple-input multiple-output (MIMO) systems. For SISO LTI systems, it reduces to a rational relation if C and P are rational relations such that $CP \neq -1$.

II. Motivation

Before rational relations can replace transfer functions, it must be determined how the algebra of rational relations differs from that of transfer functions. Rational relations are preferred because they include the free response of LTI systems and are thus more suitable for analyzing their stability and performance. Here, we briefly describe some limitations of transfer functions. Similar observations have been made in the literature on behavioral models, such as [4] and [6].

A transfer function approximates the nondeterministic relation (1) by a deterministic one (i.e. an operator). A simple way to define the transfer function is to restrict the

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[1] Postfix notation is used in (1) so that the order of composition of operators and relations agrees with their order in block diagrams drawn from left to right, from input to output. Postfix is also used in semigroup theory, and binary relations form a semigroup under composition.

[2] These rules translate into equations (13) and (14) of Theorem 2 in this paper.

[3] inverse in the semigroup sense only; the term *pseudoinverse* may be used to distinguish this from inverse in the group sense.

signal space to causal signals (i.e. $y(t) = 0$ for $t \leq 0$). Then, $\ker b(D) = \{0\}$, so $b(D)$ is injective. Since $b(D)$ is also surjective, it is bijective, and thus its converse $b(D)^{-1}$ is deterministic (i.e. an operator). This makes $a(D)b(D)^{-1}$ an operator, called the transfer function. The transfer function neglects the free response, which is noncausal. Other constructions of transfer functions (e.g. via operational calculus or the Laplace transform) similarly neglect the free response.

The free response can usually be inferred from the denominator of the transfer function, but not reliably. For example, consider the relation $r = (D-1)(D^2-1)^{-1}$, i.e. the set of all input-output pairs $(u,y) \in C_\infty^2$ such that

$$u(D-1) = y(D^2-1), \tag{2}$$

Restricting to causal signals reduces r to the transfer function $(D+1)^{-1}$, which is stable, whereas the original relation r given by (2) is unstable since the stable input $u(t) = 0$ relates to an unstable output $y(t) = e^t$. The (noncausal) free response e^t is neglected by the transfer function $(D+1)^{-1}$, which maps $u(t) = 0$ to $y(t) = 0$.

In [4], it is shown that the relation (1) is uncontrollable (i.e. no controllable state-space representation exists) if a and b have any common factors. The uncontrollable mode e^t of (2) cannot be inferred from the transfer function due to a pole-zero cancellation at 1. It is not possible to exclude uncontrollable systems such as (2) because the set of uncontrollable systems is not closed under composition or addition: series and parallel connections of controllable subsystems are not necessarily controllable.[4]

III. Module Relations

It turns out that the set of all rational relations $(Q,+,\cdot)$ defined in Section I has the same algebraic structure (plus some additional structure) as the set of all total linear relations on C_∞. This section establishes the structure of the latter, in the general context of abstract algebra, by finding the structure of the set of all total linear relations on any R-module. Other module representations of linear dynamic systems have appeared previously in [2].

It is assumed that the reader is familiar with standard algebraic systems such as semigroups, groups, semirings, rings, Bezout domains, fields, modules, and linear spaces. Lesser-known terms mentioned in Sections I, such as module relations and seminearrings, are defined and discussed in this section.

An binary relation (or simply relation) on a set M is a subset of M^2. The set of all such relations is the power set $\mathscr{P}(M^2)$. If $(u,y) \in r \in \mathscr{P}(M^2)$, we express this in infix notation as ury. For example, $1 < 2$ means $(1,2) \in <$. The composition of two relations $r_1 \in \mathscr{P}(M^2)$ and $r_2 \in \mathscr{P}(M^2)$ is the relation

$$r_1 r_2 = \{(u,y) \in M^2 : \exists x \in M, ur_1x, xr_2y\}. \tag{3}$$

Composition is associative and has identity $1 = \{(u,u) \in M^2\}$. Hence, $(\mathscr{P}(M^2),\cdot)$ is a monoid.

[4]For this reason, the set of all (SISO) rational representations defined in [6] is not algebraically closed under addition and composition of relations.

The converse r^{-1} of a relation $r \in \mathscr{P}(M^2)$ is the relation

$$r^{-1} = \{(y,u) \in M^2 : (u,y) \in r\}. \tag{4}$$

If M is an R-module over a commutative ring R, then we can define addition and negation in $\mathscr{P}(M^2)$ as follows, where $r, r_1, r_2 \in \mathscr{P}(M^2)$:

$$r_1 + r_2 = \{(u, y_1 + y_2) \in M^2 : ur_1y_1, ur_2y_2\}. \tag{5}$$

$$-r = \{(u,-y) \in M^2 : ury\} \tag{6}$$

Since $y_1 + y_2$ is addition in the group M, the addition of relations is commutative and associative. The additive identity is $0 = \{(u,0) \in M^2\}$.

The following definitions, which may be found in [3], will be useful for characterizing the properties of $(\mathscr{P}(M^2),+,\cdot)$.

Definition 1: Given a semigroup $(Q,\cdot)$ and $r \in Q$, an inverse $r^+ \in Q$ of r is an element satisfying

$$rr^+r = r, \tag{7}$$

$$r^+rr^+ = r^+. \tag{8}$$

Definition 2: A semigroup $(Q,\cdot)$ is a regular semigroup if every $r \in Q$ has an inverse. If every $r \in Q$ has a unique inverse, then $(Q,\cdot)$ is an inverse semigroup. A regular (resp. inverse) monoid is a regular (resp. inverse) semigroup with an identity.

Lemma 1: A regular semigroup is an inverse semigroup if, and only if, its idempotents commute [3].

Lemma 2: Suppose M is a module over an abelian ring R and that $S = \mathscr{P}(M^2)$ is the set of all binary relations on M. Then,

1) $(S,+)$ is a commutative inverse monoid; the identity is $0 = \{(u,0) \in M^2\}$, and for each $r \in S$, its negation $-r$ is its inverse,
2) $(S,\cdot)$ is a regular monoid; the identity is $1 = \{(u,u) \in M^2\}$, and for each $r \in S$, its converse r^{-1} is an inverse.

Proof: It follows from (5) and (6) that negation satisfies the inverse properties (7) and (8) (applied additively): $r + (-r) + r = r$ and $(-r) + r + (-r) = -r$. Hence $(S,+)$ is a regular monoid. Commutativity of $(M,+)$ implies that of $(S,+)$. This and Lemma 1 imply that $(S,+)$ is an inverse monoid. Property 2 (well-known in relation algebra) follows from Definitions 1 and 2 and equations (3), (7), and (8). ■

The *image* of a subset $M_1 \subseteq M$ under a relation $r \in \mathscr{P}(M^2)$ is denoted $M_1r = \{y \in M : \exists u \in M_1, ury\}$. If $M_1 = \{u\}$, we identify $\{u\}$ with u and write its image simply as ur. Thus, we may write ury as $ur \ni y$. If ur is also a singleton, we write ury as $ur = y$. The *range* of r is Mr, and the *domain* of r is Mr^{-1}. The *kernel* of r is $0r^{-1}$. A relation $r \in \mathscr{P}(M^2)$ is called *surjective* (or onto) if $Mr = M$ and is called *injective* (or one-to-one) if u_1ry and u_2ry imply that $u_1 = u_2$. It is *total* if its converse is surjective (i.e. $Mr^{-1} = M$), and it is *deterministic* if its converse is injective.

The set of all total relations on M is closed under composition, addition, and scaling, but not closed under converse. The same is true of all deterministic relations. A deterministic relation is called a partial function, while a total relation is called a nondeterministic (or multivalued) function. A total deterministic relation r is called a function (or operator), in which case $ury \Leftrightarrow ur = y$ for all $u,y \in M$. If the converse of a function is a function, then it is the inverse of the function. The set of all functions $M \to M$ is called $\mathrm{End}(M)$, which forms a ring under addition and composition.

If M is an R-module over an abelian ring R, then M^2 becomes a module by defining addition in M^2 by $(u_1,y_1)+(u_2,y_2) = (u_1+u_2, y_1+y_2)$ for all $u_1,y_1,u_2,y_2 \in M$ and defining scalar multiplication by $\alpha(u_1,y_1) = (\alpha u_1, \alpha y_1)$ for all $\alpha \in R$. We define a linear relation on M as a submodule of M^2. Let $\mathscr{P}_R(M^2) \subseteq \mathscr{P}(M^2)$ denote the set of all submodules of M^2, i.e. all linear relations on M. If R is a field, then the module M is called a linear space, and the linear relations are subspaces of M^2.

Let $\mathrm{Tot}_R(M) \subseteq \mathscr{P}_R(M^2)$ be the set of all total linear relations on M. A subset of $\mathrm{Tot}_R(M)$ is the set of all module endomophisms on M, denoted $\mathrm{End}_R(M) \subseteq \mathrm{End}(M)$. Whereas $\mathrm{End}_R(M)$ is a ring, we will show that $\mathrm{Tot}_R(M)$ is only a seminearring.

Definition 3: An algebraic system $(Q,+,\cdot)$ is a seminearring [5] if

1) $(Q,+)$ is a commutative monoid.
2) $(Q,\cdot)$ is a monoid.
3) For all $r,s,t \in Q$, $(r+s)t = rt+st$.

The etymology of seminearring is as follows. A semiring is a ring that may lack additive inverses, while a nearring is a ring that may lack one distributive law. Hence, a seminearring may lack additive inverses and one distributive law, as in Definition 3.

Theorem 1: Let M be a module over an abelian ring R, and let $\mathrm{Tot}_R(M)$ be the set of all total linear relations on M. Then $(\mathrm{Tot}_R(M),+,\cdot)$ is a seminearring. Also, $(\mathrm{Tot}_R(M),+)$ is an inverse monoid.

Proof: Properties 1 and 2 of Definition 3 follow from Properties 1 and 2 of Lemma 2 and the facts that $(\mathrm{Tot}_R(M),+)$ and $(\mathrm{Tot}_R(M),\cdot)$ are sub-monoids of $(\mathscr{P}(M^2),+,0)$ and $(\mathscr{P}(M^2),\cdot,1)$, respectively. That $(\mathrm{Tot}_R(M),+)$ is also an inverse monoid follows from the fact that negation is closed in $\mathrm{Tot}_R(M)$.

To prove Property 3 of Definition 3, we apply (3) and (5) to obtain, for all $u,y \in M$ and all $r,s,t \in \mathrm{Tot}_R(M)$,

$$(u,y) \in (r+s)t \Leftrightarrow \exists x_1,x_2 \in M, urx_1, usx_2, (x_1+x_2)ty \tag{9}$$

$$(u,y) \in rt+st \Leftrightarrow \exists x_1,x_2,y_1y_2 \in M, \quad urx_1, x_1ty_1, usx_2, x_2ty_2, y_1+y_2 = y \tag{10}$$

Suppose $(u,y) \in rt+st$. Adding x_1ty_1 and x_2ty_2 from (10) yields $(x_1+x_2)t(y_1+y_2)$. Setting $y_1+y_2 = y$ from (10) gives $(x_1+x_2)ty$, and thus (9) gives $(u,y) \in (r+s)t$. Hence, $(r+s)t \supseteq rt+st$. Conversely, suppose $(u,y) \in (r+s)t$ and apply (9). Since t is total, there exists $y_1 \in M$ such that x_1ty_1. Subtracting x_1ty_1 from $(x_1+x_2)ty$ in (9) gives x_2ty_2, where $y_2 = y - y_1$. Hence, the r.h.s. of (10) is satisfied and thus $(u,y) \in rt+st$. Therefore, $(r+s)t \subseteq rt+st$. Combining this with $(r+s)t \supseteq rt+st$ gives $(r+s)t = rt+st$. ■

IV. Rational Relations

Definition 4: If M is an R-module over an abelian ring R, then a Bezout domain of linear surjections (BLS) on M is a Bezout domain of R-module endomorphisms $B \subseteq \mathrm{End}_R(M)$ with the property that every nonzero $b \in B$ is surjective, i.e. $Mb = M$.

Definition 5: Given an R-module M, a BLS B on M, and $a,b \in B$, the *rational relation* a/b is defined as

$$\frac{a}{b} \equiv ab^{-1} = \{(u,y) \in M^2 : ua = yb\}. \tag{11}$$

Let $Q \subseteq \mathrm{Tot}_R(M)$ denote the set of all total rational relations:

$$Q = \{\frac{a}{b} : a \in B, b \in B\backslash\{0\}\}. \tag{12}$$

Theorem 2: Given an R-module M and a BLS B on M, the set of rational relations $(Q,+,\cdot)$ in (12) is a sub-seminearring of $(\mathrm{Tot}_R(M),+,\cdot)$. Also, $(Q,+)$ is an inverse monoid and $(Q,\cdot)$ is a regular monoid. For all $a_1,a_2 \in B$, and all $b_1,b_2,g \in B\backslash\{0\}$,

$$\mathrm{coprime}(b_1,b_2) \Rightarrow \frac{a_1}{gb_1} + \frac{a_2}{gb_2} = \frac{a_1b_2+a_2b_1}{gb_1b_2}, \tag{13}$$

$$\mathrm{coprime}(b_1,a_2) \Rightarrow \frac{a_1}{b_1g}\frac{ga_2}{b_2} = \frac{a_1a_2}{b_1b_2}, \tag{14}$$

$$\left(\frac{a_1}{b_1}\right)^{-1} = \frac{b_1}{a_1}. \tag{15}$$

Proof: Commutativity in $(B,\cdot)$ is used throughout. Since B is a Bezout domain and hence a Greatest Common Divisor (GCD) domain, every sum of rational relations can be expressed as the l.h.s. of (13), and every product can be expressed as the l.h.s. of (14).

To prove (14), suppose $(u,y) \in \frac{a_1}{b_1g}\frac{ga_2}{b_2}$. Then, by (11) and (3), there exists $x \in M$ such that $ua_1 = xgb_1$ and $xga_2 = yb_2$. Multiplying these two equations through by a_2 and b_1, respectively, and eliminating xgb_1a_2 gives $ua_1a_2 = yb_1b_2$, and thus $(u,y) \in \frac{a_1a_2}{b_1b_2}$. Hence, $\frac{a_1}{b_1g}\frac{ga_2}{b_2} \subseteq \frac{a_1a_2}{b_1b_2}$. Conversely, suppose $(u,y) \in \frac{a_1a_2}{b_1b_2}$, so that $ua_1a_2 = yb_1b_2$, and suppose b_1 and a_2 are coprime. Since B is a Bezout domain, there exists $c_1,c_2 \in B$ such that

$$1 = b_1c_1 + a_2c_2. \tag{16}$$

Since g is surjective, there exists $x \in M$ such that

$$xg = ua_1c_1 + yb_2c_2. \tag{17}$$

Multiplying (17) by b_1 and substituting $ua_1a_2 = yb_1b_2$ and (16) gives $xgb_1 = ua_1$. Similarly, $xga_2 = yb_2$. Hence $(u,y) \in \frac{a_1}{b_1g}\frac{ga_2}{b_2}$, and thus $\frac{a_1}{b_1g}\frac{ga_2}{b_2} \supseteq \frac{a_1a_2}{b_1b_2}$. Combining this with $\frac{a_1}{b_1g}\frac{ga_2}{b_2} \subseteq \frac{a_1a_2}{b_1b_2}$ gives $\frac{a_1}{b_1g}\frac{ga_2}{b_2} = \frac{a_1a_2}{b_1b_2}$.

To prove (13) for the case $g = 1$, suppose $(u,y) \in \frac{a_1}{b_1} + \frac{a_2}{b_2}$. Then there exists $y_1,y_2 \in M$ such that $ua_1 = y_1b_1$, $ua_2 = y_2b_2$, and $y_1+y_2 = y$. Multiplying $ua_1 = y_1b_1$ through by b_2 and multiplying $ua_2 = y_2b_2$ through by b_1, and adding the resulting equations gives $u(a_1b_2+a_2b_1) = (y_1+y_2)b_1b_2 = yb_1b_2$, and thus $(u,y) \in \frac{a_1b_2+a_2b_1}{b_1b_2}$. Hence, $\frac{a_1}{b_1} + \frac{a_2}{b_2} \subseteq \frac{a_1b_2+a_2b_1}{b_1b_2}$. Conversely, suppose$(u,y) \in \frac{a_1b_2+a_2b_1}{b_1b_2}$, so that $u(a_1b_2+a_2b_1) =$

yb_1b_2, and suppose b_1 and b_2 are coprime. Then there exists $d_1, d_2 \in B$ such that

$$1 = b_1d_1 + b_2d_2. \tag{18}$$

Let

$$y_1 = ua_1d_1 - ua_2d_2 + yb_2d_2, \tag{19}$$
$$y_2 = ua_2d_2 - ua_1d_1 + yb_1d_1. \tag{20}$$

Multiplying (19) by b_1 and substituting $u(a_1b_2 + a_2b_1) = yb_1b_2$ and (18) gives $y_1b_1 = ua_1$. Similarly, $y_2b_2 = ua_2$. Adding (19) and (20) and substituting (18) gives $y_1 + y_2 = y$. These last 3 results give $(u,y) \in \frac{a_1}{b_1} + \frac{a_2}{b_2}$, and thus $\frac{a_1}{b_1} + \frac{a_2}{b_2} \supseteq \frac{a_1b_2+a_2b_1}{b_1b_2}$. Combining this with $\frac{a_1}{b_1} + \frac{a_2}{b_2} \subseteq \frac{a_1b_2+a_2b_1}{b_1b_2}$ gives $\frac{a_1}{b_1} + \frac{a_2}{b_2} = \frac{a_1b_2+a_2b_1}{b_1b_2}$. Finally, multiplying this last equation by $g^{-1} = 1/g$ and applying the distributive property (Property 3 from Theorem 1), along with (11) and the fact that converse is an involution (i.e. $(gb)^{-1} = b^{-1}g^{-1}$), gives

$$\left(\frac{a_1}{b_1} + \frac{a_2}{b_2}\right)\frac{1}{g} = \frac{a_1b_2 + a_2b_1}{b_1b_2}\frac{1}{g} \tag{21}$$
$$\frac{a_1}{gb_1} + \frac{a_2}{gb_2} = \frac{a_1b_2 + a_2b_1}{gb_1b_2} \tag{22}$$

Thus, (13) is proved for a general $g \neq 0$.

Since Q is a subset of $\text{Tot}_R(M)$ and is closed under addition, multiplication, and negation, Q is a sub-seminearring of $\text{Tot}_R(M)$ and is additively an inverse monoid (since $\text{Tot}_R(M)$ is).

Finally (11) and $(ab^{-1})^{-1} = ba^{-1}$ gives (15). Every $a/b \in Q$ has a (semigroup) multiplicative inverse given by

$$\left(\frac{a}{b}\right)^+ = \begin{cases} 0 & \text{if } a = 0 \\ (a/b)^{-1} & \text{if } a \neq 0 \end{cases}. \tag{23}$$

Applying (14) and (15) verifies that these relations satisfy the properties of an inverse given by (7) and (8). Thus, $(Q,\cdot)$ is a regular monoid. ■

Note that the converse operation is only a partial unary operator in Q: if $a_1 = 0$ in (15), then the converse of a_1/b_1 exists in $\mathscr{P}(M^2)$, but not in Q.

The rules (13) and (14) for addition and multiplication in Q, which are already known from [4], generalize the corresponding rules in the field of quotients $\text{Quot}(B)$. It may be shown that if the nonzero elements of B are bijective, then the rational relations in Q are deterministic, and Q becomes a field isomorphic to $\text{Quot}(B)$. Thus, if any nonzero elements of B are not injective, then the additive and multiplicative groups of $\text{Quot}(B)$ are weakened to inverse and regular semigroups, respectively, and one distributive law is lost.

V. Rational Relations as LTI Systems

Let M be the group of smooth functions $(C_\infty,+)$ which is an R-module over $\mathbb{R}$. Since $\mathbb{R}$ is a field, this module is also a linear space. By Theorem 1, the system of all total linear relations $(\text{Tot}_R(C_\infty),+,\cdot)$ on C_∞ is a seminearring.

Let $B = \{p(D) : p \text{ a polynomial}\}$, which is a Bezout domain. Since every nonzero $p(D) : C_\infty \to C_\infty$ is linear and surjective on C_∞, B is a BLS. Let $(Q,+,\cdot)$ be the system of rational relations defined in (11) and (12). Then, by Theorem 2, $(Q,+,\cdot)$ is a seminearring.

The following examples illustrate how algebra in $(Q,+,\cdot)$ differs from that of transfer functions. The pair (u,y) in (2) belongs to the *rational relation*

$$\frac{D-1}{D^2-1} \neq \frac{1}{D+1}. \tag{24}$$

Composition of rational relations is not commutative in general. For example, the following (postfix) compositions are different:

$$\frac{1}{D-1}\frac{D-1}{D+1} = \frac{1}{D+1}, \tag{25}$$

whereas

$$\frac{D-1}{D+1}\frac{1}{D-1} = \frac{D-1}{D^2-1} \supset \frac{1}{D+1}. \tag{26}$$

A pole-zero cancellation occurs in (25), but a zero-pole cancellation does not occur in (26).

The rational relation $r = 1/(D+1)$ has no additive inverse in the group sense, but it does have one in the semigroup sense, namely $-r = -1/(D+1)$. Indeed, we have

$$\frac{1}{D+1} + \frac{-1}{D+1} = \frac{0}{D+1} \supset 0. \tag{27}$$

However, $-r$ is the unique relation that satisfies the semigroup inverse properties: $r + (-r) + r = r$ and $(-r) + r + (-r) = -r$. Thus, $(Q,+)$ is an inverse semigroup.

Although $(Q\backslash\{0\},\cdot)$ is not a group, $(Q,\cdot)$ is a regular semigroup. If the numerator of $r \in Q$ is zero, then 0 is a multiplicative inverse[3] of r. Otherwise, the unique inverse[3] r^{-1} of r is its converse, which equals its reciprocal; for example,

$$\left(\frac{D+1}{D+2}\right)^{-1} = \frac{D+2}{D+1}, \tag{28}$$

which is not an inverse in the group sense since

$$\frac{D+1}{D+2}\left(\frac{D+1}{D+2}\right)^{-1} = \frac{D+1}{D+1} \supset 1. \tag{29}$$

Composition is not left-distributive over addition, but it is right-distributive. An example of the failure of the left-distributive law is

$$0 = \frac{0}{D}0 = \frac{0}{D}(1-1) \neq \frac{0}{D}(1) + \frac{0}{D}(-1) = \frac{0}{D}. \tag{30}$$

In contrast, right distribution always holds; for example,

$$\frac{0}{D} = 0\frac{0}{D} = (1-1)\frac{0}{D} = (1)\frac{0}{D} + (-1)\frac{0}{D} = \frac{0}{D}. \tag{31}$$

Another example is discrete-time LTI systems described by difference equations. Let $M = (\mathbb{Z} \to \mathbb{R})$ be the set of all discrete-time signals. Let L denote the left-shift operator, defined as $xL(k) = x(k+1)$, where $x \in M$. Let $B = \{p(L) : p \text{ a polynomial}\}$, and define the set of rational relations Q by (11) and (12). These relations have the same advantages over transfer functions as those in the first example. For example the rational relation

$$\frac{L-2}{L-2}, \tag{32}$$

has an unstable (and noncausal) output 2^k when its input is zero. In contrast, if we restrict signals to be causal (which implies zero initial conditions), this relation reduces to a stable operator, namely the identity transfer function 1, and fails to model the uncontrollable mode 2^k.

VI. Rational Relations on the Circle Group

A geometric example illustrating Theorems 1 and 2 is given by the circle group $M = \mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is an R-module over $R = \mathbb{Z}$. Let $B = \mathbb{Z}$ and define Q by (12) and (11). Theorem 1 of Section III implies that the set of all total linear relations $(\mathrm{Tot}_{\mathbb{Z}}, +, \cdot)$ on $\mathbb{T}$ is a seminearring. Given $a \in \mathbb{Z}$ and $b \in \mathbb{Z}\backslash\{0\}$, (11) defines the rational relation

$$\frac{a}{b} = \{(u,y) \in \mathbb{T}^2 : ua = yb\}. \tag{33}$$

By Theorem 2, the system $(Q, +, \cdot)$ of all such relations is a sub-seminearring of $(\mathrm{Tot}_{\mathbb{Z}}, +, \cdot)$. These relations behave in the same way as the LTI SISO relations introduced in Section V. Analogous expressions for equations (2) and (24) through (28) in this seminearring are, respectively,

$$u2 = y6, \tag{34}$$

$$\frac{2}{6} \supset \frac{1}{3}, \tag{35}$$

$$\frac{1}{2}\frac{2}{3} = \frac{1}{3}, \tag{36}$$

$$\frac{2}{3}\frac{1}{2} = \frac{2}{6} \supset \frac{1}{3}, \tag{37}$$

$$\frac{1}{3} + \frac{-1}{3} = \frac{0}{3} \supset 0, \tag{38}$$

$$\left(\frac{3}{5}\right)^{-1} = \frac{5}{3}, \tag{39}$$

$$\frac{3}{5}\left(\frac{3}{5}\right)^{-1} = \frac{3}{3} \supset 1. \tag{40}$$

Unlike the examples of Section I, the graphs of these relations may be visualized. The module $\mathbb{T}^2$ in (33) is the surface of a torus (donut). The coordinates (u,y) of a point are its longitude and latitude: graphs of constant u are minor circles, while graphs of constant y are major circles. For example, the circle $y = 0$ could be the inner annulus (the hole) of the torus. Points are added component-wise modulo-1 (which approximates vector addition near the origin of the torus). Scaling a point or one of its coordinates by $a \in \mathbb{Z}$ means adding it to itself a times.

The graph of the relation $a/b \in Q$ in (33) appears as *windings* on the torus. The pitch of the windings is $a/b \in \mathbb{Q}$ (i.e. a/b viewed as a rational number). If a and b are coprime, then there is a single continuous winding that undergoes a minor rotations for every b major rotations. In general, a/b contains c disjoint parallel windings of pitch a/b, where c is the greatest common divisor of a and b. For example, $2/6 \in Q$ consists of two parallel windings of pitch $1/3 \in \mathbb{Q}$, one of which is the primary winding $1/3 \in Q$ which passes through the origin. This agrees with (35). Note that the winding $1/3$ and the double-winding $2/6$ are both submodules of the torus $\mathbb{T}^2$.

Equations (36) and (37) show that composition of windings (as relations) is not commutative: a cancellation (of windings) occurs in (36) but not in (37). Equation (38) shows that adding the winding $1/3$ to its additive (semigroup) inverse $-1/3$ yields 3 parallel circles having zero pitch, i.e. 3 parallel latitudinal circles including the additive identity $0/1$, which is the u-coordinate circle $y = 0$. Equations (39) and (40) show that composing $3/5$ with its multiplicative inverse yields 3 windings with a pitch of 1, which include the primary winding $1 = 1/1 \in Q$, the compositional identity.

VII. Control Applications

Consider the set Q in (12) of all rational relations representing SISO LTI DEs, defined in the first example of Section V. Let $P = a/b \in Q$ represent a plant to be controlled and suppose uPy, where $u \in C_\infty = M$ is the plant input and $y \in C_\infty$ is the plant output. The goal is to specify u so that y follows a given reference trajectory $r \in C_\infty$.

Rational relations may be used to explain why open-loop control cannot stabilize an unstable plant. Given an unstable plant $P \in Q$ and a reference $r \in C_\infty$, the control signal $u \in C_\infty$ is chosen to satisfy uPr. Subtracting uPy from this gives $0_M(P-P)e$ where $e = r - y$ is the tracking error. Equation (13) of Theorem 2 gives $P - P = a/b - a/b = 0_B/b \supset 0_Q$, and hence $0_M(P-P)e$ implies $0_M(0_B/b)e$, which is equivalent to $eb = 0_M$ by (11). Since $\ker b$ includes unstable signals, e can be unstable. This conclusion cannot be obtained from transfer functions: if the (nondeterministic) relation P is approximated by a (deterministic) transfer function $\hat{P}$, then $\hat{P} - \hat{P} = 0$, which incorrectly implies that $e = 0$.

Module relations can model general feedback control systems. Suppose that M is a module over an abelian ring R and that $\mathscr{P}(M^2)$ is the set of all binary relations on M. Suppose a relation $P \in \mathscr{P}(M^2)$ represents a plant and that $C \in \mathscr{P}(M^2)$ represents a controller. A feedback relation is defined by

$$uPy \tag{41}$$

$$eCu \tag{42}$$

$$e = r - y, \tag{43}$$

where u is the plant input, y is the plant output, r is the desired value of y, and e is the error. In the case of continuous-time nonlinear systems, $M = C_\infty$, $R = \mathbb{R}$, and P and C are relations defined by nonlinear differential equations. The goal of feedback design is to choose C so that the relation between input r and output e has desireable characteristics, usually to make e small in some sense.

Composing (42) and (41) gives $eCPy$. Adding this to $e1e$ gives $e(1+CP)(y+e)$, which implies $e(1+CP)r$ by (43). This gives $r(1+CP)^{-1}e$. Conversely, if $r(1+CP)^{-1}e$, then the definitions of addition, composition, and converse in $\mathscr{P}(M^2)$ imply the existence of $u, y \in M$ satisfying (41), (42), and (43). Thus, the goal is to design $(1+CP)^{-1}$ via C.

Note that the closed-loop relation $(1+CP)^{-1}$ is well-defined in all cases. For example, $CP=-1$ gives $(1+CP)^{-1}=0^{-1}$, which constrains r to zero while e can have any value in M. The relation 0^{-1} is partial and nondeterministic.

If C and P are total linear relations and the same is required of $(1+CP)^{-1}$, then $1+CP$ must be surjective. If C and P are rational relations representing LTI DEs, then this condition is met if $C=a_c/b_c$ is proper and $P=a_p/b_p$ is strictly proper; i.e. $\deg b>\deg a$. In this case, $(1+CP)^{-1}$ is a proper rational relation, which may be found as follows.

Let $g_1\in \mathrm{GCD}(b_c,a_p)$ and $g_2\in \mathrm{GCD}(a_c,b_p)$ so that $CP=\frac{a'_c g_2}{g_1 b'_c}\cdot\frac{g_1 a'_p}{g_2 b'_p}$, coprime$(b'_c,a'_p)$, and coprime$(a'_c,b'_p)$. Then, application of (13), (14), and (15) of Theorem 2 gives

$$(1+CP)^{-1}=\left(1+\frac{a'_c g_2}{b'_c g_1}\cdot\frac{g_1 a'_p}{g_2 b'_p}\right)^{-1} \tag{44}$$

$$=\left(1+\frac{a'_c g_2}{b'_c}\cdot\frac{a'_p}{g_2 b'_p}\right)^{-1} \tag{45}$$

$$=\left(\frac{g_2 b'_c b'_p+g_2 a'_c a'_p}{g_2 b'_c b'_p}\right)^{-1} \tag{46}$$

$$=\frac{g_2 b'_c b'_p}{g_2(b'_c b'_p+a'_c a'_p)}, \tag{47}$$

so g_2 divides the characteristic polynomial $g_2(b_c b'_p+a'_c a_p)$, and hence $\ker g_2\subseteq 0_M(1+CP)^{-1}$. Since (41), (42), and (43) are equivalent to $r(1+CP)^{-1}e$, (47) implies that every $e\in\ker g_2$ is a solution when $r=0$. If $\ker g_2$ includes unstable signals, then e can be unstable. In contrast, g_2 cancels from the transfer function approximation of $(1+CP)^{-1}$. Similarly, any uncontrollable modes in C or P (e.g. common divisors of a'_p and b'_p) appear in (47), but not in the transfer function.

To find the (r,u) relation from (41), (42), and (43), add $uP(r-e)$ to the converse of eCu (i.e. $uC^{-1}e$) and take the converse of the result to obtain $r(C^{-1}+P)^{-1}u$. Conversely, $r(C^{-1}+P)^{-1}u$ implies the existence of the realization (41), (42), and (43).

If $a_c\neq 0$, then application of (13), (14), and (15) of Theorem 2 gives

$$(C^{-1}+P)^{-1}=\left(\frac{g_1 b'_c}{g_2 a'_c}+\frac{g_1 a'_p}{g_2 b'_p}\right)^{-1} \tag{48}$$

$$=\left(\frac{g_1 b'_c b'_p+g_1 a'_p a'_c}{g_2 a'_c b'_p}\right)^{-1} \tag{49}$$

$$=\frac{g_2 a'_c b'_p}{g_1(b'_c b'_p+a'_p a'_c)}. \tag{50}$$

If $a_c=0$, then the converse of $C=a_c/b_c$ is not in Q, and so $(C^{-1}+P)^{-1}$ must be evaluated in $\mathscr{P}(M^2)$. Applying (4), (5), and (11), along with the fact that P is total gives

$$(C^{-1}+P)^{-1}=\left(\frac{b_c}{0}+P\right)^{-1} \tag{51}$$

$$=\left(\frac{b_c}{0}\right)^{-1} \tag{52}$$

$$=\frac{0}{b_c}. \tag{53}$$

This special case also agrees with (50), since $b_c=g_1 b'_c$ and since $a_c=0$ gives $g_2=b_p\in \mathrm{GCD}(0,b_p)$, $a'_c=0$, and $b'_p=1$.

Expressions for other closed-loop rational relations (such as from disturbance to error) may be found similarly as functions of the rational relations C and P.

VIII. Conclusions

In [4], feedback loops are reduced to closed-loop behaviors by eliminating *latent* variables. In Section VII, we observed that this reduction can be performed via operations (i.e. addition, multiplication, and converse) on rational relations, similar to the reduction of systems of transfer functions.

The set of (total) rational relations Q forms a seminearring. However, since every controllable rational relation is identical in form to its transfer function expressed in reduced form (i.e. with coprime numerator and denominator), it follows from Theorem 2 that that any controllable rational relation obtained from addition, composition, and converse of other rational relations may be computed by treating them as transfer functions and reducing the final transfer function. Thus, the algebra of rational relations is practically the same as the algebra of transfer functions, but rational relations model the free response.

IX. Acknowledgements

The author wishes to acknowledge the helpful comments of the reviewers, particularly for drawing the author's attention to related ideas in [4] and [6].

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