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Tuning MPC for Desired Closed-Loop Performance for MIMO Systems

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Abstract— **Model Predictive Control (MPC) is widely used in the process industries. It is an optimization-based approach, where several tuning parameters such as the penalty matrices in the cost function and the prediction and control horizons must be chosen. Tuning such parameters can be challenging as they are related to the closed-loop performance in a complex manner. This problem becomes even more complicated when tuning MPC controllers for MIMO systems. This paper addresses this problem and presents a systematic approach to determine MPC tuning parameters for MIMO systems based on a specification of the desired behaviour of the loop for small changes where the MPC controller acts as a linear controller. In this manner, the robustness of the closed loop to model mismatch can be taken into account systematically using results from robust linear control theory. The approach involves solving sequentially two semidefinite programming problems (convex optimization problems), one of which is formulated in the frequency-domain. A key feature of our approach is that the tuning parameters are determined such that in the unconstrained case they guarantee a nominal robust closed-loop performance. The approach is tested on two process control examples.**

I. INTRODUCTION

Model Predictive Control (MPC) is one of the most popular advanced process control method and used frequently in the process industries. It is a model-based control technique in which a (usually quadratic) cost function that is computed from the predicted future plant outputs and the future control moves is minimized at every time step. From the vector of control movements obtained as a solution to the above-mentioned optimization problem, only the first control input is applied to the plant and the optimization process is repeated at the next time instant. In addition to its optimizing nature, what makes MPC attractive is that:

(i) it can be easily applied to multivariable (MIMO) processes, and
(ii) input/output constraints as well as other external constraints can be easily incorporated into the overall optimization problem.

Hence, in comparison to the classical linear controllers (P, PI or PID), MPC has found wide acceptance in varying applications.

Since MPC is an optimization-based approach, in order to successfully implement an MPC several tuning parameters must be appropriately set. These include the prediction horizon, the control horizon and the penalty matrices used in the cost function. These tuning parameters are related to the process behaviour in a very complex manner and it is not straightforward to determine them. This poses a challenge and experience is needed to solve real-world problems successfully [1], [2]. Several papers have been published which discuss MPC tuning for closed-loop stability and robustness to uncertainties, however usually in a qualitative manner. With reference to the MIMO systems, the problem of MPC tuning becomes even more complicated as a compromise between speed of response, decoupling of the loops and robustness must be found.

A commonly followed procedure is to fix all the MPC tuning parameters except one, for example one could fix the prediction and the control horizons as well as the penalty matrix for the setpoint error at the beginning, leaving the penalty matrix for the change in the control movements as the only available degree of freedom. Then by varying this degree of freedom, an analysis can be performed which relates the closed-loop performance to this degree of freedom. From this analysis, tuning guidelines can be derived, see [3]-[8]. Alternatively, optimization can be applied to determine this degree of freedom, see [9]-[15]. The difficulty of determining an appropriate set of tuning parameters can be partly resolved by considering a linear quadratic regulator (LQR) design problem [16] which is equivalent to an infinite-horizon MPC. For the constrained case, Scokaert et al. [17] discuss a constrained LQR problem which leads to a finite-horizon MPC.

Another approach to the tuning of multivariable MPC controllers is to decouple the MIMO system using an external decoupler [18] and then design MPC's for the resulting individual SISO loops [19], [20] or to use the sequential loop closure method [21]. A more rigorous approach to the decoupling design using an MPC would be to modify the standard quadratic cost function. See papers by Middleton et al. [22] and Chai et al. [23].

Despite some progress made so far, there is still a need for a more systematic approach to perform MPC tuning. MPC tuning is usually performed ad-hoc based on some experience. Most of the time, several simulation runs are performed to check if the chosen tuning parameters are suitable. Linear control theory is well-established and mature. Many important results on the robust stability and performance of linear control loops have been derived. For linear plants, MPC control algorithms are linear controllers as long as the constraints are not active. This implies that in the unconstrained case MPC tuning parameters can be chosen such that a specified desired closed-loop transfer function with guaranteed performance and robustness properties is achieved. None of the above-mentioned references related MPC tuning to the closed-loop transfer function or frequency

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response. To our knowledge, only Trierweiler et al. [24] provide a formal relationship between the tuning parameters and the desired closed-loop performance, but it is mainly based on the robust performance number (RPN) which is a qualitative indicator of the ease of achieving a desired performance robustly.

In [25], we proposed a new MPC tuning approach for unconstrained SISO systems. The novelty of our approach is that we develop a formal relationship between the tuning parameters and the desired closed-loop performance. As a result of this, it is possible to choose the prediction horizon, the control horizon and the penalty weights simultaneously by solving optimization problems. The desired closed-loop performance may result from the desired step-responses or e.g. from an $H_\infty-$optimal design which guarantees robustness properties.

In this paper, we extend the above approach to the case of unconstrained MIMO systems. First, we develop a closed-loop formulation of MIMO MPC the properties of which then become easy to analyze. We equate the closed-loop representation with the desired closed-loop transfer function. After some algebraic postprocessing, two optimization problems result, one of which is solved in the frequency-domain. Solving these optimization problems sequentially directly provides the values of the penalty weights in the cost function which is the main contribution of this paper. The approach is systematic and easy to implement. To our knowledge, there exists so far no such approach that systematically determines the tuning parameters for a desired closed-loop performance and robustness of an MPC controller. The optimization problems are semidefinite programming (SDP) problems which can be solved using convex optimization.

This paper is organized as follows. We start with a preliminary discussion of MPC and discuss its closed-loop formulation in section II. In section III, we present our tuning approach and derive optimization problems for determining the weights of the quadratic performance criterion. We test our approach on some challenging examples in section IV. Finally, we conclude with some outlook on future work in section V.

II. PRELIMINARIES

A. Notations

We distinguish between a scalar variable and a vector (or a matrix) by using an underlined variable representing the latter. Let $\mathcal{S}^n$ denote a set of symmetric matrices of size $n \times n$. An underlined $\succeq 0$ indicates positive semidefiniteness of a matrix. $\mathbb{N}$ is a set of natural numbers excluding zero. $\mathbb{R}$ and $\mathbb{C}$ denote real and complex vector spaces respectively. A discrete polynomial $P(z^{-1})$ is defined as a map $f : \mathbb{C} \rightarrow \mathbb{C}$ and denoted as $P(z^{-1}) = p_0 + p_1 z^{-1} + \ldots + p_n z^{-n}$, $p_i \in \mathbb{R}$, $z \in \mathbb{C}$, $n \in \mathbb{N}$. z^{-1} is a backward shift operator such that $(1 - z^{-1})y(k) = \Delta(z^{-1})y(k) = y(k) - y(k-1)$ where Δ is the difference operator and $1/\Delta$ is a summation operator. $dim(\underline{M})$ represents the size of a matrix $\underline{M}$.

The symbols os and t_r denote the overshoot and the rise-time of a step-response.

B. Closed-Loop Formulation

A linear discrete-time plant is represented as:

$$\underline{y}(t+1) = \underline{A}^{-1}(z^{-1})\underline{B}(z^{-1})\underline{u}(t) = \bar{\underline{B}}(z^{-1})\bar{\underline{A}}^{-1}(z^{-1})\underline{u}(t), \tag{1}$$

where $\underline{A}$ and $\bar{\underline{A}}$ are diagonal matrices of polynomials obtained using the least common multiple of the denominators of the corresponding row (or column) of the overall MIMO plant transfer function. Here, $dim(\underline{A}) = n \times n$, $dim(\underline{B}) = dim(\bar{\underline{B}}) = n \times m$, $dim(\bar{\underline{A}}) = m \times m$ where $n, m \in \mathbb{R}$ are the no. of outputs and inputs respectively.

Generalized Predictive Control (GPC) [26] is a popular form of MPC. It uses a CARIMA (Controlled Auto-Regressive and Integrated Moving Average) prediction model:

$$\underline{A}(z^{-1})\underline{y}(t+k) = \underline{B}(z^{-1})\underline{u}(t+k-1) + \frac{\underline{T}(z^{-1})}{\Delta(z^{-1})}\underline{e}(t+k),$$
$$k = 1 \ldots N_p, \tag{2}$$

where $\underline{T}(z^{-1})$ is a matrix of noise filter polynomials, $\underline{e}(t)$ is a vector of white noise, N_p is the prediction horizon and $dim(\underline{T}) = n \times n$. For the sake of simplicity, we shall denote a matrix $\underline{M}(z^{-1})$ by only $\underline{M}$ henceforth. Multiplying Eq. (2) with $\bar{\underline{E}}_k\Delta$, it can be written as [2]:

$$\bar{\underline{T}}_k\underline{y}(t+k) = \bar{\underline{E}}_k\underline{B}\Delta\underline{u}(t+k-1) + \bar{\underline{F}}_k\underline{y}(t) + \bar{\underline{T}}_k\underline{E}_k\underline{e}(t+k), \tag{3}$$

where $\bar{\underline{T}}_k, \bar{\underline{E}}_k, \underline{E}_k, \bar{\underline{F}}_k$ are obtained from the Diophantine equations:

$$\underline{T} = \underline{E}_k\underline{A}\Delta + z^{-k}\underline{F}_k, \quad \bar{\underline{T}}_k = \bar{\underline{E}}_k\underline{A}\Delta + z^{-k}\bar{\underline{F}}_k,$$

where $\underline{T}\underline{E}_k^{-1} = \bar{\underline{E}}_k^{-1}\bar{\underline{T}}_k$. Assuming the future noise to be zero, that is $\underline{e}(t+k) = 0$, the prediction of $\underline{y}(t+k)$ at time step $t+k-1$ is given by

$$\hat{\underline{y}}(t+k) = \underline{J}_k\bar{\underline{E}}_k\underline{B}\Delta\underline{u}(t+k-1) + (\underline{J}_k\bar{\underline{F}}_k + \underline{L}_k)\underline{y}(t), \tag{4}$$

where

$$\underline{I} = \underline{J}_k\bar{\underline{T}}_k + z^{-k}\underline{L}_k.$$

Using the Diophantine equation:

$$\underline{J}_k\bar{\underline{E}}_k\underline{B} = \underline{G}_k\underline{I} + z^{-k}\underline{H}_k,$$

Eq. (4) can be written as:

$$\begin{aligned}\hat{\underline{y}}(t+k) = &\underbrace{\underline{G}_k\Delta\underline{u}(t+k-1)}_{\textit{free response}} + \\ &\underbrace{\underline{H}_k\Delta\underline{u}(t-1) + (\underline{J}_k\bar{\underline{F}}_k + \underline{L}_k)\underline{y}(t)}_{\textit{forced response}}.\end{aligned} \tag{5}$$

<u>Cost Function</u>: For the unconstrained case, the quadratic cost function:

$$\begin{aligned}J = &\sum_{k=N_1}^{k=N_2}\left[[\hat{\underline{y}}(t+k) - \underline{w}(t+k)]\underline{Q}^{1/2}\right]^2 + \\ &\sum_{k=1}^{k=N_u}\left[\Delta\underline{u}(t+k-1)\underline{\Lambda}^{1/2}\right]^2, \\ &\underline{Q} \in \mathbb{S}^{n(N_2-N_1+1)} \succ 0, \quad \underline{\Lambda} \in \mathbb{S}^{mN_u} \succeq 0,\end{aligned} \tag{6}$$

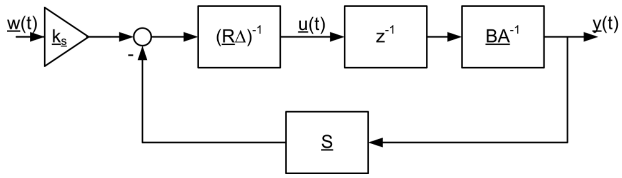


Fig. 1. MPC analysis as a closed-loop control scheme.

is minimized where N_1, N_2 denote the lower and upper prediction horizons, N_u is the control horizon, $\underline{w}(t+k)$ is a vector of future setpoints, $\underline{Q}$ and $\underline{\Lambda}$ are the penalty matrices which must be tuned. Minimizing the above cost function gives the optimal control input as:

$$\Delta\underline{u}(t+k-1) = \underbrace{(\underline{G}^T\underline{Q}\underline{G}+\underline{\Lambda})^{-1}\underline{G}^T\underline{Q}}_{\underline{K}}\cdot \qquad (7)$$
$$\left[\underline{w}(t+k) - \underline{H}_k\Delta\underline{u}(t-1) - (\underline{J}_k\underline{\bar{F}}_k + \underline{L}_k)\underline{y}(t)\right],$$

where $\underline{G}$ is a matrix containing the coefficients of the elements of $\underline{G}_k,\ k = N_1 \dots N_2$ and $\underline{G} \in \mathbb{R}^{n(N_2-N_1+1)\times m}$. For a more transparent representation, let's consider a 2×2 MIMO system. Assuming $\underline{w}(t+k) = \underline{w}(t) = \underline{w}$, the first optimal control input is given by

$$\begin{bmatrix} \Delta u_1(t) \\ \Delta u_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} k_{s,11} & k_{s,12} \\ k_{s,21} & k_{s,22} \end{bmatrix}}_{\underline{k_s}} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} -$$
$$\underbrace{\underbrace{\begin{bmatrix} k_{11} & \cdots & k_{1(2N_p')} \\ k_{21} & \cdots & k_{2(2N_p')} \end{bmatrix}}_{\underline{k}_0} \begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \\ \vdots \\ \underline{H}_{N_p'} \end{bmatrix}}_{\underline{R}_1(z^{-1})} \begin{bmatrix} \Delta u_1(t-1) \\ \Delta u_2(t-1) \end{bmatrix} -$$
$$\underbrace{\begin{bmatrix} k_{11} & \cdots & k_{1(2N_p')} \\ k_{21} & \cdots & k_{2(2N_p')} \end{bmatrix} \begin{bmatrix} \underline{J}_1\underline{\bar{F}}_1 + \underline{L}_1 \\ \underline{J}_2\underline{\bar{F}}_2 + \underline{L}_2 \\ \vdots \\ \underline{J}_{N_p'}\underline{\bar{F}}_{N_p'} + \underline{L}_{N_p'} \end{bmatrix}}_{\underline{S}(z^{-1})} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \qquad (8)$$

where $N_p' = N_2 - N_1 + 1,\ \ \underline{k}_0 \in \underline{K}(1\dots 2, 1\dots 2N_p')$,
$$k_{s,11} = k_{11} + k_{13} + \cdots + k_{1(2N_p'-1)},$$

and similarly the other elements of $\underline{k_s}$ can be determined as linear combinations of elements of $\underline{k}_0$. It is straightforward to generalize Eq. (8) to a $n\times n$ MIMO system. After rearranging the terms, a closed-loop form of MPC results as shown below and represented by Fig. (1):

$$\underline{R}(z^{-1})\Delta\underline{u}(t) = \underline{k_s}\,\underline{w} - \underline{S}(z^{-1})\underline{y}(t), \quad \underline{R} = \underline{I} + \underline{R}_1. \qquad (9)$$

C. Stability Analysis

From Eqns. (1) and (9),

$$\underline{R}\Delta\underline{u}(t) = \underline{k_s}\,\underline{w} - z^{-1}\underline{S}\underline{\bar{B}}\underline{\bar{A}}^{-1}\underline{u}(t)$$
$$\Leftrightarrow (\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})\underline{\bar{A}}^{-1}\underline{u}(t) = \underline{k_s}\,\underline{w}.$$

This implies

$$\underline{u}(t) = \underline{\bar{A}}(\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})^{-1}\underline{k_s}\,\underline{w}, \qquad (10)$$
$$\text{and } \underline{y}(t) = z^{-1}\underbrace{\underline{\bar{B}}(\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})^{-1}\underline{k_s}}_{\underline{G}_{cl,true}(z^{-1})}\,\underline{w}, \qquad (11)$$

where $\underline{G}_{cl,true}(z^{-1})$ is the true resulting closed-loop transfer function. From the above equation, the term $(\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})^{-1}$ determines the closed-loop stability. Substituting for $\underline{R}$ and $\underline{S}$ gives

$$(\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})$$
$$= \left[\underline{I} + z^{-1}\begin{bmatrix} k_{11} & \cdots & k_{1(2N_p')} \\ k_{21} & \cdots & k_{2(2N_p')} \end{bmatrix}\begin{bmatrix} \underline{H}_1 \\ \underline{H}_2 \\ \vdots \\ \underline{H}_{N_p'} \end{bmatrix}\right]\underline{\bar{A}}\Delta +$$
$$z^{-1}\begin{bmatrix} k_{11} & \cdots & k_{1(2N_p')} \\ k_{21} & \cdots & k_{2(2N_p')} \end{bmatrix}\begin{bmatrix} \underline{J}_1\underline{\bar{F}}_1 + \underline{L}_1 \\ \underline{J}_2\underline{\bar{F}}_2 + \underline{L}_2 \\ \vdots \\ \underline{J}_{N_p'}\underline{\bar{F}}_{N_p'} + \underline{L}_{N_p'} \end{bmatrix}\underline{\bar{B}}.$$

After subsituting for $\underline{H}_{1\dots N_p'}$, $\underline{\bar{F}}_{1\dots N_p'}$ and $\underline{L}_{1\dots N_p'}$, this finally leads to

$$(\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}}) = \underline{\bar{A}}\Delta +$$
$$z^{-1}\underbrace{\begin{bmatrix} k_{11} & \cdots & k_{1(2N_p')} \\ k_{21} & \cdots & k_{2(2N_p')} \end{bmatrix}}_{\textit{degrees of freedom } (\underline{k}_0)}\begin{bmatrix} z\underline{\bar{B}} - z\underline{G}_1\underline{\bar{A}}\Delta \\ z^2\underline{\bar{B}} - z^2\underline{G}_2\underline{\bar{A}}\Delta \\ \vdots \\ z^{N_p'}\underline{\bar{B}} - z^{N_p'}\underline{G}_{N_p'}\underline{\bar{A}}\Delta \end{bmatrix}. \qquad (12)$$

A properly chosen gain matrix $\underline{k}_0$ leads to a stable matrix $(\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})^{-1}$. As a simple approach one could assume a diagonal matrix $(\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})$ and use the method in [25] to determine $\underline{k}_0$ in the above equation in order to obtain a stable closed-loop system. This however does not provide the desired closed-loop performance due to the presence of the coupled terms in Eq. (11).

III. Tuning Approach

The tuning problem is tackled in two steps:

(i) determine the required $\underline{k}_0$ values in Eq. (12) for obtaining the desired closed-loop performance,

(ii) determine the MPC tuning parameters (i.e. $\underline{Q}, \underline{\Lambda}$) which correspond to these $\underline{k}_0$ values.

A. Determining the $\underline{k}_0$ Values

Let $G_{cl,desired}(z^{-1})$ be the desired closed-loop transfer function. Its diagonal elements define the desired tracking performance of the main loops while the off-diagonal elements can be specified such that they have small gains at all frequencies. It is desired to have $\underline{G}_{cl,true} \approx \underline{G}_{cl,desired}$.

Let $\underline{G}_{cl,desired2} = (\underline{R}\underline{\bar{A}}\Delta + z^{-1}\underline{S}\underline{\bar{B}})^{-1}\underline{k_s}$. Substituting in Eq. (11) and comparing the resulting equation with $\underline{G}_{cl,desired}$, we obtain

$$\underline{G}_{cl,desired2} = \underline{\bar{B}}^{-1}\underline{G}_{cl,desired}. \qquad (13)$$

Since $\underline{G}_{cl,desired2} = (\underline{R}\bar{\underline{A}}\Delta + z^{-1}\underline{S}\bar{\underline{B}})^{-1}\underline{k_s}$, we have

$$(\underline{R}\bar{\underline{A}}\Delta + z^{-1}\underline{S}\bar{\underline{B}}) \cdot \underline{G}_{cl,desired2} = \underline{k_s}. \tag{14}$$

This implies that we have in the frequency-domain

$$\left[\begin{array}{c} \underline{R}(\omega)\bar{\underline{A}}(\omega)\Delta(\omega)+ \\ e^{-j\omega T_s}\underline{S}(\omega)\bar{\underline{B}}(\omega) \end{array} \right] \cdot \underline{G}_{cl,des.2}(\omega) - \underline{k_s} = \underline{0}, \tag{15}$$

where $\underline{R}(\omega) = \underline{R}(e^{-j\omega T_s})$ and T_s is the sampling time of the discrete system. Equivalently, Eq. (15) can be formulated as

$$\left| \left[\begin{array}{c} \underline{R}(\omega)\bar{\underline{A}}(\omega)\Delta(\omega)+ \\ e^{-j\omega T_s}\underline{S}(\omega)\bar{\underline{B}}(\omega) \end{array} \right] \cdot \underline{G}_{cl,des.2}(\omega) - \underline{k_s} \right| = \underline{0}. \tag{16}$$

Eq. (16) is relaxed into an inequality constraint as below:

$$\left| \left[\begin{array}{c} \underline{R}(\omega)\bar{\underline{A}}(\omega)\Delta(\omega)+ \\ e^{-j\omega T_s}\underline{S}(\omega)\bar{\underline{B}}(\omega) \end{array} \right] \cdot \underline{G}_{cl,des.2}(\omega) - \underline{k_s} \right| \le \underline{\epsilon}_k^{\frac{1}{2}}, \tag{17}$$

where our goal is to minimize $\|\underline{\epsilon}_k\|_2$. Discretizing the frequency range and comparing the elements on both sides of Eq. (17), we have

$$[Re_{ij\omega}]^2 + [Im_{ij\omega}]^2 \le \epsilon_{ij\omega}, \ \ \epsilon_{ij\omega} \ge 0, \ \ \epsilon_{ij\omega} \in \underline{\epsilon}_k,$$

where $Re_{ij\omega}$ refers to the real part of an element (i,j) of the left-hand side of Eq. (15) at the frequency $\omega\ rad/s$ while $Im_{ij\omega}$ refers to the imaginary part of the same. The above inequality can be transformed into a linear matrix inequality (LMI) as follows:

$$\left[\begin{array}{ccc} 1 & Re_{ij\omega} & 0 \\ Re_{ij\omega} & \epsilon_{ij\omega} & Im_{ij\omega} \\ 0 & Im_{ij\omega} & 1 \end{array} \right] \succeq 0. \tag{18}$$

So the optimization problem for determining $\underline{k}_0$ results as:

$$\begin{array}{l} min_{\,\underline{k}_0,\underline{\epsilon}_k} \quad \underline{\epsilon}_k^T\underline{\epsilon}_k \quad \text{subject to:} \\ Ineq.\ (18)|_{\omega=\omega_l}, \ \ \forall \omega_l \in [0,\omega_1], \ \omega_1 < \dfrac{\pi}{T_s}. \end{array} \tag{19}$$

Eq. (19) is a SDP problem and hence can be solved using a convex optimization method. A solution to the optimization problem with small $\underline{\epsilon}_k$ values implies that the obtained $\underline{k}_0$ values achieve $\underline{G}_{cl,desired}$. This values can then be used to determine the penalty matrices $\underline{Q}$, $\underline{\Lambda}$ in the cost function.

Analysis of the optimization problem: Eq. (17) can be written as a minimization problem as follows:

$$min_{\underline{k}_0} \left\| \left[\begin{array}{c} \underline{R}(\omega)\bar{\underline{A}}(\omega)\Delta(\omega)+ \\ e^{-j\omega T_s}\underline{S}(\omega)\bar{\underline{B}}(\omega) \end{array} \right] \cdot \underline{G}_{cl,des.2}(\omega) - \underline{k_s} \right\|_2 .$$

Substituting Eq. (13), we get

$$min_{\underline{k}_0} \left\| \left[\begin{array}{c} \underline{R}(\omega)\bar{\underline{A}}(\omega)\Delta(\omega)+ \\ e^{-j\omega T_s}\underline{S}(\omega)\bar{\underline{B}}(\omega) \end{array} \right] \bar{\underline{B}}^{-1}(\omega)\underline{G}_{cl,des.}(\omega) - \underline{k_s} \right\|_2 .$$

Using Eq. (11), we get

$$min_{\underline{k}_0} \left\| \underline{k_s}\underline{G}_{cl,true}^{-1}(\omega)\underline{G}_{cl,desired}(\omega) - \underline{k_s} \right\|_2 .$$

Simplifying further, we have

$$min_{\underline{k}_0} \left\| \underline{k_s}\underline{G}_{cl,true}^{-1}(\omega)\left[\underline{G}_{cl,desired}(\omega) - \underline{G}_{cl,true}(\omega)\right] \right\|_2$$

$$\Leftrightarrow min_{\underline{k}_0}\ ||\underline{k_s}\underline{G}_{cl,true}^{-1} \underbrace{\left[\underline{G}_{cl,desired} - \underline{G}_{cl,true}\right]}_{\text{error term}} ||_2. \tag{20}$$

If $\underline{G}_{cl,true} \approx \underline{G}_{cl,desired}$, in solving the minimization problem (19), the error term is minimized weighted by $\underline{k_s}\underline{G}_{cl,true}^{-1}$. So if the frequency range is $\omega_1 \approx 3\omega_B$, where ω_B is the largest bandwidth of all the diagonal elements of $\underline{G}_{cl,desired}$, this weighting emphasizes the region around the open-loop gain crossover frequency.

B. Determining the Penalty Matrices $\underline{Q}$, $\underline{\Lambda}$

This step is identical to that described in [25]. For simplicity, we assume $N_u = 1$. For a given $\underline{k}_0 \in \underline{K}$, we have

$$\begin{array}{l} (\underline{G}^T\underline{Q}\underline{G} + \underline{\Lambda})^{-1}\underline{G}^T\underline{Q} = \underline{k}_0 \\ \Rightarrow \underline{G}^T\underline{Q} = (\underline{G}^T\underline{Q}\underline{G} + \underline{\Lambda})\underline{k}_0 \\ \Rightarrow \underline{G}^T\underline{Q} - (\underline{G}^T\underline{Q}\underline{G} + \underline{\Lambda})\underline{k}_0 = \underline{0}. \end{array} \tag{21}$$

The above equation results in a set of linear equality constraints in the coefficients of $\underline{Q}$ and $\underline{\Lambda}$:

$$\begin{array}{l} f_i(q_{xy},\lambda_{vw}) = 0, \ \ i = 1\ldots(m\cdot n\cdot(N_2 - N_1 + 1)), \\ q_{xy} \in \underline{Q}, \ \lambda_{vw} \in \underline{\Lambda}, \end{array}$$

which can be relaxed into inequality constraints:

$$-\epsilon_i \le f_i(q_{xy},\lambda_{vw}) \le \epsilon_i. \tag{22}$$

We formulate the optimization problem for determining $\underline{Q}$, $\underline{\Lambda}$ as:

$$\begin{array}{l} min_{\underline{Q},\,\underline{\Lambda},\,\underline{\epsilon}_{k0}} \quad \underline{\epsilon}_{k0}^T\underline{\epsilon}_{k0} \quad \text{subject to:} \\ -\epsilon_i \le f_i(q_{xy},\lambda_{vw}) \le \epsilon_i, \\ \forall\ i = 1\ldots m\cdot n\cdot(N_2 - N_1 + 1), \\ \underline{Q} \succ 0, \ \underline{\Lambda} \succeq 0, \\ q_{xy} \in \underline{Q}, \ \lambda_{vw} \in \underline{\Lambda}, \ \epsilon_i \in \underline{\epsilon}_{k0}. \end{array} \tag{23}$$

Eq. (23) is a SDP problem and hence is a convex optimization problem. A solution with small $\underline{\epsilon}_{k0}$ values implies that the obtained matrices $\underline{Q}$, $\underline{\Lambda}$ satisfy Eq. (21) which in turn results in achieving $\underline{G}_{cl,desired}$. If small $\underline{\epsilon}_{k0}$ values do not result, we can increase the number of degrees of freedom by increasing the term $N_2 - N_1 + 1$ and repeat the optimization.

It is possible to obtain diagonal or semi-diagonal $\underline{Q}$, $\underline{\Lambda}$ by augmenting Eq. (23) with additional linear constraints.

IV. EXAMPLES

We formulate our optimization problems in YALMIP [27], a MATLAB-based toolbox and we use the SeDuMi solver [1], which can be accessed from YALMIP, to solve the convex optimization problems. In some cases, the discrete-time plant models were obtained from their continuous transfer functions using the MATLAB-based System Identification Toolbox (V7.3). The optimization problems were solved on an Intel Core2 Duo CPU with 1.96 GB RAM. We would like to point out that different feasible solutions may result on different machines and with different optimization solvers.

[1] freely available at http://sedumi.ie.lehigh.edu/

A. A Simplified Distillation Column

We consider a simplified distillation column model [28, p. 100] which is a good example of an ill-conditioned statically coupled plant. The example has been proposed as a case study in linear robust control in [29]. The continuous plant is discretized with a sampling time of $T_s = 1\ s$ and the discrete-time plant is obtained as below:

$$(1 - 0.9868z^{-1})\underline{I}\,\underline{y}(t) = z^{-1}\begin{bmatrix} 1.1629 & -1.1444 \\ 1.4331 & -1.4516 \end{bmatrix}\underline{u}(t).$$

The following closed-loop behaviour, (t_r, os), is specified:

$$w_1 \to y_1 : (10\ s, 10\%),\ w_2 \to y_2 : (10\ s, 20\%).$$

In order to achieve minimal coupling between different loops, the behaviour of channels $w_1 \to y_2$ and $w_2 \to y_1$ can be specified considering a high-pass filter having relatively low gain in the low-frequency region. These correspond to the following closed-loop transfer matrix ($\underline{G}_{cl,desired}$):

$$\begin{bmatrix} \frac{0.0334(1+0.8978z^{-1})}{1-1.661z^{-1}+0.724z^{-2}} & \frac{0.001(1-0.98z^{-1})}{1-0.3333z^{-1}} \\ \frac{0.001(1-0.98z^{-1})}{1-0.3333z^{-1}} & \frac{0.0245(1+0.9325z^{-1})}{1-1.764z^{-1}+0.811z^{-2}} \end{bmatrix}.$$

We choose the prediction and the control horizons as $N_1 = 1$, $N_2 = 6$ and $N_u = 1$ respectively. The frequency range $\omega = [0, 0.4]\ rad/s$ with 30 grid-points spaced at $0.0138\ rad/s$ was chosen for solving the optimization problem (19).

Solving the optimzation problem (19), we obtained:

$$\underline{k}_0 = \left[\begin{matrix} 0.0782 & -0.1178 & 0.2114 & -0.1993 & 0.2529 & -0.1739 & \cdots \\ 0.0765 & -0.1204 & 0.2095 & -0.2016 & 0.2497 & -0.1766 & \cdots \end{matrix}\right.$$
$$\left.\begin{matrix} 0.3272 & -0.1910 & 0.3827 & -0.1809 & 0.4374 & -0.1704 \\ 0.3232 & -0.1941 & 0.3778 & -0.1839 & 0.4316 & -0.1732 \end{matrix}\right].$$

Solving the optimization problem (23) gives the required values of $\underline{Q}$, $\underline{\Lambda}$. Since, $dim(\underline{Q}) = 12 \times 12$, it is not possible to present it here. $\underline{\Lambda}$ results as:

$$\underline{\Lambda} = \begin{bmatrix} 323.8076 & -319.2207 \\ -319.2207 & 314.7139 \end{bmatrix}.$$

In Fig. (2) the desired closed-loop performance and the obtained closed-loop performance after solving the optimization problem (23) are compared. Observe that only the magnitude plots are shown for the off-diagonal elements. Fig. (3) shows the closed-loop responses with the MPC controller to step changes in w_1 and w_2. We have decoupled plant responses meeting the desired closed-loop specifications.

B. A Glass-Tube Manufacturing Process

We consider a glass-tube manufacturing process discussed in [30]. The model is discretized with a sampling time of $T_s = 10\ s$. The discrete-time plant matrices are given as below:

$$\begin{aligned} A(1,1) =& 1 - 2.426z^{-1} + 2.11z^{-2} - 0.7784z^{-3} + \\ & 0.1035z^{-4} + 0.001929z^{-5} - 0.0008307z^{-6}, \\ A(2,2) =& 1 - 2.522z^{-1} + 2.118z^{-2} - 0.5919z^{-3}, \\ B(1,1) =& -0.0279 + 0.0034z^{-1} + 0.02124z^{-2} - 0.002867z^{-3}, \\ B(1,2) =& 0.0044 - 0.004874z^{-1} + 0.0005004z^{-2} + \\ & 0.0005999z^{-3} - 9.312 \cdot 10^{-5}z^{-4} - 9.94 \cdot 10^{-6}z^{-5}, \\ B(2,1) =& -0.0239 + 0.03974z^{-1} - 0.01646z^{-2}, \\ B(2,2) =& -0.0031 + 0.002665z^{-1}. \end{aligned}$$

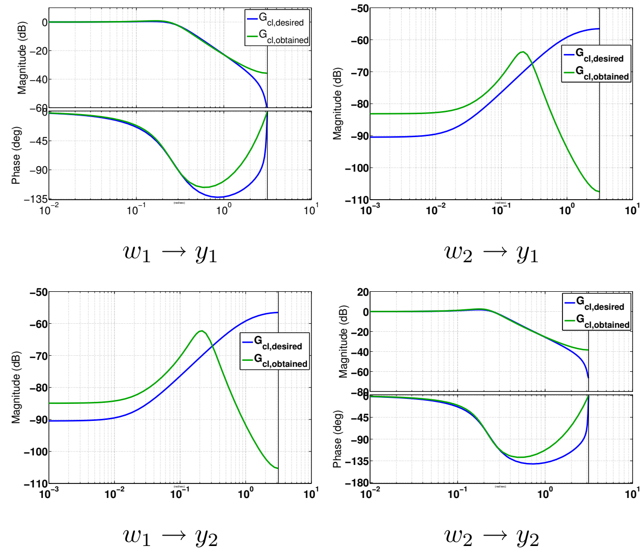


Fig. 2. Ex. (IV-A) - Bode plot of desired vs. obtained performances.

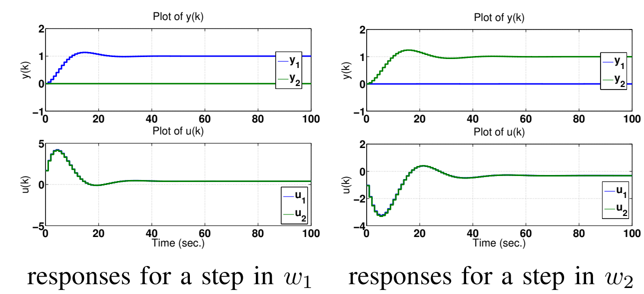


Fig. 3. Ex. (IV-A) - Plant responses to a step change in w_1 & w_2.

Owing to technical reasons, y_2 is measured with a time delay of $214\ s$. As a result of this and considering additional robustness margins, the second-loop is made relatively slower compared to the first loop. Hence, the following closed-loop behaviour, (t_r, os), is desired:

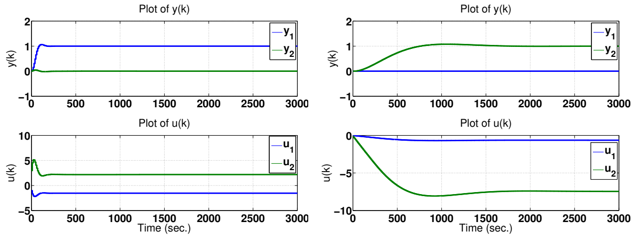
$$w_1 \to y_1 : (90\ s, 5\%),\ w_2 \to y_2 : (560\ s, 10\%).$$

The requirement for the minimal coupling between different loops can be specified similarly as in the previous example. These correspond to the following closed-loop transfer matrix ($\underline{G}_{cl,desired}$):

$$\begin{bmatrix} \frac{0.054178(1+0.8478z^{-1})}{(1-1.51z^{-1}+0.61z^{-2})} & \frac{0.01(1-0.98z^{-1})}{1-0.3333z^{-1}} \\ \frac{0.01(1-0.98z^{-1})}{1-0.3333z^{-1}} & \frac{z^{-21}0.002953(1+0.9697z^{-1})}{(1-1.906z^{-1}+0.9119z^{-2})} \end{bmatrix}.$$

Note the additional time delay in the loop $w_2 \to y_2$ whose desired transfer function has a rise time of $350\ s$. We choose the prediction and the control horizons as $N_1 = 1$, $N_2 = 10$ and $N_u = 1$. We choose the frequency range $\omega = [0, 0.03]\ rad/s$ with 30 grid-points spaced at $0.0010\ rad/s$ for solving the optimization problem (19). Since, $dim(\underline{Q}) = 20 \times 20$, it is not possible to present it here. $\underline{\Lambda}$ results as:

$$\underline{\Lambda} = \begin{bmatrix} 0.0789 & -0.0048 \\ -0.0048 & 0.0008 \end{bmatrix}.$$



responses for a step in w_1 responses for a step in w_2

Fig. 4. Ex. (IV-B) - Plant responses to a step change in w_1 & w_2.

Fig. (4) shows the closed-loop responses to step changes in w_1 and w_2. The desired closed-loop specifications are almost met. The Bode plots showing the comparison of the desired and the obtained closed-loop performances are not shown owing to the lack of space.

V. Conclusions and Future Work

We have presented a new systematic approach to determine MPC tuning parameters for unconstrained MIMO systems. The tuning approach is based upon the specification of the closed-loop performance in the frequency-domain. Then, two optimization problems are solved. First, the degrees of freedom, that is the feedback gain matrix $\underline{k}_0$, is determined using frequency response approximation. In the next step, we solve an another optimization problem to determine the penalty matrices $\underline{Q}$, $\underline{\Lambda}$ which correspond to the obtained matrix $\underline{k}_0$. Both the optimization problems are SDP problems which can be solved using convex optimization solvers.

So far, we have performed the tuning with $N_u = 1$. If we set $N_u > 1$, Eq. (23) becomes bilinear in the unknown coefficients of $\underline{Q}$, $\underline{\Lambda}$ and $\underline{K}$ and hence is no longer a SDP problem. The resulting bilinear problem can be solved using global optimization approaches which is currently under investigation. We intend to extend the approach to larger MIMO systems with some additional robust performance criteria, in particular bounds on the closed-loop transfer matrices that result from the necessary robustness to plant-model mismatch, an issue for which no systematic approach so far is available in the MPC literature.

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