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Loop Gain Adjustment for Second Order Sliding Modes

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Abstract— **In this paper, a method of adjustment of the loop gain for second order sliding modes is proposed. This approach is based on the equivalent gain k_n concept. Selection of the loop gain is realized via introduction of a compensator which is designed with the use of the frequency domain methods. The proposed approach involves the describing function method. An example of design, simulations and experimental results are presented.**

Keywords: **Sliding-Modes, Frequency domain analysis.**

I. INTRODUCTION

Frequency domain analysis of Second Order Sliding Modes (SOSM) [1], [2], [3] gives a new incite on the performance and design methods of control systems that utilize these principles. In particular, this approach allows for the introduction of such methods of system performance enhancement as the chattering attenuation [4] and nonideal disturbance rejection [5]. These two problems are a result of the presence of parasitic dynamics in the system and are analyzed with the use of such frequency domain methods as the Describing Function method [6] and the Locus of a Perturbed Relay System (LPRS) [7].

Nonideal disturbance rejection is analyzed for the averaged motion with the use of the concept of the equivalent gain k_n. This problem is solved through increasing k_n [5], which is done via introduction of a compensator.

In this paper, a method of adjustment of the loop gain for second order sliding modes is proposed. This approach is based on the equivalent gain k_n concept. Selection of loop gain is realized via introduction of a compensator which is designed with the use of the frequency domain methods. The proposed approach involves the describing function method. An example of design, simulations and experimental results are presented.

The structure of this paper is as follows. Section II contains the problem statement. Section III presents the Describing Function analysis of the Twisting and Sub-Optimal algorithms. Section IV contains the design method for the equivalent gain: in a general form and for each of the considered algorithms. In Section V, an example of design is presented, with the comparison between the theoretical results, simulations and experimental results.

II. PROBLEM STATEMENT

Consider the control loop as in Fig. 1, where $W(s)$ comprises the principal and parasitic dynamics of the plant, actuator and measurement sensor. The Second Order Sliding Mode is the controller and D is an external disturbance.

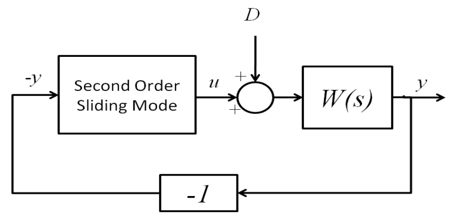

Fig. 1. General Diagram

Due to the presence of parasitic dynamics in the system controlled by the Second Order Sliding Mode algorithm, which is revealed through the chattering [2] and [3], the performance of the averaged dynamics is deteriorated causing nonideal disturbance rejection [5].

The concept of the equivalent gain is applied to the steady state analysis of systems controlled by the Second Order Sliding Modes with disturbance and parasitic dynamics present. Similar to [2] and [8], for averaged motion analysis the SOSM is replaced with equivalent gain, k_n, and the control loop is analyzed as a linear loop. With this methodology applied, the final value theorem can be used for the steady state analysis. Also, the non-ideal disturbance rejection can be mitigated via the increase of the equivalent gain.

Let the serially connected linear compensator W_c be aimed at the increase of the equivalent gain. Replacement of the SOSM algorithm with the equivalent gain and the introduction of W_c in the control scheme Fig. 1 leads to the following diagram.

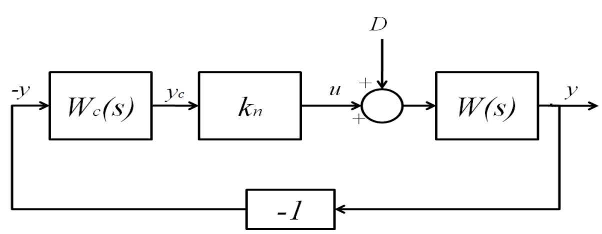

Fig. 2. General Diagram Linearized

Applying the final value theorem to the system Fig. 2 we can find the output value in the steady state via the following formula:

$$y_{ss} = \frac{D}{k_n + 1} \tag{1}$$

from which one can see that the increase of the equivalent gain will lead to the decrease of the error in the steady state.

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The proposed compensator W_c, has the transfer function as in [5]:

$$W_c(s) = \frac{\frac{s}{\delta\omega_0} + 1}{\frac{s}{\omega_0} + 1} \tag{2}$$

where $\delta > 0$ and ω_0 is the characteristic frequency (reciprocal of the time constant).

The main goal of this work is to develop a way of increasing the equivalent gain through a linear compensator design W_c, with the objective of partial disturbance rejection in SOSM systems.

For the design of compensator W_c and computing of the equivalent gain, the frequency domain methods are used.

III. DESCRIBING FUNCTION FOR HOSM

This section contains some results on the describing function analysis of the Twisting and Sub-optimal algorithms from [2] and [3], which apply to the stated purpose of the present analysis.

A. Twisting algorithm and its Describing Function

The twisting algorithm is defined as

$$u = -c_1 sign(y) - c_2 sign(\dot{y}), \tag{3}$$

where $c_1 > c_2 > 0$ are parameters of the algorithm.

Considering (3), we can present the system with the Twisting algorithm as the following diagram, where it is assumed that $D = 0$.

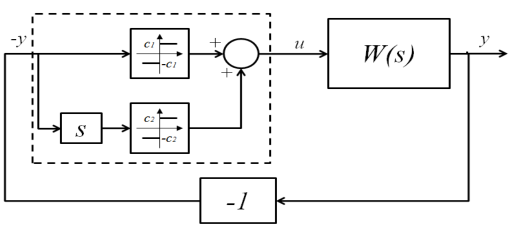

Fig. 3. Diagram of twisting algorithm.

The describing function for the Twisting algorithm can be derived (considering Fig. 3) as follows [1]:

$$N = N_1 + sN_2 \approx \frac{4c_1}{\pi a_1} + j\Omega\frac{4c_2}{\pi a_2} = \frac{4}{\pi a_1}(c_1 + jc_2) \tag{4}$$

where a_1 is the amplitude of the input to the nonlinearity (amplitude of output y), $a_2 = \Omega a_1$ and Ω is the frequency of $y(t)$.

Now, to obtain the periodic solution in the system $y(t)$ (or chattering parameters) the harmonic balance equation must be considered and solved:

$$W(j\omega) = -\frac{1}{N(a_1)}, \tag{5}$$

where $W(j\omega)$ is the frequency response (Nyquist plot) of the plant. Then considering (4), one can rewrite the harmonic balance equation as follows:

$$\frac{1}{N(a_1)} = \frac{\pi a_1}{4}\frac{c_1 - jc_2}{(c_1^2 + c_2^2)} \tag{6}$$

Therefore, both the amplitude a_1 and the frequency Ω of chattering can be found from the point of intersection of the Nyquist plot of the plant and the negative reciprocal describing function of the Twisting algorithm (as in Fig. 4).

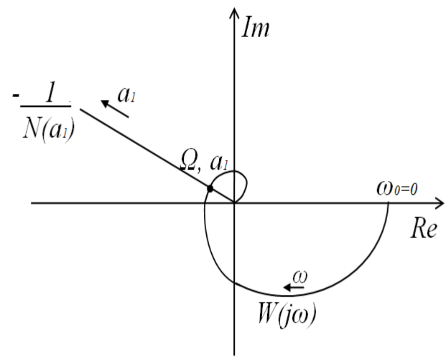

Fig. 4. DF analysis of Twisting

Once we have obtained the parameters of chattering , we can now compute the equivalent gains of the relays (with respect to the propagation of constant or averaged components of the motion) using the following formula:

$$\begin{aligned} k_{nTw1} &= \frac{2c_1}{\pi a_1} \\ k_{nTw2} &= \frac{2c_2}{\pi a_1 \Omega} \end{aligned} \tag{7}$$

The model of the system, in which the relays are replaced with equivalent gains, describes the dynamics with respect to the averaged motions in the system. It will be used below for analysis of propagation of constant (or slow varying) control inputs and disturbances through the system.

B. Sub-Optimal algorithm and its Describing Function

The Sub-optimal algorithm is defined as follows:

$$u = -c \cdot sign(y - \beta y_{Mi}) \tag{8}$$

where c (control magnitude) and β (anticipation parameter) are the controller parameters and y_{Mi} is the latest "singular point" of y, i.e., the value of y at the most recent time instant t_{Mi} $(i = 1, 2, ...)$ where $\dot{y}(t) = 0$. This algorithm can be presented as a relay with variable hysteresis - as shown in Fig. 5.

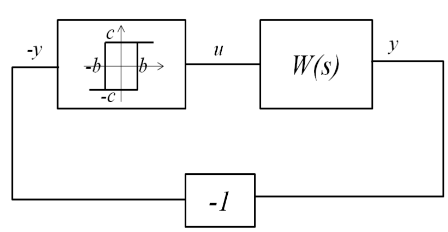

Fig. 5. Diagram of Sub-Optimal algorithm

However, the describing function for Sub-Optimal is the describing function of a relay with hysteresis the value of

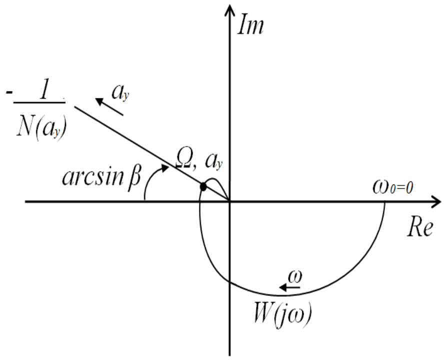

Fig. 6. DF analysis of Sub-Optimal

which is actually unknown. But assuming that during a periodic motion, the extreme values of the output coincide, in magnitude, with the amplitude of the oscillation, we can obtain the describing function of the algorithm as follows:

$$N(a_y) = \frac{4c}{\pi a_y}\sqrt{1-\frac{b^2}{a_y^2}} + j\frac{4bc}{\pi a_y^2} \tag{9}$$

where $b = \beta y_{mi}$ and $a_y = y_M$. After some manipulations (9) can be rewritten as follows:

$$N(a_y) = \frac{4c}{\pi a_y}\left(\sqrt{1-\beta^2} + j\beta\right) \tag{10}$$

Like in the case of the Twisting algorithm, the parameters of chattering in the system with the Sub-optimal algorithm can be found from the harmonic balance equation $W(j\omega)N(a_1) = -1$, which can be rewritten as follows (see also Fig. 6):

$$-\frac{1}{N(a_y)} = -\frac{\pi y_M}{4c}\left(\sqrt{1-\beta^2} - j\beta\right) \tag{11}$$

The obtained equation can be easily solved for the frequency of chattering, after which the amplitude a_y is obtained as follows:

$$a_y = \frac{4cW(j\Omega)}{\pi} \tag{12}$$

The equivalent gain for the Sub-Optimal algorithm k_{nsub} is given by the following formula:

$$\begin{aligned} k_n^* &= \frac{2c}{\pi a_y\sqrt{1-\beta^2}} \\ k_{nsub} &= k_n^*\,(1+\beta) \end{aligned} \tag{13}$$

The replacement of the algorithm with the equivalent gain provides one with the model of propagation of constant or slow varying control inputs and disturbances through the system.

IV. Way of computing the equivalent gain

The main idea was to find an expression that relates the equivalent gain value with the compensator, W_c, particularly with the design parameters, δ and ω_0. Then with help of the harmonic balance equation we obtained the next equation

$$W_c(j\omega)W(j\omega) = -\frac{1}{N(a)} \tag{14}$$

where transfer function of W_c is presented above. Now considering that the negative reciprocal of the describing function for SOSM has real part and imaginary part, equation (14) can be written as

$$\begin{aligned} &[ReW_cReW - ImW_cImW(j\omega)] + \\ &\quad j\,[ReW_cImW + ImW_cReW] \\ &\qquad = -Re\frac{1}{N(a)} - jIm\frac{1}{N(a)} \end{aligned} \tag{15}$$

ReW and ImW are real part and imaginary part of system(plant+parasitic dynamics), respectively. For W_c (2) real part and imaginary part are given as

$$\begin{aligned} ReW_c &= \frac{\frac{\omega^2}{\delta\omega_0^2}+1}{\frac{\omega^2}{\omega_0^2}+1} \\ ImW_c &= \frac{\frac{\omega}{\delta\omega_0}-\frac{\omega}{\omega_0}}{\frac{\omega^2}{\omega_0^2}+1} \end{aligned}$$

the characteristic frequency ω_0 (reciprocal of the time constant) of the compensator is selected from the following consideration: the mid-frequency of the compensator $\omega_{mid} = \sqrt{\delta\omega_0^2}$ must coincide with the mid-frequency of the range between the maximal frequency of the plant input and the frequency of chattering:

$$\begin{aligned} \omega_{mid} = \sqrt{\delta\omega_0^2} &= \sqrt{\omega_{max}\Omega} \\ \omega_0 &= \sqrt{\frac{\omega_{max}\Omega}{\delta}} \end{aligned} \tag{16}$$

where ω_{max} is the maximal frequency of the system bandwidth and Ω is the frequency of chattering.

Therefore, equating the real and imaginary parts of (15) and considering that $N(a)$ can be written as $N(k_n)$, we obtain the following two equations

$$\frac{ReW\Omega_0^2 + \varphi^2 ReW - ImW\Omega_0\varphi\frac{1}{x} + ImW\varphi x}{x^2\Omega_0^2+\varphi^2} = -Re\frac{1}{N(k_n)} \tag{17}$$

$$\frac{ImW\Omega_0^2 + \varphi^2 ImW + ReW\Omega_0\varphi\frac{1}{x} - ReW\varphi x}{x^2\Omega_0^2+\varphi^2} = -Im\frac{1}{N(k_n)} \tag{18}$$

where $ReW = ReW(j\Omega_0)$, $ImW = ImW(j\Omega_0)$, $x = \sqrt{\delta}$ and $\varphi = \sqrt{\omega_{max}\Omega}$. Then, replacing right part of (17) and (18) by the negative reciprocal of the describing function of a SOSM, there is a set of equations which can be solved for finding the two variables desired, δ and k_n.

Two SOSM, Twisting and Sub-optimal, are considered in this work. The equations corresponding to this algorithms are shown in following subsection.

A. Adjustment of equivalent gain for Twisting

Considering the right-hand part of (4) and the equivalent gain (7) the negative reciprocal of the describing function for Twisting as a function of k_{nTw} is:

$$-\frac{1}{N(k_{nTw})} = \frac{2c_1^2}{4k_{nTw}\left(c_1^2+c_2^2\right)} - j\frac{2c_1c_2}{4k_{nTw}\left(c_1^2+c_2^2\right)} \quad (19)$$

Then replacing (19) in (17) and (18) we can solve the set of equations as shown below. The value of k_{nTw} is obtained from (17).

$$k_{nTw} = \frac{2\Omega_0^2 c_1^2 x^2 + 2\varphi^2 c_1^2}{4\left(c_1^2+c_2^2\right)\left[ReW\Omega_0^2 + \varphi^2 ReW - ImW\Omega_0\varphi\frac{1}{x} + ImW\Omega\varphi x\right]} \quad (20)$$

Via replacing k_{nTw} in (18), δ(or x^2) is obtained as

$$\begin{aligned}&\left[\frac{c_2}{c_1}ImW\Omega_0 c_1\varphi - ReW\Omega_0\varphi\right]x^2\\ &+\left[ImW\Omega_0^2 + ImW\varphi^2 + \frac{c_2}{c_1}ReW\Omega_0^2 + \frac{c_2}{c_1}ReW\varphi^2\right]x\\ &+\left[ReW\varphi - \frac{c_2}{c_1}ImW\Omega_0\varphi\right] = 0 \quad (21)\end{aligned}$$

In summary, the adjustment of the equivalent gain can be done via the following steps:

1) With the steady state value (1) we compute the equivalent gain k_{nTw} which corresponds to the desired steady state.
2) Select some initial value for frequency Ω_0 (will become mid frequency between $\Omega - 1$decade and Ω) and compute ReW and ImW
3) Find the value of δ through (21) and the values of step 2, and replace them in k_{nTw}
4) Compare the value of k_{nTw} obtained with the value desired at step 1, if k_{nTw} not is satisfactory, return to step 2 to find the desired k_{nTw}.

B. Adjustment of equivalent gain for Suboptimal

For the case of Sub-optimal the developed approach is similar to the case of the Twisting algorithm and the iterations are similar; the only difference is the use of respective equations for k_{nsub} and δ.

Considering (11), (13) and (17), write the expression for k_{nsub}

$$k_{nsub} = -\frac{1}{2}\frac{(1+\beta)\,\Omega_0^2 x^2 + (1+\beta)\,\varphi^2}{ReW\Omega_0^2 + \varphi^2 ReW - ImW\Omega_0\varphi\frac{1}{x} + ImW\Omega_0\varphi x} \quad (22)$$

and the expression for calculating δ is

$$\begin{aligned}&\left[ReW\Omega_0\varphi - BImW\Omega_0\varphi\right]x^2\\ &-\left[ImW\Omega_0^2 + ImW\varphi^2 + BReW\Omega_0^2 + BReW\varphi^2\right]x\\ &+\left[BImW\Omega_0\varphi - ReW\Omega_0\varphi\right] = 0 \quad (23)\end{aligned}$$

where $B = \frac{\beta}{\sqrt{1-\beta^2}}$.

As one can see, for the Sub-optimal algorithm k_{nsub} does not depend on parameter c. Therefore, we can conclude that c does not "participate" in the disturbance rejection. The only parameter that has an effect on that is β. Unlike in the Sub-optimal algorithm, in the Twisting algorithm the two parameters, c_1 and c_2, have effect on the disturbance rejection.

In the following section we present an example of the equivalent gain adjustment.

V. Test Example

Consider the following mass-spring-damper system of Education Control Products ECP, see Fig. 7

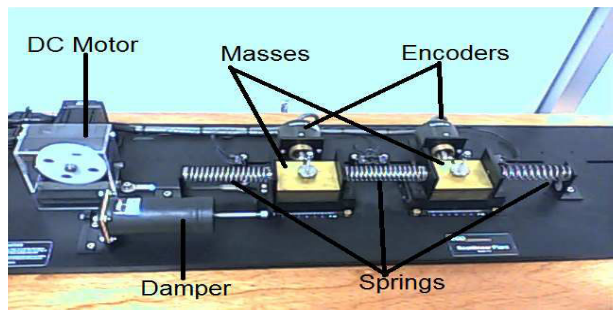

Fig. 7. Mass-Spring-Damper system, ECP

The model of system of Fig. 7 is shown below

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\frac{k_1+k_2}{m_1}x_1 - \frac{c_1}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{1}{m_1}u_a\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= \frac{k_2}{m_2}x_1 - \frac{k_2+k_3}{m_2}x_3\\ y &= x_3\end{aligned} \quad (24)$$

The model is divided in two parts, with the objective of having the principal and parasitic dynamics, x_1 and x_2 are the dynamics of actuator(parasitic dynamics), u_a is the actuator input generated by the SOSM. The input of the plant is x_1 and, x_3 and x_4 are the position and velocity of mass, respectively, then the output y is the position of the mass x_3. The maximum displacement of y is ± 3 (cm).

The values of the parameters of system ECP are, considering Fig.7 from left to right: damper coefficient $c_1 = 15$, spring constant $k_1 = 800[\frac{N}{m}]$, mass $m_1 = 1.28[kg]$, spring constant $k_2 = 800[\frac{N}{m}]$, mass $m_2 = 1.05[kg]$, and spring constant $k_3 = 450[\frac{N}{m}]$.

The objective of control is to keep the position of the mass y in the equilibrium point despite the presence of disturbance using the Twisting algorithm and Sub-optimal algorithm. The equilibrium point is the position $y = 0$ (m).

The model of mass-spring-damper (24) can be obtained as the next block diagram which is the control diagram of the test, where u_a is the actuator input, u is actuator output and control input of the plant.

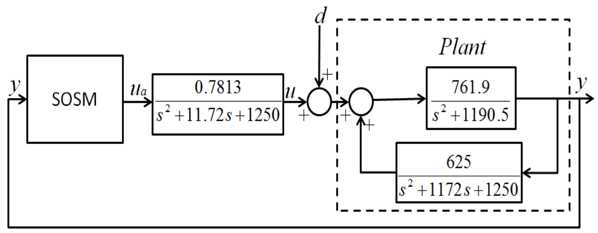

Fig. 8. Block Diagram with mass-spring-damper system

Then, computed the equivalent gain k_n of SOSM, the steady state error of the output y for the disturbance d is

$$y_{ss} = \frac{952380 \cdot d}{1011900 + 595.2381 \cdot k_n} \tag{25}$$

So that increasing the equivalent gain k_n of SOSM the effect of the disturbance d in the plant can be attenuated.

The SOSM controllers used in the test are

$$u_{Tw} = -8 \cdot sign(y) - 6 \cdot sign(\dot{y}) \tag{26}$$
$$u_{sub} = -10 \cdot sign(y - 0.5y_{Mi}) \tag{27}$$

The maximal frequency of the system input is $w_{max} = 50.1(rad/sec)$, the amplitude of the oscillations, the frequency of oscillations and the equivalent gain, for both u_{Tw} and u_{sub}, are shown in Fig. 9 and Fig. 10, respectively.

Let $D = 1(cm)$ as the disturbance in the system. The steady state values using (25) of the output of system are:

$$y_{ssTw} = -8.3(mm)$$
$$y_{ssSub} = -7.2(mm)$$

where y_{ssTw} is the steady state value for system controlled by Twisting and y_{ssSub} is the steady state value for Sub-optimal.

Now, applying the proposed algorithm to the computing of the equivalent gain, we start with Step 1. The desired values are $y_{ssTw} = -7.1(mm)$, 15 percent down of the uncompensated system and $y_{ssSub} = -5.7(mm)$, 20 percent down of the uncompensated system. Therefore, the equivalent gain value required for the Twsiting and Sub-optimal algorithms are $k_{nTw} \approx 550$ and $k_{nsub} \approx 1100$.

Step 2: the frequencies selected for finding ReW and ImW are as follows: for Twisting $\Omega_{Tw} = 31(rad/sec)$ and for Sub-optimal $\Omega_{Sub} = 32(rad/sec)$. Step 3: the values obtained in Step 2 give the following result: $\delta_{Tw} = 7.6830$ for Twisting and $\delta_{Sub} = 4.6253$ for Sub-optimal. The corresponding compensators are

$$W_{c_{Tw}}(s) = \frac{0.007989s + 1}{0.06138s + 1} \tag{28}$$
$$W_{c_{Sub}}(s) = \frac{0.01043s + 1}{0.472s + 1} \tag{29}$$

Step 4: the Fig. 9 and Fig. 10 show the equivalent gains and oscillations parameters obtained with Wc_{Tw} and Wc_{sub}, respectively.

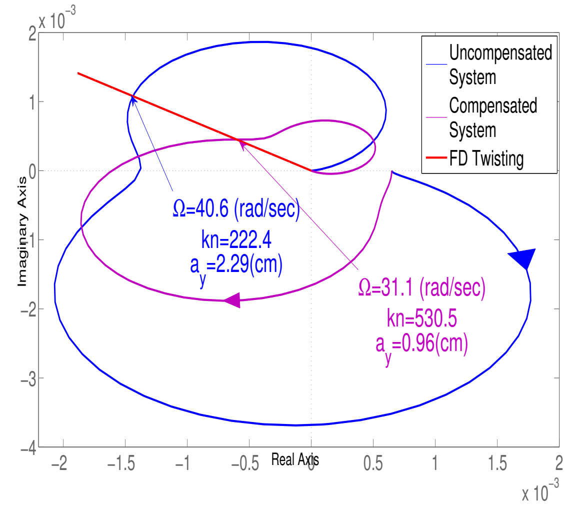

Fig. 9. DF analysis for uncompensated and compensated system controlled for Twisting

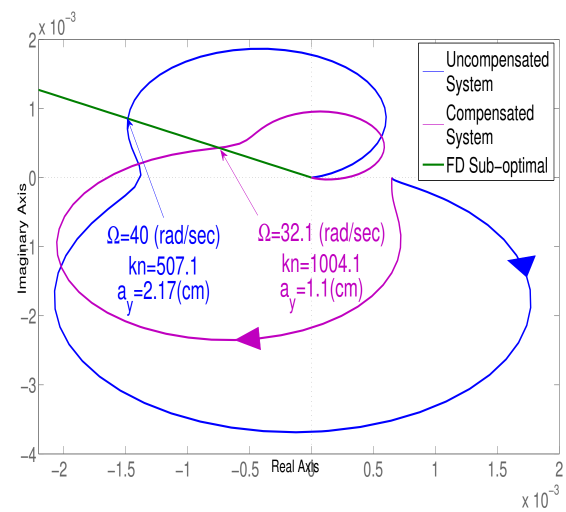

Fig. 10. DF analysis for uncompensated and compensated system controlled for Sub-optimal

In tables I and II the values of Fig. 9 and Fig. 10, respectively, are compared with the desired values of steps 1 and 2.

It can be seen from the tables that the result of adjustment of k_{nTw} and k_{nSub} has a good approximation, because the desired values are similar to the computed values. The percentage of steady state error desired for each algorithm is the percentage most big reach with the proposed algorithms.

The results of the simulations of steady oscillatory modes(chattering) of the system controlled by Twisting and Sub-optimal are in the table III.

A. Exprimental results

The disturbance constant is applied inclining the system, see Fig. 11, then with the disturbance about the system, the experiment is the next: the first 5 seconds the system without

TABLE I

Comparison Table of Adjustment of k_{nTw}

Values	$y_{ssTw}(mm)$	k_{nTw}	$\Omega_{Tw}(rad/sec)$
Desired	-7.1	550	31
Calculated (FD)	-7.2	530.5	31.1

TABLE II

Comparison Table of Adjustment of k_{nsub}

Values	$y_{ssSub}(mm)$	k_{nSub}	$\Omega_{Sub}(rad/sec)$
Desired	-5.7	1100	32
Calculated	-5.9	1040.6	32.1

compensator is present, finally the compensator is activated in the interval from 5 to 10 seconds. The experimental results are present for both algorithms in the Fig. 12.

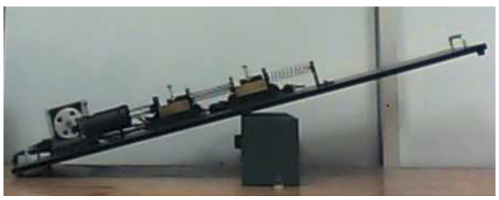

Fig. 11. Disturbed system

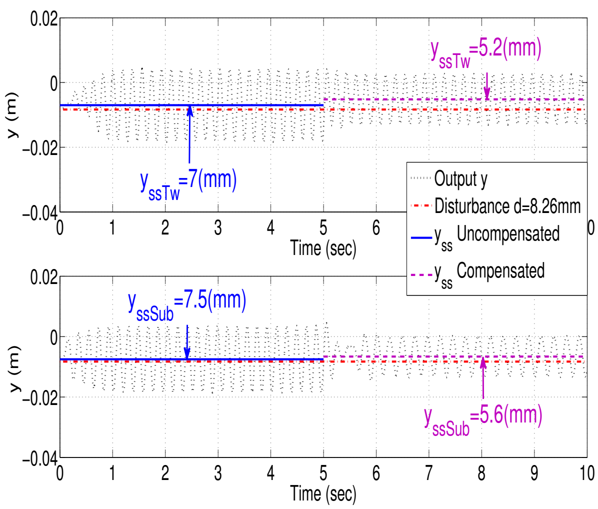

Fig. 12. Experimental results of steady state for Twisting (Above) and Sub-optimal (below)

As can be seen, the experimental results show the effect of compensator about the steady state of the system, there are difference between the theoretical results, obtained with DF, with the simulation and experimental results but the values are close.

VI. Conclusions

The design of the equivalent gain for SOSM controlled systems aimed at the adjustment of the loop gain for disturbance rejection is proposed. This is done with the use of a compensator. The way proposed allow to choose the

TABLE III

Comparison Table of Adjustment

Values	$y_{ssTw}(mm)$	$y_{ssSub}(mm)$
Simulation	-5.4	-7
Experiment	-5.2	-6.6

chattering frequency and steady state error considering the operation desired of the system.

The provided example of design and simulations show a good approximation between the theoretically predicted performance, the performance assessed via simulations and the experimental results.

This work has a logical direction of development: the approximate method of analysis used in the present work can be replaced with the exact LPRS (Locus of a perturbed relay systems) method; and the same methodology can be extended to other second-order sliding mode control algorithms.

VII. Acknowledgments

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