2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Unfalsified Adaptive Control With Weak Cost-detectability

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Abstract— **In this paper, the unfalsified adaptive control [1] is reestablished with a new term, weak cost-detectability, replacing the cost-detectability. The weak cost-detectability implies that a bounded property of a cost function for the final controller in a switching algorithm is sufficient for stability of an underlying adaptive control system, but the converse may not be true. With the weak cost-detectability instead of the cost-detectability, more choices of cost functions can be employed in the unfalsified adaptive control. Some of the choices introduced in this paper are shown to be greater than the cost functions suggested in [2] but have the same limit values. This property is achieved by seeking a better fictitious reference signal for the purpose of developing greater cost functions while, in [2], a particular choice of fictitious reference signal is used. An example is provided where a weakly cost-detectable cost function leads to faster convergence in the switching algorithm than a cost-detectable cost function.**

I. Introduction

The unfalsified adaptive control [1] exploits collected data in a real-time experiment rather than employs any assumption on a plant and disturbance signals. Based on the concept of the unfalsified control [3], the unfalsified adaptive control stabilizes a system with an unknown plant and unknown disturbance signals using ε-hysteresis algorithm [4] whenever there exists a feasible controller in a candidate controller set, provided that a plant-independent cost function of the switching algorithm is cost-detectable.

A candidate controller set in the unfalsified adaptive control can contain infinite number of controllers but there is a restriction, which is called the SCLI assumption [5], imposed on the candidate controllers, i.e. each candidate controller has to be causally left invertible and the causal left inverse has to be incrementally stable. In order to expand the range of controllers that can be placed in a candidate controller set, the matrix fraction description method is employed in [6] and [7]. If candidate controllers can be factored into incrementally stable factors, then the adaptive control system is reorganized in a way that a new reference signal is injected between the factors of the candidate controllers. These new controllers in the forms of their factors satisfy the SCLI condition. Further, in [2], the factorization requirements are removed and the unfalsified adaptive control is reestablished with no restriction on candidate controllers. The newly-developed fictitious reference signals exist for any controller and cost-detectable cost functions are designed using one of the fictitious reference signals. However, since the fictitious reference signal used in the cost functions in [2] is a particular choice among all possible fictitious reference signals, the cost functions may not efficiently indicate stability of closed-loop systems composed of a plant and candidate controllers.

In this paper, based on the unfalsified adaptive control in [2], all the possible fictitious reference signals are taken into consideration and the smallest fictitious reference signal in the truncated $\mathfrak{L}_2$-norm is sought. This smallest one replaces the fictitious reference signal in the cost functions in [2], which makes the cost functions increase faster. Due to the improved increase of the cost functions, it is expected that the convergence of the switching algorithm in a finite number of switches [4] is achieved faster. Fast convergence means fast stabilization and a shorter transient response. However, in doing so, two problems arise.

First, the cost functions lose cost-detectability by replacing the fictitious reference signal chosen in [2] with the smallest fictitious reference signal in the truncated $\mathfrak{L}_2$-norm. If the cost function for the final controller is bounded, the adaptive control system is stable. But the converse may not be true. This property is called weak cost-detectability and the cost-detectability would be recovered if the converse were guaranteed to be true. Fortunately, weak cost-detectability is shown to be a sufficient property for the cost functions for the purpose of stability of the adaptive control system. Thus, the unfalsified adaptive control is reestablished with the weak cost-detectability instead of the cost-detectability.

The second problem is that in order to obtain the smallest fictitious reference signal in the truncated $\mathfrak{L}_2$-norm, optimization problems should be solved. Even with perfect knowledge of candidate controllers, finding the optimal solutions in real time must be a rough task. In order to resolve this problem, two remedies are considered, a suboptimal solution and finding a solution on a discrete-time basis. If there is a suboptimal solution that is easy to obtain, then it can be substituted for the smallest fictitious reference signal in the cost functions, provided that it is greater than the fictitious reference signal in [2] in the truncated $\mathfrak{L}_2$-norm. If solving an optimization problem takes some time and the computation time is not negligible, the obtained optimal or suboptimal solution can be contributed later on top of cost functions with the fictitious reference signal in [2]. The details are discussed in Section III. Consequently, the improvement in the unfalsified adaptive control with a weakly cost-detectable cost function in this paper depends on how close the suboptimal values of the optimization problems are to the optimal values and how often and fast the solutions are obtained.

The paper is organized as follows. In Section II, an adaptive switching control system is carefully described and the unfalsified adaptive control in [2] is summarized. In Section

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III, the unfalsified adaptive control is reestablished with the newly-defined weak cost-detectability and weakly cost-detectable cost functions using fictitious reference signals are introduced. Conclusion follows in Section IV.

II. Background of Unfalsified Adaptive Control

In this section, the main result of the unfalsified adaptive control in [2] is summarized.

A. Notations and Adaptive Control System Formulation

The norm $\|\cdot\|$ is the $\mathfrak{L}_2$-norm and denote by $\mathfrak{L}_2^m$ the $\mathfrak{L}_2$ space of m-dimensional functions of time, i.e. $\mathfrak{L}_2^m = \{x : [0,\infty) \mapsto \mathbb{R}^m | \ \|x\| < \infty\}$. Define a truncated version of the $\mathfrak{L}_2$-norm

$$\|x\|_t \triangleq \sqrt{\int_0^t x^T(\tau)x(\tau)d\tau}$$

for any function of time x and denote the extended space of $\mathfrak{L}_2^m$ by $\mathfrak{L}_{2e}^m = \{x : [0,\infty) \mapsto \mathbb{R}^m | \ \|x\|_t < \infty, \forall t \in [0,\infty)\}$.

Definition 1: (Stability) A mapping (or a system) $H : \mathfrak{L}_{2e}^{m_i} \mapsto \mathfrak{L}_{2e}^{m_o}$ is said to be stable if there exist constants α_h, $\beta_h \geq 0$ such that for any given input signal $x \in \mathfrak{L}_{2e}^{m_i}$

$$\|Hx\|_t \leq \alpha_h \|x\|_t + \beta_h \quad \text{for } \forall t \geq 0.$$

Otherwise, H is said to be unstable.

Consider an adaptive control system in Fig. 1, which is a mapping from two system-input signals, i.e. a reference signal $w = \begin{bmatrix} q^T & r^T & s^T \end{bmatrix}^T \in \mathfrak{L}_{2e}^{(m_y+m_r+m_u)}$ and a disturbance signal $d \in \mathfrak{L}_{2e}^{m_d}$, to an observed system-output signal $z = \begin{bmatrix} u^T & y^T \end{bmatrix}^T$. The reference signal w is known and the disturbance signal d is unknown. The plant $P : \mathfrak{L}_{2e}^{m_u} \times \mathfrak{L}_{2e}^{m_d} \mapsto \mathfrak{L}_{2e}^{m_y}$ is an unknown mapping from u and d to y. When the plant P has a state, its initial condition at time 0 is also unknown. Then, the input-output relationship of P can be expressed by

$$\mathbb{Z}_P(d) = \left\{ x_z = \begin{bmatrix} x_u \\ x_y \end{bmatrix} \middle| x_u \in \mathfrak{L}_{2e}^{m_u}, x_y = P(x_u, d) \right\}$$

whose element is one possible experimental data over a time interval $[0,\infty)$.

A candidate controller set $\mathbb{C}$ contains N number of candidate controllers. For any given $C \in \mathbb{C}$, the candidate controller $C : \mathfrak{L}_{2e}^{m_r} \times \mathfrak{L}_{2e}^{m_y} \mapsto \mathfrak{L}_{2e}^{m_u}$ is a mapping from controller-input signals , denoted by $r_C \in \mathfrak{L}_{2e}^{m_r}$ and $y_C \in \mathfrak{L}_{2e}^{m_y}$, to a controller-output signal, denoted by $u_C \in \mathfrak{L}_{2e}^{m_u}$. Further, denote by $z_C = \begin{bmatrix} y_C^T & r_C^T & u_C^T \end{bmatrix}^T$ the input and the output signals of C. If C has a state, we choose one initial state. Then, the candidate controller C can be expressed by input-output relationship

$$\mathbb{Z}_C = \Big\{ x_{z_C} = \begin{bmatrix} x_{y_C}^T & x_{r_C}^T & x_{u_C}^T \end{bmatrix}^T \\ \Big| x_{y_C} \in \mathfrak{L}_{2e}^{m_y}, x_{r_C} \in \mathfrak{L}_{2e}^{m_r}, x_{u_C} = C(x_{r_C}, x_{y_C}) \Big\}.$$

A switching algorithm selects a candidate controller at each selecting time from the candidate controller set $\mathbb{C}$ and keeps its controller-output signal delivered to the loop of the adaptive control system until the next selecting time.

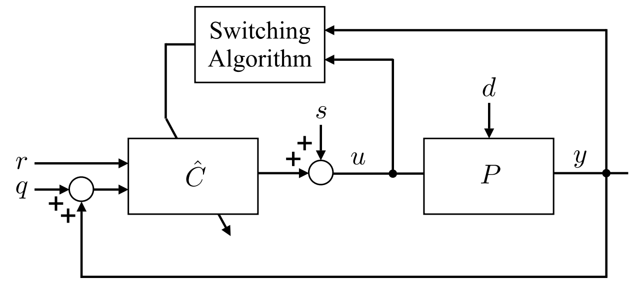


Fig. 1. An adaptive control system

Denote by $\hat{C}$ the sequence of controllers that are chosen and connected in the loop of the adaptive control system by the switching algorithm and let $\hat{C}_t$ denote the candidate controller that is connected in the loop of the adaptive control system at time $t \geq 0$. When a candidate controller C is selected by the switching algorithm, the input signal of P is given by the output signal of C as shown in Fig. 2 (a) until the next selecting time. Thus, the input-output signal of C is obtained by $z_C(t) = \begin{bmatrix} y_C(t)^T & r_C(t)^T & u_C(t)^T \end{bmatrix}^T = \begin{bmatrix} y(t)^T + q(t)^T & r(t)^T & u(t)^T - s(t)^T \end{bmatrix}^T$ for any time $t \geq 0$ satisfying $\hat{C}_t = C$.

When C is not connected in the loop of the adaptive control system, C makes a closed-loop system with a subcontroller K as shown in Fig. 2 (b). The subcontroller K is designed to stabilize C in the closed loop. Although K in Fig. 2 is depicted to use only the output signal of C, actually K is allowed to use not only the output signal of C but also every information on C with perfect knowledge of C. If C is stable itself, K can be given as a zero subcontroller whose output signal is 0 for $\forall t \geq 0$. The role of the subcontroller is to build a stable mapping, as in Definition 1, from $\begin{bmatrix} r^T & q^T + y^T \end{bmatrix}^T$ to a signal anywhere in the closed-loop system of C and K while the candidate controller is not in the loop of the adaptive control system. Thus, the output signal of C does not blow out while it is disconnected from the input signal of P.

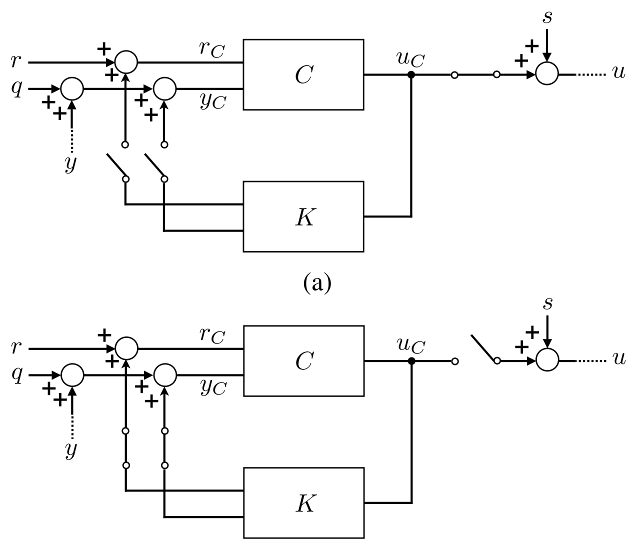
B. Unfalsified Adaptive Control

Definition 2: (Fictitious reference signal) Given the candidate controller set $\mathbb{C}$ in Section II-A, fictitious reference signals for a candidate controller $C \in \mathbb{C}$ are defined by

$$\tilde{w}(x_{z_C}, x_z) \triangleq \begin{bmatrix} \tilde{q}(x_{z_C}, x_z) \\ \tilde{r}(x_{z_C}, x_z) \\ \tilde{s}(x_{z_C}, x_z) \end{bmatrix} \triangleq \begin{bmatrix} x_{y_C} - x_y \\ x_{r_C} \\ x_u - x_{u_C} \end{bmatrix}$$

for $\forall x_{z_C} \in \mathbb{Z}_C$ and $\forall x_z = \begin{bmatrix} x_u^T & x_y^T \end{bmatrix}^T \in \mathfrak{L}_{2e}^{(m_u+m_y)}$. Denote by $\tilde{w}(x_{z_C}, x_z, t)$ the evaluated value of the signal $\tilde{w}(x_{z_C}, x_z)$ at time $t \geq 0$.

For any given $C \in \mathbb{C}$, $x_{z_C} \in \mathbb{Z}_C$, and $x_z \in \mathbb{Z}_P(d)$, the fictitious reference signal $\tilde{w}(x_{z_C}, x_z)$ is a hypothetical signal that would have exactly reproduced the input-output signal $x_{z_C} = \begin{bmatrix} x_{y_C}^T & x_{r_C}^T & x_{u_C}^T \end{bmatrix}^T$ of C and the input-output signal $x_z = \begin{bmatrix} x_u^T & x_y^T \end{bmatrix}^T$ of P had the fictitious reference



(a)

(b)

Fig. 2. A candidate controller and its subcontroller (a) when the candidate controller is selected and connected in the adaptive control system (b) when it is not connected

signal been injected into a fictitious system in Fig. 3, i.e. $\begin{bmatrix} x_q^T & x_r^T & x_s^T \end{bmatrix}^T = \tilde{w}(x_{z_C}, x_z)$.

Remark 1: Alternative types of fictitious reference signals are used in [5], [6], and [7] under various assumptions.

If a controller $C \in \mathbb{C}$ has one degree of freedom and is SCLI [5], then there exists $x_{z_C} \in \mathbb{Z}_C$ such that a fictitious reference signal has $\tilde{q}(x_{z_C}, x_z, t) = \tilde{s}(x_{z_C}, x_z, t) = 0$ for $\forall x_z = \begin{bmatrix} x_u^T & x_y^T \end{bmatrix}^T \in \mathfrak{L}_{2e}^{(m_u+m_y)}$ and $\forall t \geq 0$, which is the unique fictitious reference signal in [5]. If a controller $C \in \mathbb{C}$ has one degree of freedom and can be factored into incrementally stable factors, then the fictitious reference signals for C can be represented by one signal, which is the virtual reference signal in [6].

If a candidate controller C is connected on the loop of the adaptive control system in Section II-A at time $t \geq 0$, then it is clear that $z_C(t) = \begin{bmatrix} y(t)^T + q(t)^T & r(t)^T & u(t)^T - s(t)^T \end{bmatrix}^T$, from which, together with the definition of the fictitious reference signal, it follows that

$$\tilde{w}(z_C, z, t) = \begin{bmatrix} q(t)^T & r(t)^T & s(t)^T \end{bmatrix}^T = w(t).$$

If C is not connected on the loop of the adaptive control system, then its corresponding subcontroller K makes a closed-loop system with C and stabilizes C so that a mapping from $\begin{bmatrix} r^T & q^T + y^T \end{bmatrix}^T$ to z_C is stable and, hence, a mapping from $\begin{bmatrix} r^T & q^T & z^T \end{bmatrix}^T$ to $\tilde{w}(z_C, z)$ is stable. Therefore, a mapping from $\begin{bmatrix} w^T & z^T \end{bmatrix}^T$ to $\tilde{w}(z_C, z)$ is always stable whether or not the candidate controller is connected in the loop of the adaptive control system.

The observed signals in the adaptive control system in Section II-A and the fictitious reference signals are the same as the data from experiments on fictitious systems for each C in Fig. 3 with the fictitious reference signal $\tilde{w}(z_C, z)$ as an input

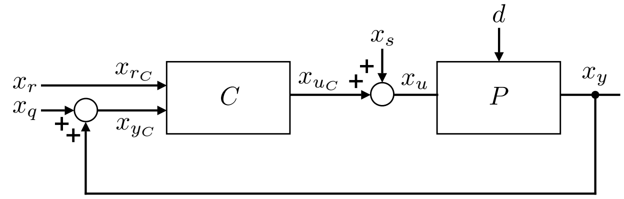


Fig. 3. A candidate controller and a corresponding fictitious system

signal. Based on this data, the fictitious systems are assessed by a mapping $V : \mathbb{C} \times \mathfrak{L}_{2e}^{(m_r+m_u+m_y)} \times \mathfrak{L}_{2e}^{(m_u+m_y)} \mapsto \mathfrak{L}_{2e}^{1}$ that is called a cost mapping. For any given $C \in \mathbb{C}$, $x_{z_C} \in \mathbb{Z}_C$, and $x_z \in \mathbb{Z}_P(d)$, the fictitious system (P, C) in Fig. 3 is evaluated by $V(C, x_{z_C}, x_z)$. Denote by $V(C, x_{z_C}, x_z, t)$ the evaluated value of $V(C, x_{z_C}, x_z)$ at time $t \geq 0$. The cost mapping V is designed to be causal, which means that $V(C, x_{z_C}, x_z, t)$ depends only on C, $x_{z_C}(\tau)$, and $x_z(\tau)$ for $\forall \tau \in [0, t]$.

Definition 3: (Feasibility) Given the plant P and the disturbance signal d in the adaptive control system in Section II-A, together with a cost mapping V, a controller C is said to be a feasible controller if there exist constants $\alpha_f \geq 0$ such that for any given $x_{z_C} \in \mathbb{Z}_C$ and $x_z \in \mathbb{Z}_P(d)$

$$V(C, x_{z_C}, x_z, t) \leq \alpha_f \quad \text{for } \forall t \geq 0.$$

The adaptive control problem is said to be feasible if the candidate controller set $\mathbb{C}$ contains at least one feasible controller.

Whether a controller is a feasible controller or not depends on the plant and the disturbance signal in the experiment conducted from time 0 to ∞.

Given the observed signal z_C and z in the adaptive control system in Section II-A and a cost mapping V, the ε-Hysteresis Switching Algorithm [4] is employed.

Algorithm 1: (ε-Hysteresis Switching Algorithm)

$$\hat{C}_t = \arg\min_{C \in \mathbb{C}} \left\{ V(C, z_C, z, t) - \varepsilon \delta_{C\hat{C}_{t^-}} \right\}$$

where $\varepsilon > 0$ is a constant, δ_{ij} is the Kronecker's δ, and $\hat{C}_{t^-} = \lim_{\tau \uparrow t} \hat{C}_\tau$.

Convergence of the switching algorithm in a finite number of switches is stated in the following lemma.

Lemma 1: (Convergence)[5] Consider the adaptive control system in Section II-A, together with a cost mapping V and Algorithm 1. Suppose that 1) $V(C, x_{z_C}, x_z, t)$ is nondecreasing in time t and 2) the candidate controller set $\mathbb{C}$ contains at least one feasible controller (Definition 3). Then, the number of switches is finite and $V(C_f, z_{C_f}, z, t)$ remains bounded as t increases to infinity where C_f is the final controller in the controller sequence and z_{C_f} is the input-output signal of C_f.

Definition 4: (Cost-detectability) Given the reference signal w and the candidate controller set $\mathbb{C}$ in the adaptive control system in Section II-A, together with a cost mapping V, the pair $(V, \mathbb{C})$ is said to be cost-detectable if, for every sequence of switched controllers $\hat{C}$ with finitely many switches and the accordingly observed system-output signal z, the following statements are equivalent:

1) The function $V(C_f, z_{C_f}, z, t)$ is bounded as t increases to infinity where C_f is the final controller in the controller sequence $\hat{C}$ and z_{C_f} is the input-output signal of C_f.

2) There exist constants α_c, $\beta_c \geq 0$ such that

$$\|z\|_t \leq \alpha_c \|w\|_t + \beta_c \quad \text{for } \forall t \geq 0.$$

An example of the cost-detectable cost function is

$$V_0(C, x_{z_C}, x_z, t) = \max_{0 \leq \tau \leq t} \frac{\|x_z\|_\tau}{\|\tilde{w}(x_{z_C}, x_z)\|_\tau + \rho} \tag{1}$$

for $\forall t \geq 0$ where ρ is a positive constant. If we use the cost mapping V_0 in (1) in Algorithm 1, the performance of a candidate controller C is assessed by $V_0(C, z_C, z)$ which is calculated using $\tilde{w}(z_C, z)$. Note that $\tilde{w}(z_C, z)$ is a particular selection of the fictitious reference signal for C given z.

The main result of the unfalsified adaptive control in [2] follows.

Lemma 2: [2] Consider the adaptive control system in Section II-A, together with a cost mapping V and Algorithm 1. Suppose that 1) $V(C, x_{z_C}, x_z, t)$ is nondecreasing in t, 2) the adaptive control problem is feasible (Definition 3), and 3) the pair $(V, \mathbb{C})$ is cost-detectable (Definition 4). Then, there exist constants α_u, $\beta_u \geq 0$ such that

$$\|z\|_t \leq \alpha_u \|w\|_t + \beta_u \quad \text{for } \forall t \geq 0.$$

III. Unfalsified Adaptive Control With Weak Cost-detectability

In this section, the unfalsified adaptive control is reestablished with weak cost-detectability which is newly defined in the following.

Definition 5: (Weak Cost-detectability) Given the reference signal w and the candidate controller set $\mathbb{C}$ in the adaptive control system in Section II-A, together with a cost mapping V, the pair $(V, \mathbb{C})$ is said to be weakly cost-detectable if, for every sequence of switched controllers $\hat{C}$ with finitely many switches and the accordingly observed system-output signal z, the statement 1) implies the statement 2) in the following:

1) The function $V(C_f, z_{C_f}, z, t)$ is bounded as t increases to infinity where C_f is the final controller in the controller sequence $\hat{C}$ and z_{C_f} is the input-output signal of C_f.

2) There exist constants α_w, $\beta_w \geq 0$ such that

$$\|z\|_t \leq \alpha_w \|w\|_t + \beta_w \quad \text{for } \forall t \geq 0.$$

Clearly, the weak cost-detectability is less restrictive than the cost-detectability in Definition 4.

The main theorem of the unfalsified adaptive control with the weak cost-detectability assumption is presented in the following.

Theorem 1: Consider the adaptive control system in Section II-A, together with a cost mapping V and Algorithm 1. Suppose that 1) $V(C, x_{z_C}, x_z, t)$ is nondecreasing in t, 2) the adaptive control problem is feasible (Definition 3), and 3) the pair $(V, \mathbb{C})$ is weakly cost-detectable (Definition 5). Then, there exist constants α_a, $\beta_a \geq 0$ such that

$$\|z\|_t \leq \alpha_a \|w\|_t + \beta_a \quad \text{for } \forall t \geq 0.$$

Proof. Lemma 1 and Definition 5 complete the proof. □

Theorem 1 has a weaker assumption on the cost mapping than Lemma 2 and, hence, a larger range of cost mappings can be considered in the unfalsified adaptive control.

Consider a cost mapping V_1 defined by

$$V_1(C, x_{z_C}, x_z, t) = \max_{0 \leq \tau \leq t} \frac{\|x_z\|_\tau}{\gamma(C, x_z, \tau) + \rho} \tag{2}$$

for $\forall C \in \mathbb{C}$, $\forall x_{z_C} \in \mathbb{Z}_C$, $\forall x_z \in \mathfrak{L}_{2e}^{(m_u+m_y)}$, and $\forall t \geq 0$ where ρ is a positive constant and

$$\gamma(C, x_z, t) = \inf_{x \in \mathbb{Z}_C} \|\tilde{w}(x, x_z)\|_t \tag{3}$$

for $\forall t \geq 0$ with the fictitious reference signal $\tilde{w}(x, x_z)$ defined in Definition 2. Since we only consider causal candidate controllers, $\|\tilde{w}(x, x_z)\|_t$ is solely determined by $x(\tau)$ and $x_z(\tau)$ for $\forall \tau \in [0, t]$. Hence, $\gamma(C, x_z, t)$ can be obtained using the data up to time t.

The cost function for $C \in \mathbb{C}$ in (2) is always greater than or at least equal to the cost function for C in (1) since every possible fictitious reference signal, including $\tilde{w}(x_{z_C}, x_z)$, is considered in (3). From the fact that both cost functions for C have the same limit and the cost function for C in (2) is greater than or equal to the cost function for C in (1), it is expected that the switching algorithm in the unfalsified adaptive control with the cost mapping V_1 in (2) converges faster. Later, Example 1 illustrates this phenomenon.

The optimization problem in (3) can be rewritten as

$$\gamma(C, x_z, t)^2 = \inf_{x_{\tilde{q}} \in \mathfrak{L}_{2e}^{m_y}, x_{\tilde{r}} \in \mathfrak{L}_{2e}^{m_r}} \|x_{\tilde{q}}\|_t^2 + \|x_{\tilde{r}}\|_t^2 + \|x_u - C(x_{\tilde{r}}, x_{\tilde{q}} + x_y)\|_t^2,$$

which can be interpreted as a tracking problem (e.g. [8]) or a model predictive control problem (e.g. [9]). If we have perfect knowledge of C, as we usually do, then, fundamentally, we should be able to obtain $\gamma(C, x_z, t)$ with no problem. However, mathematically, an optimization problem is not an easy task depending on the structure of C and, thus, a suboptimal value can be employed instead of $\gamma(C, x_z, t)$. In the adaptive control system in Section II-A, one input-output signal pair of C is observed and the corresponding fictitious reference signal $\tilde{w}(z_C, z)$ is obtained from mere subtraction between observed signals. Hence, we already have one candidate for a suboptimal value in our hands, which is the case using the cost mapping V_0 in (1).

In the aspect of numerical analysis, $\gamma(C, x_z, t)$ may not be calculated on a real-time basis in an experiment due to computation time or may not be obtained continuously in time. One way to deal with these cases is to consider an alternative cost mapping V_2 defined by

$$\begin{aligned} &V_2(C, x_{z_C}, x_z, t) \\ &\quad = \max\{V_a(C, x_{z_C}, x_z, t), V_b(C, x_{z_C}, x_z, t)\} \end{aligned} \tag{4}$$

for $\forall C \in \mathbb{C}$, $\forall x_{z_C} \in \mathbb{Z}_C$, $\forall x_z \in \mathfrak{L}_{2e}^{(m_u+m_y)}$, and $\forall t \geq 0$ where

$$V_a(C, x_{z_C}, x_z, t) = \max_{0\leq\tau\leq t} \frac{\|x_z\|_\tau}{\|\tilde{w}(x_{z_C}, x_z)\|_\tau + \rho}$$
$$V_b(C, x_{z_C}, x_z, t) = \max_{1\leq n\leq N} \frac{\|x_z\|_{T_n}}{\gamma(C, x_z, T_n) + \rho}$$

for $\forall t \geq 0$ with a positive constant ρ and a sequence of time points $\{T_1, T_2, \cdots\}$ at which $\gamma(C, x_z, t)$ is computed.

In (4), the cost mapping V_a is the same as the cost mapping V_0 in (1) and the cost mapping V_b is the improvement in the cost mapping given the limited ability to solve the optimization problem. Thus, the improvement depends on how close the suboptimal solution is to the optimal solution and how often and fast the optimization problem is solved.

The two cost mappings V_1 in (2) and V_2 in (4) are proved to be weakly cost-detectable in the following theorem.

Theorem 2: Given the reference signal w, the candidate controller set $\mathbb{C}$, the input-output signal z_C of C for $\forall C \in \mathbb{C}$, the observed input-output signal z of P in the adaptive control system in Section II-A, together with cost mappings V_1 in (2) and V_2 in (4), the pairs $(V_1, \mathbb{C})$ and $(V_2, \mathbb{C})$ are weakly cost-detectable (Definition 5).

Proof. Suppose that there are finite number of switches and denote the final controller and the final switching time by C_f and $t_f < \infty$, respectively. Also, suppose that there exists a constant $\alpha \geq 0$ such that $V_1(C_f, z_{C_f}, z, t) \leq \alpha$ for $\forall t \geq 0$. Then, from (1) and (2), it is clear that

$$V_0(C_f, z_{C_f}, z, t) \leq V_1(C_f, z_{C_f}, z, t) \leq \alpha$$

for $\forall t \geq 0$. Since the pair $(V_0, \mathbb{C})$ is cost-detectable, it follows, from Definition 4, that there exist constants α_c, $\beta_c \geq 0$ such that

$$\|z\|_t \leq \alpha_c \|w\|_t + \beta_c \quad \text{for } \forall t \geq 0.$$

Therefore, the pair $(V_1, \mathbb{C})$ is weakly cost-detectable. Similarly, the pair $(V_2, \mathbb{C})$ can be shown to be weakly cost-detectable. □

From Theorem 1 and 2, it can be concluded that the unfalsified adaptive control with Algorithm 1 and the cost mapping V_1 in (2) or V_2 in (4), achieves stability in the adaptive control system in Section II-A, provided that the feasibility assumption is satisfied.

Example 1: Suppose that the unknown plant P is described by a state-space model

$$\dot{x} = x + u$$
$$y = x + d$$

with $x(0) = 0$ and $d(t) = 0$ for $\forall t \geq 0$. The candidate controller set is given by $\mathbb{C} = \{C_i, i = 1, 2\}$ where

$$C_1 : u_{C_1} = 2(r_{C_1} - y_{C_1})$$
$$C_2 : u_{C_2} = (1 + r_{C_2})(r_{C_2} - y_{C_2})$$

and the reference signal is given by

$$w(t) = \begin{bmatrix} q(t) & r(t) & s(t) \end{bmatrix}^T = \begin{cases} \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T & \text{for } 0 \leq t < 500 \\ \begin{bmatrix} 0 & -2 & 0 \end{bmatrix}^T & \text{for } t \geq 500 \end{cases}.$$

Algorithm 1 is performed with $\varepsilon = 0.2$ and C_1 initially connected in the loop.

When the cost-detectable cost mapping V_0 in (1) is employed in Algorithm 1, the result is shown in Fig. 4. The controller C_1 stabilizes P and the controller C_2 acts like a stabilizing controller until $t = 500$. After r changes to -2 at $t = 500$, the destabilizing property of C_2, which is currently connected in the loop at $t = 500$, is exposed and causes a fast increase in the cost function for C_2 right after $t = 500$. Accordingly, C_2 is switched off and stays disconnected. Apparently, the destabilizing property of C_2 is not shown in the cost function for C_2 when C_2 is not connected in the loop.

On the other hand, when the weakly cost-detectable cost mapping V_1 in (2) is employed in Algorithm 1, the result is shown in Fig. 5. For the cost function for C_2, instead of the minimal value $\gamma(C_2, z, t)$ in (3), a suboptimal value $\|\tilde{w}(\bar{x}, z)\|_t$ with $\bar{x} = \begin{bmatrix} y & 0 & -y \end{bmatrix}^T \in \mathbb{Z}_{C_2}$ is employed. The cost functions for C_1 and C_2 corresponding to V_1 shown in Fig. 5 (b) are greater than the cost functions for C_1 and C_2 corresponding to V_0 shown in Fig. 4 (b). Especially, the cost function for C_2 grows very high relative to the cost function for C_1 even when C_2 is not connected in the loop and, hence, the switching stops early and the switching algorithm keeps the controller C_1 in the loop as shown in Fig. 5 (a). □

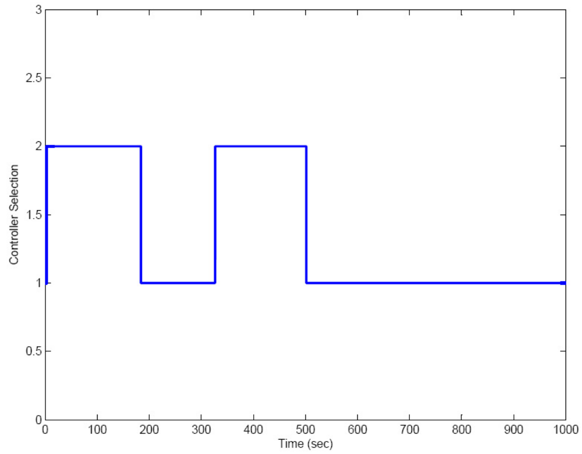
Clearly, the weak cost-detectability is less restrictive than the cost-detectability and, hence, a larger range of cost mappings can be considered with the weak cost-detectability, e.g. the cost mapping V_1 in (2). What can happen with a weakly cost-detectable cost function but can not happen with a cost-detectable cost mapping in the unfalsified adaptive control is that, based on collected data, the adaptive control system behaves like a stable system but the final controller turns out to be a destabilizing controller, which means that the cost function for the final controller goes to infinity as time goes to infinity. Example 2 illustrates this situation. However, this case does not happen if the unfalsified adaptive control employs Algorithm 1 and the feasibility assumption is satisfied.

Example 2: Consider the adaptive control system in Example 1 with a candidate controller set $\mathbb{C} = \{C_2\}$ and a reference signal $w(t) = \begin{bmatrix} q(t) & r(t) & s(t) \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 0 \end{bmatrix}^T$ for $\forall t \geq 0$. Algorithm 1 employs the weakly cost-detectable cost mapping V_1 in (2). Since there is only one candidate controller, switching does not occur. In V_1, instead of the minimal value $\gamma(C_2, z, t)$ in (3), a suboptimal value $\|\tilde{w}(\bar{x}, z)\|_t$ with $\bar{x} = \begin{bmatrix} y & 0 & -y \end{bmatrix}^T \in \mathbb{Z}_{C_2}$ is employed.

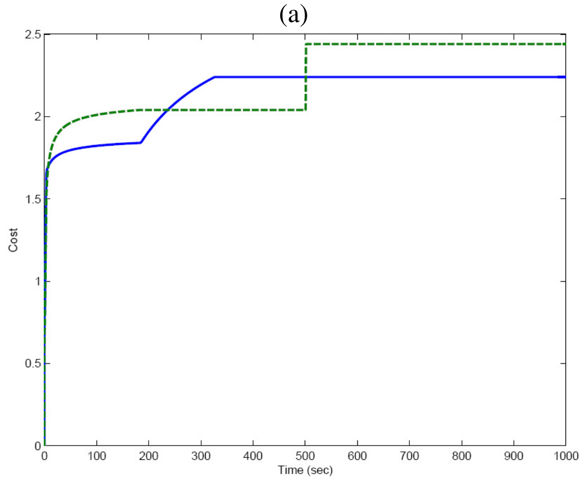
Then, it can be shown that $\|w\|_t^2 = 4t$, $\|z\|_t^2 = 18t - 36(1 - e^{-2t}) + \frac{45}{2}(1 - e^{-4t})$, and $\|\tilde{w}(\bar{x}, z)\|_t^2 = 9(1 - e^{-4t})$ for $\forall t \geq 0$, from which it follows that $\|z\|_t \leq 3\|w\|_t + 1$ for $\forall t \geq 0$ but $V_1(C_2, z_{C_2}, z, t) \to \infty$ as $t \to \infty$. □

IV. Conclusion

The assumptions on cost functions in the unfalsified adaptive control are loosened by replacing the cost-detectability with the weak cost-detectability that is newly-defined in this paper. The weak cost-detectability means that if a cost

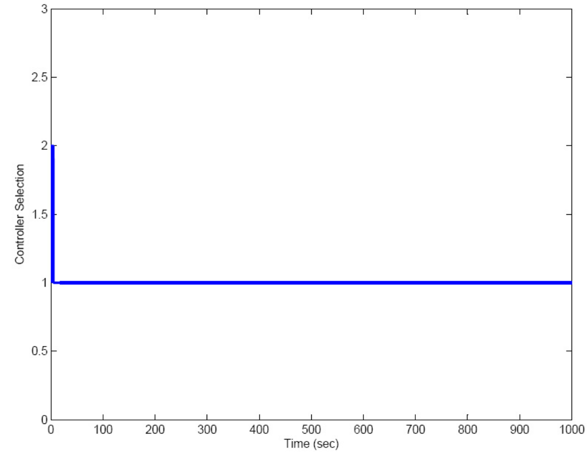


(a)

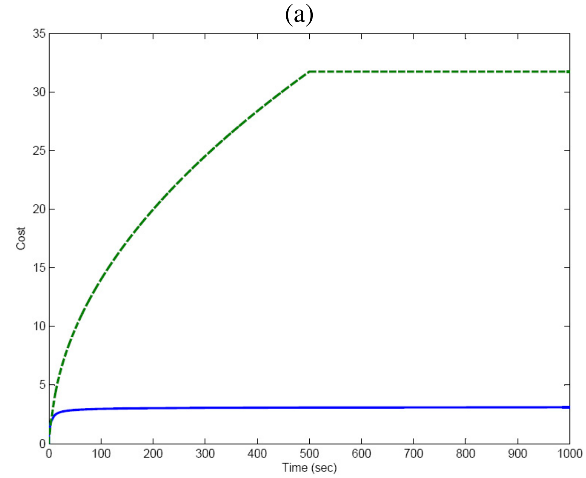


(b)

Fig. 4. The system behavior with the cost mapping V_0 in Example 1 (a) Controller index i (b) Cost functions for C_1 (Solid line) and C_2 (Dashed line).



(a)



(b)

Fig. 5. The system behavior with the cost mapping V_1 in Example 1 (a) Controller index i (b) Cost functions for C_1 (Solid line) and C_2 (Dashed line).

function for the final controller is bounded, the adaptive control system is stable but the converse may not be true. The cost-detectability would be recovered if the converse were guaranteed to be true. The cost functions introduced in this paper have the largest possible values by employing the best fictitious reference signal while, in [2], a particular choice of fictitious reference signal is employed. The improvement in the cost functions leads to fast convergence of the switching algorithm. The best fictitious reference signal is obtained by solving an optimization problem with perfect knowledge of candidate controllers. If the optimization problem is not easy, suboptimal solutions can be employed. Also, the optimization problem can be solved in a discrete-time basis instead of a real-time basis.

V. ACKNOWLEDGMENTS

The author gratefully acknowledges L. D. and dear professors M. G. S. and R. R. B.

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