2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Global Practical Stabilization for Integrator Chain with Actuator Saturation and Input Additive Disturbances

Haijun Fang

Abstract— **In this paper, we revisit the problem of disturbance rejection for an integrator chain system with actuator saturation and input additive disturbances. The goal is to design a nonlinear state feedback law to globally practically stabilize the closed-loop system, which means that any closed-loop system trajectory will converge to any arbitrarily small closed set where the origin is inside. The numerical examples will show the effectiveness of the proposed control design.**

I. Introduction and Preliminary Results

In this paper, we will revisit the problem of global practical stabilization for an integrator chain system with actuator saturation and input additive disturbances. The system we will consider in this paper is

$$\begin{aligned} \dot{x}_n &= x_{n-1} \\ \dot{x}_{n-1} &= x_{n-2} \\ &\vdots \\ \dot{x}_2 &= x_1 \\ \dot{x}_1 &= \sigma(u+\nu), \end{aligned}$$

where $u \in \mathbf{R}$ is the control input and ν is the disturbance whose magnitude is bounded, *i.e.*, $|\nu| < d$, $d > 0$. σ is the standard saturation function,

$$\sigma(u) = \text{sign(u)} \min(1, |u|). \tag{1}$$

Stabilization of the integrator chain system with actuator saturation had been a popular research topic. In [6], it had been proved that the global stabilization can not be achieved via linear state feedback law. Hence, in [4], a linear state feedback law is introduced by applying low gain design technique to achieve semi-global stabilization, *i.e.*, for any pre-defined arbitrarily large set, any system trajectory starting from this set will converge to the origin asymptotically. A nonlinear state feedback law has been explicitly constructed to achieve the global stabilization in [7], which means that any system trajectory will be driven to the origin as time goes to ∞.

As the problem of stabilization has been solved for the integrator chain system with actuator saturation, system robustness has been a naturally arisen problem for the integrator chain system with not only actuator saturation but also disturbances. In this paper, we will consider that there are input additive disturbances in the system and propose a nonlinear state feedback law to achieve the global practical stabilization, *i.e.*, any system trajectory will converge to a pre-defined arbitrarily small closed set where the origin resides inside in a finite time. Actually, this problem has been already addressed by some researchers. In [5], semi-global practical stabilization problem is solved for (1) through linear state feedback law. The same system is considered in [3] and global practical stabilization is achieved. However, the proposed control law needs to solve an state dependent Algebraic Riccati equation online and it will add complexity in the real system implementation. In our recent work [1] and [2], we propose a parameterized state feedback law for the system (1) with 2 and 3 dimensions respectively. For any large disturbance, all system trajectories will converge to a pre-defined arbitrarily small closed set with the origin inside through tuning the parameter. In this paper, we will extend the main result in [1] and [2] to the general case (1). The proposed control law in this paper is in the following format,

$$\begin{aligned} u(t) &= -\left(f_1x_1(t) + f_2x_2(t) + \sigma_{D_3}(f_3x_3(t) + \right. \\ &\quad \left. \cdots + \sigma_{D_n}(f_nx_n(t)))\right), \qquad (2) \\ f_1 &= \frac{\sqrt{2}}{2\epsilon}, f_2 = \frac{1}{\epsilon}\sqrt{\frac{\sqrt{2}\epsilon(1+d+D_3)+1}{2}}, \\ f_i &= \frac{1}{\epsilon}, D_i > 1, \epsilon \in (0,1), \end{aligned}$$

$\sigma_{D_i}(u), i \in [3, n]$ is defined as

$$\sigma_{D_i}(u) = \text{sign(u)} \min(D_i, |u|). \tag{3}$$

When $D_i = 1$, $\sigma_{D_i}(u)$ becomes the standard saturation function, $\sigma(u)$. Before showing the preliminary results in [1] and [2], we will show some notations first.

For a given positive definite matrix $P \in \mathbf{R}^{2\times2}$ and a positive scalar ρ, denote

$$\varepsilon(P, x_0, \rho) := \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : (x-x_0)^{\mathrm{T}} P (x-x_0) \leq \rho \right\},$$

x_1 and x_2 are states of (1) and $x_0 \in \mathbf{R}^2$ is a constant vector.

Let ϵ and $\bar{d}$ are positive numbers, define

$$\begin{aligned} \mathcal{L}(x_0, \bar{d}) &:= \left\{ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : \right. \\ &\quad \left. -\bar{d} - 1 \leq \begin{bmatrix} f_1 & f_2 \end{bmatrix} (x - x_0) \leq \bar{d} + 1 \right\}, \\ P(\epsilon, \bar{d}) &:= \begin{bmatrix} \frac{\sqrt{\sqrt{2}\epsilon(1+\bar{d})+1}}{\epsilon(1+\bar{d})} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \sqrt{\frac{\sqrt{2}\epsilon(1+\bar{d})+1}{2}} \end{bmatrix}, \\ L_i(a, b) &:= \{x_i : a \leq x_i \leq b, a < b, i \in [1, n]\}, \end{aligned}$$

x_i is the state of (1), $i \in [1, n]$. $\mathcal{I}$ and 0 denote respectively the two dimensional identity matrix and the two dimensional zero vector.

H. Fang is with MKS Instrument Inc., Rochester, NY 14586, USA haijunfang@ieee.org

978-1-4577-0079-8/11/$26.00 ©2011 AACC

Lemma 1: [1] Consider the system (1) with $n = 2$, the control law (2) becomes

$$u = -(f_1x_1 + f_2x_2).$$

Choose a ρ^* such that $\varepsilon(P(\epsilon, d), 0, \rho^*)$ is the smallest set with

$$\mathcal{L}(0,d) \bigcap \varepsilon(\mathcal{I}, 0, 2\epsilon^2(1+d)^2 d) \subset \mathcal{L}(0,d) \bigcap \varepsilon(P_{\epsilon,d}, 0, \rho^*).$$

Denote $\Omega(0, d, \epsilon) = \mathcal{L}(0,d) \bigcap \varepsilon(P(\epsilon,d), 0, \rho^*)$. Under the proposed state feedback law, any system trajectory will enter into $\Omega(0, d, \epsilon)$ in a finite time. Also, as ϵ approaches to 0, $\Omega(0, d, \epsilon)$ shrinks towards the origin.

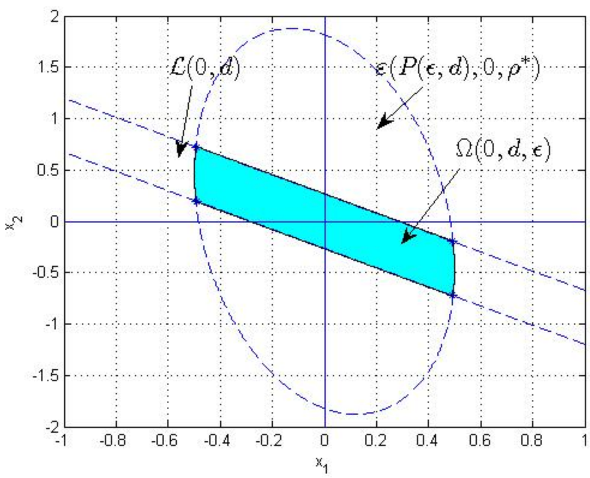

Fig. 1. $\varepsilon(P(\epsilon, d), 0, \rho^*), \mathcal{L}(0, d)$ and $\Omega(0, d, \epsilon)$

Lemma 2: [2] Consider the system of (1) with $n = 3$ and then the control law becomes

$$u = -(f_1x_1 + f_2x_2 + \sigma_{D_3}(f_3x_3)).$$

For any given d, there exists an $\epsilon_3^* \in (0, 1)$, $D_3^* > 0$ and $k_3^* > 1$ such that, for any $\epsilon \in (0, \epsilon_3^*)$ and $D_3 > D_3^*$, all trajectories of the closed-loop third order system (1) will enter χ_3 in a finite time and remain there thereafter, where

$$\chi_3 = \left\{ x \in \mathbf{R}^3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Omega(0, d + D_3, \epsilon), \right. \left. x_3 \in L_3(-k_3^* D_3 \epsilon, k_3^* D_3 \epsilon) \right\}.$$

Obviously, χ_3 will shrink to the origin as ϵ approaches to 0.

The remainder of the paper is organized as follows. Section II shows the global practical stabilization of the considered system. Section III will demonstrate the effectiveness of the proposed control law by numerical examples. The concluding remark will be drawn in Section IV.

II. Main Results

In this section, we will address the global practical stabilization of the system (1) under the control law (2). We will first consider the fourth order system of (1) and then extend the result to the general case by using the method of mathematical induction.

Theorem 1: Consider the system (1) with the control law (2), for any given d and any given arbitrarily small set $\chi_n \in \mathbf{R}^n$ containing the origin inside its interior, there exists an $\epsilon_n^* \in (0, 1)$ and $D_i^* > 0, i \in [3, n]$, such that, for any $\epsilon \in (0, \epsilon_n^*)$ and $D_i > D_i^*$, all trajectories of the closed-loop system (1) will enter χ_n in a finite time and remain there thereafter, and χ_n will shrink to the origin as t goes to ∞.

Proof: First, we will consider the case of $n = 4$. Then the system (1) becomes

$$\begin{aligned} \dot{x}_4 &= x_3 \\ \dot{x}_3 &= x_2 \\ \dot{x}_2 &= x_1 \\ \dot{x}_1 &= \sigma(u + \nu), \end{aligned} \quad (4)$$

and the control law (2) becomes

$$u = -\left(f_1x_1 + f_2x_2 + \sigma_{D_3}\left(f_3x_3 + \sigma_{D_4}(f_4x_4)\right)\right). \quad (5)$$

For the system (4), we will prove that there exists $k_3 > 1$ and $k_4^* > 1$ such that x_3 and x_4 of (4) will be bounded in $L_3(-k_3(D_3 + D_4)\epsilon, k_3(D_3 + D_4)\epsilon)$ and $L_4(-k_4^* D_4 \epsilon, k_4^* D_4 \epsilon)$. By Theorem 1 of [1], we know that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ will enter $\in \Omega(0, d + D_3, \epsilon)$ in a finite time, therefore we will assume that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is already inside $\Omega(0, d + D_3, \epsilon)$ in the following analysis. Since $|\sigma_{D_4}(f_4x_4)| < D_4$, if $x_3 > \frac{1}{f_3}(D_3 + D_4) = (D_3 + D_4)\epsilon$, then $\sigma_{D_3}(f_3x_3 + \sigma_{D_4}(f_4x_4)) = D_3$. By the proof of Theorem 1 in [2], we can prove that x_3 will finally return to $(D_3 + D_4)\epsilon$ in a finite time after it is over $(D_3 + D_4)\epsilon$. As $x_3 = (D_3 + D_4)\epsilon$ and $x_2 > 0$, then x_3 will deviate from $(D_3 + D_4)\epsilon$ toward to ∞. Denote the time x_3 starts to increase from $(D_3 + D_4)\epsilon$ as t_0. Once again, by following the proof of Theorem 1 in [2], there exists T_2 and δ_2, which are functions of D_3 and d and can be denoted as

$$T_2 = g_2(D_3, d) > 0, \quad \delta_2 = l_2(D_3, d) > 0,$$

such that if x_3 keeps above $(D_3 + D_4)\epsilon$ during the time $[t_0, t_0 + T_2]$, $x_2 < -\delta_2 \epsilon$ at the time $t > t_0 + T_2$. Since x_2 becomes negative when $t \in (t_0, t_0 + T_2)$, x_3 will start to decrease until it reaches to $(D_3 + D_4)\epsilon$ again. Since $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Omega(0, d + D_3, \epsilon)$, there exists $k_2 > 0$ such that $|x_2| < k_2\epsilon$. Because $\dot{x}_3 = x_2$, the maximal increase for x_3 starting from $(D_3 + D_4)\epsilon$ will be less than $T_2 k_2 \epsilon$. Symmetrically, we can conclude the same result if $x_3 < -(D_3 + D_4)\epsilon$. Select $k_3 > 1 + \frac{T_2 * k_2}{D_3 + D_4}$, then we can prove that $x_3 \in L_3(-k_3(D_3 + D_4)\epsilon, k_3(D_3 + D_4)\epsilon)$.

Next, we will show that $x_4 \in L_4(-k_4^* D_4\epsilon, k_4^* D_4\epsilon)$. If $x_4 > \frac{D_4}{f_4} = D_4\epsilon$, $\sigma_4(f_4 x_4) = D_4$. Hence, the control law (2) becomes

$$u = -\left(f_1x_1 + f_2x_2 + \sigma_{D_3}\left(f_3x_3 + D_4\right)\right). \quad (6)$$

By the state transformation

$$\begin{array}{rcl} \tilde{x}_1 &=& x_1 \\ \tilde{x}_2 &=& x_2 \\ \tilde{x}_3 &=& x_3 + \frac{D_4}{f_3}, \end{array} \quad (7)$$

the dynamic of $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ will be

$$\begin{array}{rcl} \dot{\tilde{x}}_3 &=& \tilde{x}_2 \\ \dot{\tilde{x}}_2 &=& \tilde{x}_1 \\ \dot{\tilde{x}}_1 &=& \sigma(-\left(f_1\tilde{x}_1 + f_2\tilde{x}_2 + \sigma_{D_3}\left(f_3\tilde{x}_3\right)\right) + \nu). \end{array} \quad (8)$$

By Theorem 1 of [2], there exists a $k_3^* > 1$ such that $\tilde{x}_3$ will enter into the set $[-k_3^* D_3\epsilon, k_3^* D_3\epsilon]$ in a finite time, which means that $x_3 \in [(-k_3^* D_3 - D_4)\epsilon, (k_3^* D_3 - D_4)\epsilon]$. So choose $D_4 > k_3^* D_3$, then x_3 will be negative if it is inside the set $L_3((-k_3^* D_3 - D_4)\epsilon, (k_3^* D_3 - D_4)\epsilon)$. Since $\dot{x}_4 = x_3$, it can be shown that, as x_3 becomes negative, x_4 will decrease until it reaches to $\frac{D_4}{f_4} = D_4\epsilon$. As $x_4 = D_4\epsilon$ and $x_3 > 0$, we know that x_4 will deviate from $D_4\epsilon$ toward to ∞ for a period of time. In this case, the system (4) can be transformed into (8). Since $D_4 > k_3^* D_3 > D_3$, then (8) becomes

$$\begin{array}{rcl} \dot{\tilde{x}}_3 &=& \tilde{x}_2 \\ \dot{\tilde{x}}_2 &=& \tilde{x}_1 \\ \dot{\tilde{x}}_1 &=& \sigma(-\left(f_1\tilde{x}_1 + f_2\tilde{x}_2 + D_3\right) + \nu), \end{array}$$

we already know that, withing the time period, T_2, $\tilde{x}_2 = x_2$ will become negative from being positive, and also $\tilde{x}_2 = x_2 < -\delta_2\epsilon$ if $\tilde{x}_3$ is kept positive, which means that $x_3 \geq (-D_4 + D_3)\epsilon$. In the above analysis, we have shown that $x_3 \in L_3(-k_3(D_3 + D_4)\epsilon, k_3(D_3 + D_4)\epsilon)$. Therefore, the maximal time for x_3 to be negative is

$$T_3 = T_2 + \frac{k_3(D_3 + D_4)}{\delta_2} = g_3(D_3, D_4, d) > 0.$$

T_3 is a constant if D_3, D_4 and d are chosen. Furthermore, the maximal deviation from $D_4\epsilon$ for x_4 can be estimated by $T_3k_3(D_3 + D_4)\epsilon$. Select $k_4^* > T_3k_3(D_3 + D_4)$, it can be proved that the maximal deviation is proportional to ϵ if D_4, D_3 and d are set. And this result holds if $x_4 < -D_4\epsilon$. Hence, x_4 will enter into the set $L_4(-k_4^* D_4\epsilon, k_4^* D_4\epsilon)$ in a finite time. So far, we have proved that, if d is given and D_3 and D_4 are correctly selected, any trajectory of (4) will enter into an arbitrarily small close set χ_4 with the origin inside by tuning the parameter ϵ, where

$$\begin{array}{rcl} \chi_4 &=& \left\{ x \in \mathbf{R}^3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Omega(0, d + D_3, \epsilon), \right. \\ && x_3 \in L_3(-k_3(D_3 + D_4)\epsilon, k_3(D_3 + D_4)\epsilon), \\ && \left. x_4 \in L_4(-k_4^* D_4\epsilon, k_4^* D_4\epsilon) \right\}. \end{array}$$

In the following, we will prove the main result for the general system (1).

We will use mathematical induction method. Therefore, we assume that for the following system

$$\begin{array}{rcl} \dot{x}_{m-1} &=& x_{m-2} \\ \dot{x}_{m-2} &=& x_{m-3} \\ &\vdots& \\ \dot{x}_2 &=& x_1 \\ \dot{x}_1 &=& \sigma(u + \nu), m > 4, \end{array}$$

with the control

$$\begin{array}{rcl} u &=& -\left(f_1x_1(t) + f_2x_2(t) + \sigma_{D_3}(f_3x_3(t) + \cdots \right. \\ && \left. + \sigma_{D_{m-1}}(f_{m-1}x_{m-1}(t)))\right), \end{array} \quad (9)$$

and

$$\begin{array}{rcl} \dot{x}_m &=& x_{m-1} \\ \dot{x}_{m-1} &=& x_{m-2} \\ &\vdots& \\ \dot{x}_2 &=& x_1 \\ \dot{x}_1 &=& \sigma(u + \nu), m > 4, \end{array} \quad (10)$$

with the control

$$\begin{array}{rcl} u &=& -\left(f_1x_1(t) + f_2x_2(t) + \sigma_{D_3}(f_3x_3(t) + \cdots \right. \\ && \left. + \sigma_{D_m}(f_mx_m(t)))\right), \end{array} \quad (11)$$

if d is given and $D_i, i \in [3, m-1]$, are correctly selected, then there exist $k_i > 1, k_i^* > 1, i \in [3, m-1]$ and $k_{m-2}^* > 1$ such that any system trajectory of (9) will enter into the set χ_{m-1},

$$\begin{array}{l} \chi_{m-1} = \left\{ x \in \mathbf{R}^{m-1} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Omega(0, d + D_3, \epsilon), \right. \\ x_i \in L_i(-k_i(D_i + D_{i+1})\epsilon, k_i(D_i + D_{i+1})\epsilon), \\ i \in [3, m-2], \\ \left. x_{m-1} \in L_{m-1}(-k_{m-1}^* D_{m-1}\epsilon, k_{m-1}^* D_{m-1}\epsilon) \right\}, \end{array}$$

and if $\sigma_{i+1}(f_{i+1}x_{i+1} + \cdots + \Sigma_{m-1}(f_{m-1}x_{m-1})) > D_{i+1}$ or $\sigma_{i+1}(f_{i+1}x_{i+1} + \cdots + \sigma_{m-1}(f_{m-1}x_{m-1})) < -D_{i+1}$, there exists a closed set in which $x_i \leq (-D_{i+1} + k_i^* D_i)\epsilon < 0$ or $x_i \geq (D_{i+1} - k_i^* D_i)\epsilon > 0$, and x_i will enter into the set in the time which is less than

$$T_i = g_i(D_3, D_4, \cdots, D_i, D_{i+1}, d) > 0, i \in [3, m-2].$$

Obviously, $T_i > T_{i-1}$. Moreover, if $D_m > k_{m-1}^* D_{m-1}$, there exists $k_i > 1, i \in [3, m-1]$, $k_m^* > 1$ and $k_m > 1$ such that any system trajectory of (10) will enter into the set χ_m,

$$\begin{array}{l} \chi_m = \left\{ x \in \mathbf{R}^m : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Omega(0, d + D_3, \epsilon), \right. \\ x_i \in L_i(-k_i(D_i + D_{i+1})\epsilon, k_i(D_i + D_{i+1})\epsilon), i \in [3, m-1], \\ \left. x_m \in L_m(-k_m^* D_m\epsilon, k_m^* D_m\epsilon) \right\}. \end{array}$$

If $x_m > D_m\epsilon$ or $x_m < -D_m\epsilon$, x_{m-1} will be less than $-D_m + k_{m-1}^* D_{m-1}$ from being positive or be greater than

$D_m - k^*_{m-1}D_{m-1}$ from being negative within the time, T_{m-1},

$$T_{m-1} = g_{m-1}(D_3, D_4, \cdots, D_{m-1}, D_m, d).$$

As ϵ becomes smaller, χ_{m-1} and χ_m will shrink to the origin. Then we will consider the $m+1$-dimensional system (1) and it is

$$\begin{aligned} \dot{x}_{m+1} &= x_m \\ \dot{x}_m &= x_{m-1} \\ &\vdots \\ \dot{x}_2 &= x_1 \\ \dot{x}_1 &= \sigma(u+\nu), \end{aligned} \quad (12)$$

and the control u is

$$\begin{aligned} u &= -\left(f_1x_1(t) + f_2x_2(t) + \sigma_{D_3}(f_3x_3(t) + \cdots \right. \\ &\quad \left. +\sigma_{D_{m+1}}(f_{m+1}x_{m+1}(t)))\right). \end{aligned} \quad (13)$$

In the next, we will prove that there exists a $k_m > 1$ such that x_m of (12) will enter into the set $L_m(-k_m(D_m + D_{m+1})\epsilon, k_m(D_m + D_{m+1})\epsilon)$ in a finite time. We will prove it for the case of $x_m > (D_m + D_{m+1})\epsilon$ and the case of $x_m < -(D_m + D_{m+1})\epsilon$ can be symmetrically proved. Since $|\sigma_{m+1}(f_{m+1}x_{m+1})| \leq D_{m+1}$, if $x_m > (D_m + D_{m+1})\epsilon$, then $\sigma_m(f_mx_m + \sigma_{m+1}(f_{m+1}x_{m+1})) = D_m$. By the state transformation

$$\begin{aligned} \tilde{x}_1 &= x_1 \\ \tilde{x}_2 &= x_2 \\ &\cdots \\ \tilde{x}_{m-1} &= x_{m-1} + D_m, \end{aligned} \quad (14)$$

the dynamic of $\tilde{x}_i, i \in [1, m-1]$ will become

$$\begin{aligned} \dot{\tilde{x}}_{m-1} &= \tilde{x}_{m-2} \\ \dot{\tilde{x}}_{m-2} &= \tilde{x}_{m-3} \\ &\cdots \\ \dot{\tilde{x}}_2 &= \tilde{x}_1 \\ \dot{\tilde{x}}_1 &= \sigma(u+\nu), \end{aligned}$$

and the control law is

$$\begin{aligned} u &= -\left(f_1\tilde{x}_1(t) + f_2\tilde{x}_2(t) + \sigma_{D_3}(f_3\tilde{x}_3(t) + \cdots \right. \\ &\quad \left. +\sigma_{D_{m-1}}(f_{m-1}\tilde{x}_{m-1}(t)))\right). \end{aligned}$$

By the assumption, we know that $\tilde{x}_{m-1}$ will enter into the set $[-k^*_{m-1}D_{m-1}\epsilon, k^*_{m-1}D_{m-1}\epsilon]$ in a finite time and remain there as long as $x_m > (D_{m-1} + D_m)\epsilon$. Since $D_m > k^*_{m-1}D_{m-1}$, x_{m-1} will be negative if $x_{m-1} \in L_{m-1}((-k^*_{m-1}D_{m-1} - D_m)\epsilon, (k^*_{m-1}D_{m-1} - D_m)\epsilon)$, which means that x_m will decrease until it reaches to $(D_m + D_{m+1})\epsilon$. As $x_m = (D_m + D_{m+1})\epsilon$ and $x_{m-1} > 0$, by the assumption, we have that the maximal time for x_{m-1} to be less than $(-D_m + k^*_{m-1}D_{m-1})\epsilon$ is T_{m-1}. And then the maximal deviation for x_m from $D_m + D_{m+1}$ will be $T_{m-1}k_{m-1}(D_{m-1} + D_m)\epsilon$. Let

$$k_m = 1 + \frac{T_{m-1}k_{m-1}(D_{m-1} + D_m)}{D_m + D_{m+1}},$$

then it can be proved that x_m will enter into the set $L_m(-k_m(D_m + D_{m+1})\epsilon, k_m(D_m + D_{m+1})\epsilon)$.

In the next, we will show that there exists a $k^*_{m+1} > 1$ such that x_{m+1} of (12) will enter into the set $L_{m+1}(-k^*_{m+1}D_{m+1}\epsilon, k^*_{m+1}D_{m+1}\epsilon)$ in a finite time and remain there thereafter. As $x_{m+1} > D_{m+1}\epsilon$, $\sigma_{m+1}(f_{m+1}x_{m+1}) = D_{m+1}$. Then by the assumption, x_m will enter into the set $L_m((-k^*_mD_m - D_{m+1})\epsilon, (k^*_mD_m - D_{m+1})\epsilon)$ as long as $x_{m+1} > D_{m+1}\epsilon$. Choose $D_{m+1} > k^*_mD_m$, then x_m will be negative if $x_m \in L_m((-k^*_mD_m - D_{m+1})\epsilon, (k^*_mD_m - D_{m+1})\epsilon)$. Since $\dot{x}_{m+1} = x_m$, x_{m+1} will decrease until it reaches to $D_{m+1}\epsilon$. If $x_{m+1} = D_{m+1}\epsilon$ and $x_m > 0$, x_{m+1} will increase from $D_{m+1}\epsilon$ and stop increasing until $x_m < 0$. we know that the maximal time for x_{m-1} entering into the the set where $x_{m-1} < (-D_m + k^*_{m-1}D_{m-1})\epsilon$ is T_{m-1}. Then the maximal time T_m for x_m being negative is

$$T_m = T_{m-1} + \frac{k^*_m(D_m + D_{m+1})}{D_m - D_{m-1}}.$$

Hence the maximal increase for x_m from $D_{m+1}\epsilon$ will be $k_m(D_m + D_{m+1})T_m\epsilon$. Choose $k^*_{m+1} > 1 + \frac{k^*_m(D_m+D_{m+1})T_m}{D_{m+1}}$, then x_{m+1} will enter into the set $L_{m+1}(-k^*_{m+1}D_{m+1}\epsilon, k^*_{m+1}D_{m+1}\epsilon)$. Also, as $D_i, i \in [3, m+1]$ are chosen and d is given, all time $T_i, i \in [3, m]$ will be fixed. So any trajectory will enter into the set χ_{m+1}

$$\begin{aligned} \chi_{m+1} = \Big\{ & x \in \mathbf{R}^{m+1} : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \Omega(0, d + D_3, \epsilon), \\ & x_i \in L_i(-k_i(D_i + D_{i+1})\epsilon, k_i(D_i + D_{i+1})\epsilon), \\ & i \in [3, m], \\ & x_{m+1} \in L_{m+1}(-k^*_{m+1}D_{m+1}\epsilon, k^*_{m+1}D_{m+1}\epsilon) \Big\}, \end{aligned}$$

and χ_{m+1} will shrink toward to the origin as ϵ is decrease to 0.

III. EXAMPLES

We consider the 5-th order system of (1). Let $D_5 = 10$, $D_4 = 5$ and $D_3 = 2$, and choose the disturbance, $d = 0.1 * sin(t)$. Set the initial condition as $\begin{bmatrix} 0.01 & 0.01 & 0.01 & 0.01 & 0.01 \end{bmatrix}^{\mathrm{T}}$. Fig.2-4 and Fig.3-5 show the simulation of the states $x_i, i \in [1, 5]$ and the control input with $\epsilon = 0.001$ and $\epsilon = 0.0005$ respectively. It can be seen that the states will driven into a smaller state space set as ϵ decreases. The first picture in Fig. 3 and Fig. 5 shows the general picture of the control input. The second picture in these figures shows the control input at the beginning time of the simulation. It can be seen that the control input is beyond the saturation limit and then decrease to be within the saturation limit very quickly and stay there forever.

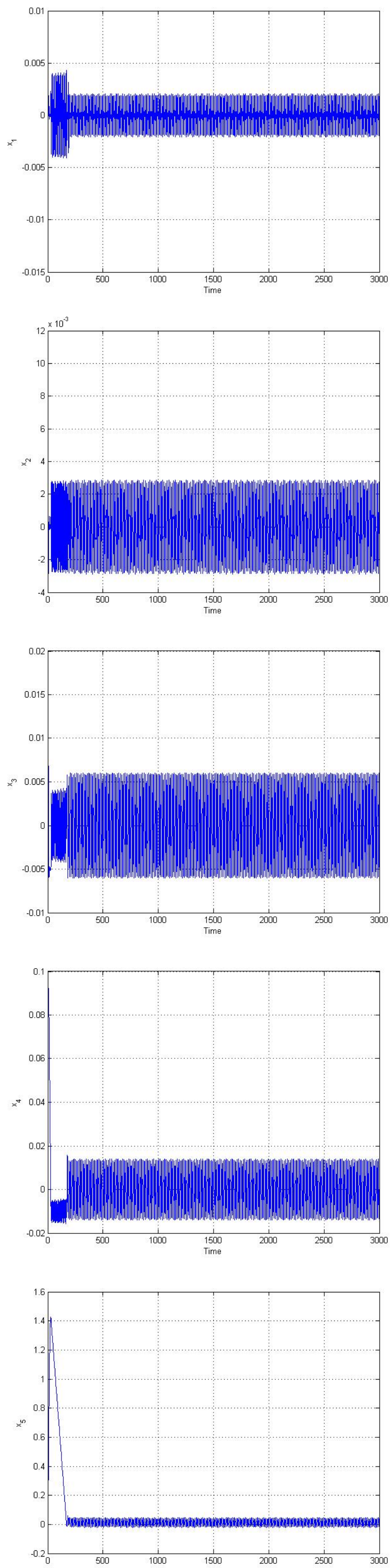

Fig. 2. the state $x_i, i \in [1, 5]$, $\epsilon = 0.001$

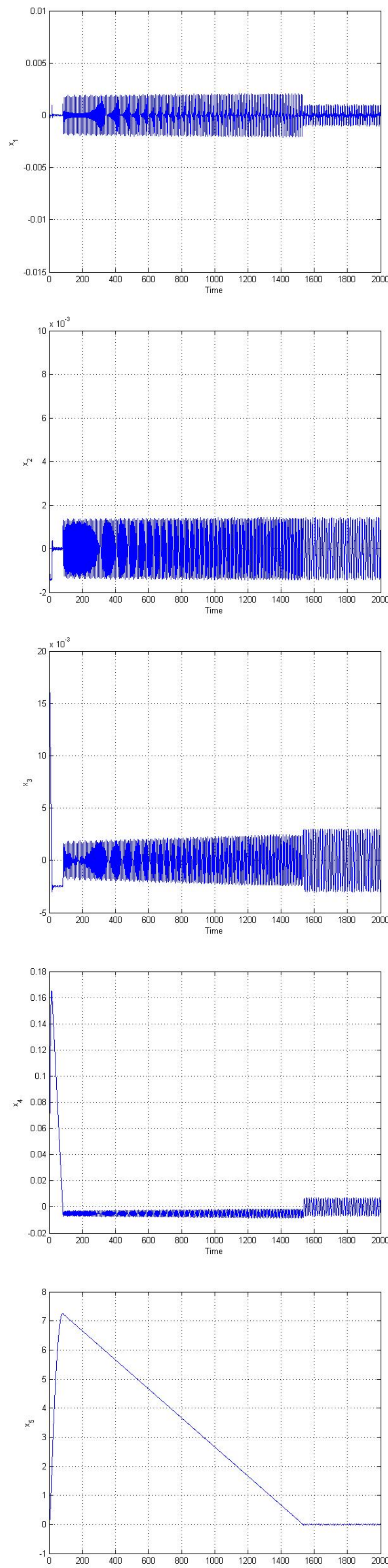

Fig. 3. the state $x_i, i \in [1, 5]$, $\epsilon = 0.0005$

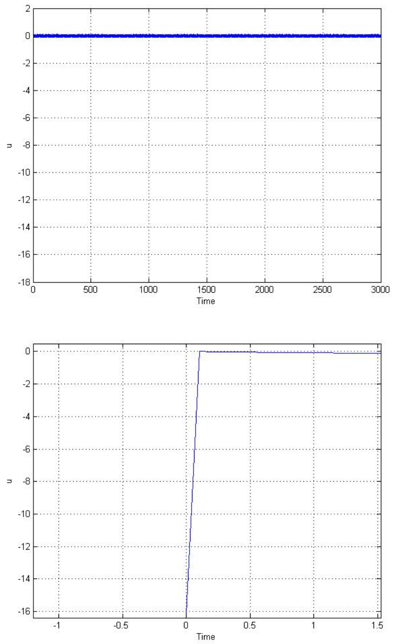

Fig. 4. the control input u, $\epsilon = 0.001$.

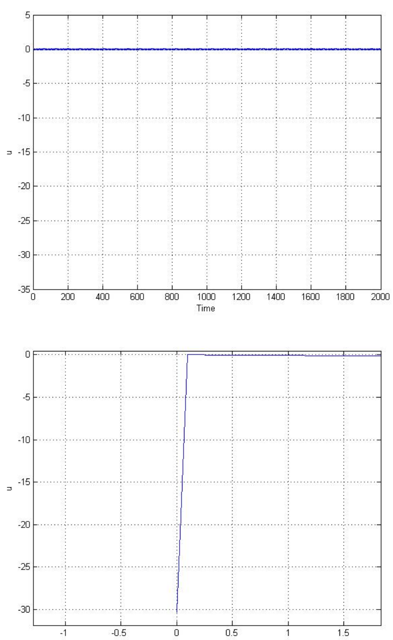

Fig. 5. the control input u, $\epsilon = 0.0005$.

IV. Conclusions

In this paper, global practical stabilization is achieved for an integrator chain system with actuator saturation and input additive disturbances. Under the proposed control law, any system trajectory will be driven into a pre-defined arbitrarily small closed set with the origin inside in a finite time and remain there thereafter.

References

[1] H. Fang and Z. Lin, "Global practical stabilization of planar linear systems in the presence of actuator saturation and input additive disturbance," *IEEE Transactions on Automatic Control*, vol. 51, no. 7, pp. 1177-1184, 2006.

[2] H. Fang, " Global Practical Stabilization for Third Order Integrator Chain with Actuator Saturation and Input Additive Disturbances", *Proceedings of 49th IEEE Conference on Decision and Control*, pp. 5222-5226, 2010.

[3] Z. Lin, "Global control of linear systems with saturating actuators," *Automatica*, Vol. 34, pp. 897-905, 1998.

[4] Z. Lin, "Low Gain Feedback, Lecture Notes in Control and Information Sciences," Vol. 240, Spring-Verlag, London, 1998.

[5] A. Saberi, Z. Lin and A.R. Teel, "Control of Linear Systems with Saturating Actuators," *IEEE Transactions on Automatic Control*, Vol. 41, pp. 368-378, 1996.

[6] Sussmann, H. J. and Y. Yang, "On the stabilizability of multiple integrators by means of bounded feedback controls" *Proc. of the 30th IEEE Conference on Decision and Control*, pp. 70-73, 1991.

[7] A.R.Teel, "Global stabilization and restricted tracking for multiple integrators with bounded controls," *Systems & Control Letters*, Vol. 18, pp. 165-171, 1992.