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# Strobing Optimization in a Mobile Sensor System Associated with the Pursuit Problem

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*Abstract*— **Strobing optimization in a pursuit problem with dynamic linear objects in discrete time involving mobile sensor of radar or sonar type is considered. This problem is formally represented as that of maximizing confidence probability of the pursuer-target system state. Through recasting the latter problem into that of stochastic optimal control, a recursive procedure is obtained that provides an optimal solution.**

## I. INTRODUCTION

Mobile sensors of radar or sonar type are pervasive in the technological systems [1], [2], and the latter type is also found in nature [3]. In many cases these sensors utilize strobing [4] as the key element for improving the signal-to-noise ratio in the measuring channel. However, there is a paucity of theoretical support for the design of optimal strobing action.

In the present work it is shown that the strobing optimization under Gaussian noise measurement - a standard assumption in tracking problems [5] - can be recast into the problem of confidence probability maximization for Gaussian random vector $\theta \sim \mathcal{N}(m, \gamma)$ observable against an additive background noise in discrete time. In the simplest case of scalar random state this problem was first considered in [6]. Discrete in nature, this problem was reduced in [6] to a continuous time time-terminal optimal control one. The latter was accomplished through the use of continuous approximation and Bellman's dynamic programming technique, with optimal control serving as a tool for obtaining a suboptimal discrete time problem solution. However, no boundedness guarantee was provided for the deviation of the latter solution from the true optimal one.

In this paper, a similar problem of a step-by-step maximization of confidence probability for a dynamic linear object in discrete time is considered. The problem is approached directly, i.e. without continuous approximation and the use of Bellman's equation. Using the technique proposed, a true optimal solution is obtained.

## II. MOTIVATING EXAMPLE

Consider a geometrically two-dimensional pursuer-target system in a Cartesian plane $XY$ depicted in Fig. 1, where $\mathrm{P}_0$ and $\mathrm{E}_0$ are the initial positions of a pursuer $P$ and a target $E$, respectively, and $\mathrm{P}_0$ is located at the origin of the system $XY$. Suppose the pursuer $P$ and the target $E$

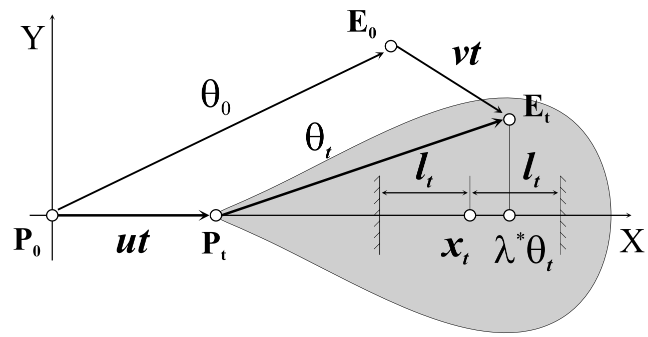


Fig. 1. Example geometry of the pursuer-target system in the case of constant pursuer velocity.

move in the plane $XY$ with constant vector velocities $u$ and $v$, respectively, where $u$ is directed along the $X$-axis. Let the pursuer $P$ have a sensor of a radar or a sonar type with radiation pattern directed along the $X$-axis as well.

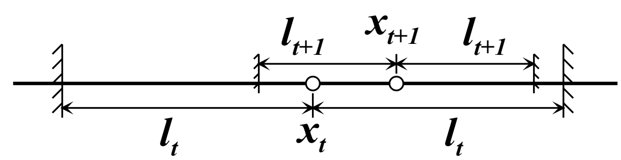


Fig. 2. Strobe markers and strobe gates in a nested configuration.

Now we note that, as shown in Fig.2, a set of strobe markers – the values indicating the beginning and the end of a strobing gate – can be viewed as forming a sequence, in discrete time $t = 0, 1, 2, \ldots,$ of the endpoints of nested gates, i.e. intervals $[x_t - l_t, x_t + l_t]$, located on the symmetry axis ( $X$-axis) of a radiation pattern. Here, $x = (x_t)_{t=0,1,2,\ldots}$ is a sequence of the centers of intervals in the relative coordinate system attached to the pursuer $P$.

The latter setting gives rise to two problems:
i) how to locate the centers $x_t$ of the strobe intervals, and
ii) is it possible to optimize, in some sense, the process of localization of these centers.
To answer these questions we first note that, given the target $E$, the projection of its current position $\mathrm{E}_t$, described by a state-vector $\theta_t \in R^2$ in the relative coordinate system, onto $X$-axis must belong to the strobe interval $[x_t - l_t, x_t + l_t]$,

$$\text{i.e.} \qquad x_t - l_t \le \lambda^* \theta_t \le x_t + l_t, \tag{1}$$

where $\lambda \in R^2$ is a unit vector directed along the $X$-axis and symbol $*$ means transposition.

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Now it is clear how to formulate an optimization problem: we need to select the centers of nested strobe intervals to maximize the current confidence probability

$$P\{\lambda^*\theta_t \in [x_t - l_t,\, x_t + l_t]\}, \quad t = 0, 1, 2, \dots\ . \qquad (2)$$

The problem formulated in this example is solved in Section 5 where it is shown that the solution is arrived at on the basis of a general theoretical result presented in the next section. Two different cases are resolved: when the target velocity vector $v$ is known and when it is an unknown random two-component vector. In both cases it is assumed that the pursuer velocity vector $u$ is known.

## III. Statement of the Problem

Let on a probability space $(\Omega, F, P)$ a linear unobservable vector-process $\theta \triangleq (\theta_t)_{t=0,1,2,\dots}$ be given that describes an evolution of state of the dynamical system with Gaussian initial condition $\theta_0 \sim \mathcal{N}(m, \gamma)$:

$$\theta_{t+1} = a(t)\theta_t + b(t), \qquad t = 0, 1, 2, \dots, \qquad (3)$$

where $\theta_t \in R^n$, $a(t) \in R^{n\times n}$, $b(t) \in R^n$, and parameters $m \in R^n$, $\gamma \in R^{n\times n}$ are given.

Let the observation process $\xi \triangleq (\xi_t)_{t=1,2,\dots}$ have the representation

$$\xi_{t+1} = A(t)\theta_t + B(t)W_{t+1}, \qquad \text{with} \quad \xi_0 = 0, \qquad (4)$$

where $\xi_t \in R^k$, $A(t) \in R^{k\times n}$, $B(t) \in R^{k\times r}$, and $W \triangleq (W_t)_{t=1,2,\dots}$ is a sequence of independent Gaussian vectors from $R^r$, independent on $\theta_0$, with zero mean and a unit covariance matrix. Denoting

$$\varphi(t) \triangleq \Phi(t, 0) \quad \text{and} \quad h(t) \triangleq \sum_{1\le\tau\le t} \Phi(t, \tau)b(\tau - 1), \qquad (5)$$

$$\text{where} \quad \Phi(t, s) \triangleq \prod_{s\le\tau<t} a(\tau) \quad \text{with} \quad \Phi(t, t) = E,$$

permits introducing a solution of equation (3) in the form

$$\theta_t = \varphi(t)\theta_0 + h(t). \qquad (6)$$

Here $E$ is an identity matrix.
Then, to evaluate the state vector $\theta_t$ one clearly needs to evaluate the initial condition $\theta_0$.

The latter remark permits formulating the following optimization problem: select a sequence $x \triangleq (x_t)_{t=0,1,2,\dots}$ of the centers of nested intervals (Fig. 2)

$$[x_{t+1} - l_{t+1},\, x_{t+1} + l_{t+1}] \subseteq [x_t - l_t,\, x_t + l_t] \qquad (7)$$

to maximize payoff function represented by the confidence probability

$$P\{\lambda^*\theta_0 \in [x_t - l_t,\, x_t + l_t]\}. \qquad (8)$$

Here, a decreasing sequence of the strobe marker end points $l = (l_t)_{t=0,1,2,\dots}$, a constant vector $\lambda$, and a value $x_0$ are assumed to be given.

## IV. Problem Solution

Let us now reformulate the optimization problem introduced above as a stochastic control problem according to the following recursive procedure.

**Step 1.** Introduce control of the centers $x$ of strobing intervals. For this purpose note that the condition (7) implies the inequality

$$l_{t+1} - l_t \le x_{t+1} - x_t \le -(l_{t+1} - l_t),$$

which, in turn, yields

$$x_{t+1} = x_t + \alpha_t, \qquad (9)$$

where $\alpha \triangleq (\alpha_t)_{t=0,1,2,\dots}$ is a control sequence satisfying the restrictions

$$\alpha_t \in [-\Delta l_t,\, \Delta l_t], \quad \text{with} \quad \Delta l_t \triangleq -(l_{t+1} - l_t). \qquad (10)$$

**Step 2.** Recast a step-by-step payoff function (8) into the form

$$J(\alpha, t) = P\{|x_t - \lambda^*\theta_0| \le l_t\} \to \sup_{\alpha_t},$$

or

$$J(\alpha, t) = \mathbf{E}\,\mathbf{E}\left(I\{|x_t - \lambda^*\theta_0| \le l_t\}/\mathcal{F}_t^\xi\right) \to \sup_{\alpha_t}, \qquad (11)$$

where $I\{\cdot\}$ is an indicator function, $\mathbf{E}$ is an expectation symbol, and $\mathcal{F}_t^\xi = \sigma\{\xi_s,\, s \le t\}$ is a $\sigma$-algebra generated by observations $\xi_1, \dots, \xi_t$.

**Step 3.** "Dynamisize" random state vector $\theta_0$. For this purpose introduce matrix-functions

$$A_1(t) \triangleq A(t)\Phi(t, 0) \quad \text{and} \quad A_0(t) \triangleq A(t)h(t), \qquad (12)$$

and set formally $\theta_0 = \theta_0(t)$. Then, the evolution of $\theta_0$ can be represented as

$$\theta_0(t + 1) = \theta_0(t), \qquad \text{with} \quad \theta_0(0) \sim \mathcal{N}(m, \gamma), \qquad (13)$$

and, using the notation introduced above, the observation process takes the form

$$\xi_{t+1} = A_0(t) + A_1(t)\theta_0(t) + B(t)W_{t+1}, \quad \xi_0 = 0. \qquad (14)$$

Now, for the two-component partially observable process $(\theta_0(t), \xi_t)_{t=0,1,2,\dots}$ generated by (13) and (14), Kalman filter describing an evolution of

$$m_t = \mathbf{E}\left(\theta_0/\mathcal{F}_t^\xi\right), \quad \gamma_t = \mathbf{E}\left[(\theta_0 - m_t)(\theta_0 - m_t)^*/\mathcal{F}_t^\xi\right],$$

takes the form [7]

$$\begin{cases} m_{t+1} = m_t + \gamma_t A_1^*(t) D_t^+ \widetilde{W}_{t+1}, & m_0 = m, \\ \gamma_{t+1} = \gamma_t - \gamma_t A_1^*(t)[D_t D_t^*]^+ A_1(t)\gamma_t, & \gamma_0 = \gamma, \end{cases} \qquad (15)$$

where

$$D_t \triangleq [B(t)B^*(t) + A_1(t)\gamma_t A_1^*(t)]^{1/2}, \qquad (16)$$

and $\widetilde{W} \triangleq (\widetilde{W}_t)_{t=1,2,\dots}$ is an innovation process independent of $\mathcal{F}_t^\xi$ represented by a sequence of independent Gaussian vectors. Here a superscript plus stands for pseudo-inversion.

**Step 4.** Rewrite (11) in the integral form. For this purpose note that the conditional distribution of $\theta_0$, which has the form

$$P\{\theta_0 \le \vartheta/\mathcal{F}_t^\xi\} \sim \mathcal{N}(m_t, \gamma_t),$$

is normal, with parameters $m_t, \gamma_t$. Hence

$$P\{\lambda^*\theta_0 \le \vartheta/\mathcal{F}_t^\xi\} \sim \mathcal{N}(\lambda^* m_t, \sigma_t^2), \tag{17}$$

where $\sigma_t^2 = \lambda^*\gamma_t\lambda$.

Now, due to (17), the payoff function (11) can be represented as

$$J(\alpha,t) = \mathbf{E}\left\{\frac{1}{\sqrt{2\pi}\sigma_t}\int_{x_t-l_t}^{x_t+l_t}\exp\left(-\frac{(z-\lambda^* m_t)^2}{2\sigma_t^2}\right)dz\right\} \tag{18}$$

or, putting $r = (z-\lambda^* m_t)/\sigma_t$, as

$$J(\alpha,t) = \mathbf{E}\left\{\frac{1}{\sqrt{2\pi}}\int_{(x_t-\lambda^* m_t-l_t)/\sigma_t}^{(x_t-\lambda^* m_t+l_t)/\sigma_t}\exp\left(-\frac{r^2}{2}\right)dr\right\}. \tag{19}$$

**Step 5.** Bring (19) to an analytically tractable form. For this purpose introduce a process

$$y_t \triangleq x_t - \lambda^* m_t, \quad \text{with} \quad y_0 = x_0 - \lambda^* m. \tag{20}$$

Then $y_{t+1} = x_{t+1} - \lambda^* m_{t+1}$ and hence

$$y_{t+1} - y_t = (x_{t+1} - x_t) - \lambda^*(m_{t+1} - m_t) =$$
$$= \alpha_t - \lambda^*\gamma_t A_1^*(t) D_t^+ \widetilde{W}_{t+1}.$$

Finally,

$$y_{t+1} = y_t + \alpha_t - \varepsilon_{t+1}, \tag{21}$$

where

$$\alpha_t = x_{t+1} - x_t \quad \text{and} \quad \varepsilon_{t+1} \triangleq \lambda^*\gamma_t A_1^*(t) D_t^+ \widetilde{W}_{t+1}.$$

Here $\varepsilon_{t+1}$, $(t = 0,1,2,\dots)$, are independent Gaussian random values with zero mean and a covariance

$$\varrho_t^2 \triangleq \lambda^*\gamma_t A_1^*(t)(D_t D_t^*)^+ A_1(t)\gamma_t\lambda.$$

Then, the payoff, in terms of the processes $y_t$ and $\varepsilon_{t+1}$, takes the form

$$J(\alpha,t) =$$
$$= \mathbf{EE}\left\{\frac{1}{\sqrt{2\pi}}\int_{(y_{t-1}+\alpha-\varepsilon_t-l_t)/\sigma_t}^{(y_{t-1}+\alpha-\varepsilon_t+l_t)/\sigma_t}\exp\left(-\frac{r^2}{2}\right)dr\Big/ y_{t-1} = y\right\}. \tag{22}$$

Replacing the variable $r$ in (22) by $r_1/\sigma_t$ and averaging over $\varepsilon_t \sim \mathcal{N}(0, \varrho_{t-1}^2)$, we obtain

$$J(\alpha,t) =$$
$$= C\int_{-\infty}^{\infty}\int_{y+\alpha-r_2-l_t}^{y+\alpha-r_2+l_t}\exp\left(-\frac{r_1^2}{2\sigma_t^2}\right)\exp\left(-\frac{r_2^2}{2\varrho_{t-1}^2}\right)dr_1 dr_2, \tag{23}$$

where $C \triangleq 1/(2\pi\sigma_t\varrho_{t-1})$.

Next, introducing the unit step function

$$\chi(\theta) \triangleq \begin{cases} 1, & \theta \ge 0, \\ 0, & \theta < 0, \end{cases}$$

rewrite the integral representation (23) of the payoff function as

$$J(\alpha,t) =$$
$$= C\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi(r_1-y-\alpha+r_2+l_t)\chi(y+\alpha-r_2+l_t-r_1)\times$$
$$\times\exp\left\{-\frac{1}{2}\left(\frac{r_1^2}{\sigma_t^2}+\frac{r_2^2}{\varrho_{t-1}^2}\right)\right\}dr_1 dr_2. \tag{24}$$

Thus, (24) reduces the problem of calculating the value of the payoff function $J(\alpha,t)$ in (11) to simply integrating the 2-dimensional Gaussian density over a $2l_t$-wide strip shifted by $y$ with respect to the origin, as illustrated in Fig. 3–5.

**Step 6.** To complete the task of optimal control calculation, we note that minimization of the payoff function (24) must be carried out through compensating shift $y$. This is accomplished by setting the control variable $\alpha_t$ as

$$\alpha_t = \begin{cases} -y_t, & \text{if} \quad |y_t| \le \Delta l_t = l_t - l_{t+1}, \\ -\Delta l_t \operatorname{sign} y_t, & \text{if} \quad |y_t| > \Delta l_t. \end{cases} \tag{25}$$

**Step 7.** Finally, returning back to the original optimization problem, for given $\alpha_t$, the value $x_{t+1}$ of the center of the next strobe interval is calculated by formula (9).

## V. Example Solution

In this example, consisting of two cases, the target dynamics in the relative coordinate system is described by equation

$$\theta_{t+1} = \theta_t + v - u, \tag{26}$$

where $v$ and $u$ are 2-dimensional vectors, $\theta_0 \sim \mathcal{N}(m,\gamma)$, and the observation process is described by (4). Here and later $t = 0,1,2,\dots$. The first case considers known constant target and pursuer velocities, while the second case assumes that the target velocity is a random Gaussian vector with known mean and variance and the pursuer velocity value is a known function of time.

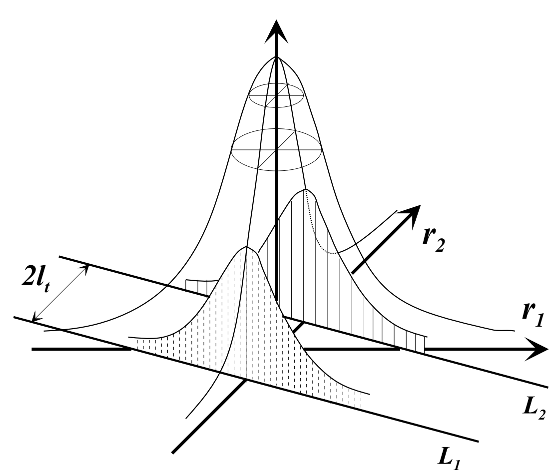


Fig. 3. Gaussian integrand in the integral representation of the payoff function.

### A. Case 1: Constant Velocities

With the target and the pursuer velocities $v$ and $u$, respectively, being known constant vectors, it follows from (26) that

$$\theta_t = \theta_0 + (v - u)t, \tag{27}$$

and hence the inequality (1) can be rewritten as

$$x_t - \beta t - l_t \leq \lambda^* \theta_0 \leq x_t - \beta t + l_t,$$

where $\beta = \lambda^*(v - u)$ is a projection of the target relative velocity onto the $X$-axis. Nesting requirement for the left ends of strobe intervals gives

$$x_t - \beta t - l_t \leq x_{t+1} - \beta(t+1) - l_{t+1},$$

or

$$x_t + \beta - l_t \leq x_{t+1} - l_{t+1}. \tag{28}$$

Analogously, for the right ends of strobe intervals we have the condition (Fig. 6)

$$x_{t+1} + l_{t+1} \leq x_t + \beta + l_t. \tag{29}$$

Following the procedure described in the previous section, introduce, analogously to (9), a control variable $\alpha_t$ as

$$\alpha_t \triangleq x_{t+1} - (x_t + \beta)$$

or

$$x_{t+1} = x_t + \beta + \alpha_t. \tag{30}$$

It follows from (28) and (29) that $\alpha_t$ satisfies the restriction (10). As it was mentioned in Section 1, our objective is to maximize the confidence probability (2).

Note that in contrast to the statement considered in Section 2 involving the time independent stochastic variable $\theta_0$ in (8), the confidence probability (2) contains a time-dependent variable $\theta_t$ rather then $\theta_0$. Therefore, unlike (9), equation (30) contains a shift constant $\beta$ arising due to this time dependence. However, as it will be shown below, this does not complicate solving the problem posed in the present example.

Indeed, direct comparison of (27) with (6) yields $\varphi(t)$ being identity matrix and $h(t) = (v - u)t$. Therefore, due to (5), $\Phi(t, 0)$ is identity matrix as well. Consequently, according to (12), $A_0(t) = A(t)(v-u)t$ and $A_1(t) = A(t)$. Hence, the observation process $\xi$ of (4) reduces to the form (14). Now, setting formally $\theta_0 = \theta_0(t)$ as in Section 3, the Kalman filter equations (15) and the distribution (17) become applicable to $(\theta_0(t), \xi_t)_{t=0,1,2,\ldots}$ . It follows from (17) that the conditional distribution of $\lambda^*\theta_t = \lambda^*\theta_0 + \beta t$ is given by

$$P\{\lambda^*\theta_t \leq \vartheta / \mathcal{F}_t^\xi\} \sim \mathcal{N}(\lambda^* m_t + \beta t, \sigma_t^2), \tag{31}$$

i.e. remains normal with parameters $\lambda^* m_t + \beta t$ and $\sigma_t^2 = \lambda^* \gamma_t \lambda$.

Next, representing the payoff function (2) in the form (11) as

$$J(\alpha, t) = \mathbf{E}\,\mathbf{E}\left(I\{|x_t - \lambda^*\theta_t| \leq l_t\}/\mathcal{F}_t^\xi\right) \to \sup_{\alpha_t}, \tag{32}$$

rewrite it, in view of (31) and (32), in integral form (19) as

$$J(\alpha, t) = \mathbf{E}\left\{\frac{1}{\sqrt{2\pi}} \int_{(x_t - \lambda^* m_t - \beta t - l_t)/\sigma_t}^{(x_t - \lambda^* m_t - \beta t + l_t)/\sigma_t} \exp\left(-\frac{r^2}{2}\right) dr\right\}. \tag{33}$$

As in Step 5 of the procedure in the previous section, introduce the process

$$y_t = x_t - \lambda^* m_t - \beta t, \quad \text{with} \quad y_0 = x_0 - \lambda^* m, \tag{34}$$

that due to (30) satisfies (21). Then the representations (30), (33), and (34) provide the possibility to directly repeat all calculations at Step 5 of the procedure in the previous section. This yields an optimal control law of the form (25).

### B. Case 2: Gaussian Target Velocity, Varying Pursuer Velocity Magnitude

Assume now that the target vector velocity is Gaussian with $v \sim \mathcal{N}(m_v, \gamma_v)$, where $m_v$ and $\gamma_v$ are given, and the pursuer velocity is a known vector-function of time $u_t$ directed along the $X$-axis.

For brevity, introduce the following notation. Denote the relative velocity of the target with respect to the pursuer by

$$\nu_t \triangleq v - u_t \sim \mathcal{N}(m_\nu(t), \gamma_v), \quad \text{with} \quad m_\nu \triangleq m_v - u_t,$$

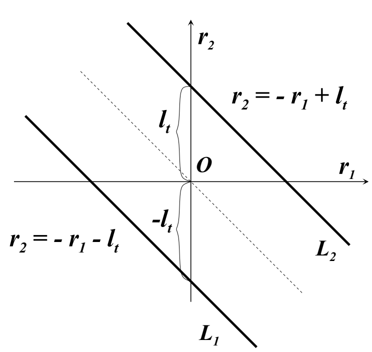


Fig. 4. Non-shifted strip in the 2-dimensional domain of integration of the payoff function.

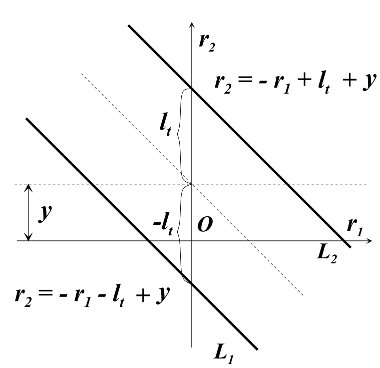


Fig. 5. Shifted by $y$ strip in the 2-dimensional domain of integration of the payoff function.

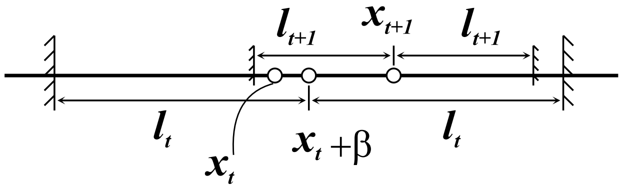


Fig. 6. Strobe markers with constant shift $\beta$ arising due to propagation of the system state time dependence into the strobing gate.

and introduce a block-vector

$$\zeta_t \triangleq \begin{bmatrix} \theta_t \\ - \\ \nu_t \end{bmatrix}.$$

Then

$$\nu_{t+1} = \nu_t + \big(u_t - u_{t+1}\big), \quad \text{with} \quad \nu_0 = v - u_0. \tag{35}$$

Due to (26) and (35), we have

$$\zeta_{t+1} = a\zeta_t + g_t, \tag{36}$$

with block-matrix $a$ and a block-vector $g_t$

$$a \triangleq \begin{bmatrix} E & | & E \\ - & | & - \\ 0 & | & E \end{bmatrix}, \quad g_t \triangleq \begin{bmatrix} \bar{0} \\ ---- \\ u_t - u_{t+1} \end{bmatrix},$$

$$\text{under} \quad \zeta_0 \sim \mathcal{N}\left( \begin{bmatrix} m \\ - \\ m_\nu \end{bmatrix}, \begin{bmatrix} \gamma & | & 0 \\ - & | & - \\ 0 & | & \gamma_v \end{bmatrix} \right),$$

where $0 \in R^{2\times 2}$ and $\bar{0} \in R^2$ are a matrix and a vector, respectively, with zero elements.
In these notations the observation process (4) takes the form

$$\xi_{t+1} = A_1(t)\zeta_t + B(t)W_{t+1}, \tag{37}$$

with a block matrix $A_1(t) = [A(t)|0]$. Further on, Kalman filter for processes (36), (37) takes the form analogous to (15), namely

$$\begin{cases} m_{t+1} = g_t + am_t + a\gamma_t A_1^*(t)D_t^+\widetilde{W}_{t+1}, & m_0 = \widetilde{m}, \\ \gamma_{t+1} = a\gamma_t a^* - a\gamma_t A_1^*(t)[D_t D_t^*]^+ A_1(t)\gamma_t a^*, & \gamma_0 = \widetilde{\gamma}, \end{cases} \tag{38}$$

where

$$\widetilde{m} = \begin{bmatrix} m \\ --- \\ m_\nu(0) \end{bmatrix}, \quad \widetilde{\gamma} = \begin{bmatrix} \gamma & | & 0 \\ - & | & - \\ 0 & | & \gamma_v \end{bmatrix},$$

and $D_t$ is defined as in (16). Here

$$m_t = \mathbf{E}\Big(\zeta_t/\mathcal{F}_t^\xi\Big) = \begin{bmatrix} m_\theta(t) \\ --- \\ m_\nu(t) \end{bmatrix},$$

$$\gamma_t = \mathbf{E}\Big[(\zeta_t - m_t)(\zeta_t - m_t)^*/\mathcal{F}_t^\xi\Big] = \begin{bmatrix} \gamma_\theta(t) & | & \gamma_{\theta\nu}(t) \\ --- & | & --- \\ \gamma_{\nu\theta}(t) & | & \gamma_\nu(t) \end{bmatrix}.$$

Then, the conditional distributions take the form

$$P\{\zeta_t \le \vartheta/\mathcal{F}_t^\xi\} \sim \mathcal{N}(m_t, \gamma_t),$$
$$P\{\lambda^*\theta_t \le \vartheta/\mathcal{F}_t^\xi\} \sim \mathcal{N}(\lambda^* m_\theta(t), \sigma_t^2), \tag{39}$$

where $\sigma_t^2 = \lambda^*\gamma_\theta(t)\lambda$.

Due to (39), as in Step 4 of the previous section, rewrite the payoff function (2) in the form (18), substituting $m_\theta(t)$ instead of $m_t$. Then, introducing, as above, a projection $\beta_t$ of the target relative velocity $\nu_t$ onto the $X$-axis, which, unlike in the Case 1, will be a Gaussian random process, we have

$$\beta_t = \lambda^*\nu_t \sim \mathcal{N}\big(\lambda^*(m_v - u_t), \lambda^*\gamma_v\lambda\big). \tag{40}$$

It is obvious that the restrictions (28), (29), and equation (30), with $\beta = \beta_t$ and $\alpha_t$ satisfying (10), remain valid. Using the change of variable $r = (y - \lambda^* m_\theta(t))/\sigma_t$, the integral representation (18) of the payoff function could be rewritten in the form (19) with $m_\theta(t)$ instead of $m_t$, as in Step 4 of the procedure in the previous section. Following Step 6 of that procedure, introduce the process

$$y_t = x_t - \lambda^* m_\theta(t), \quad \text{with} \quad y_0 = x_0 - \lambda^* m. \tag{41}$$

Then, in view of (30),

$$y_{t+1} - y_t = x_{t+1} - x_t - \lambda^*\big[m_\theta(t+1) - m_\theta(t)\big] =$$
$$= \beta_t + \alpha_t - \lambda^*\big[m_\theta(t+1) - m_\theta(t)\big],$$

where the difference in square brackets satisfies Kalman filter (38), which in this case takes the form

$$m_\theta(t+1) - m_\theta(t) = m_\nu(t) + G_t\widetilde{W}_{t+1}.$$

Here $G_t$ is a matrix consisting of the first two rows of matrix
$a\gamma_t A_1^*(t)D_t^+$ of (38). Hence

$$y_{t+1} = y_t + \alpha_t + \beta_t - \lambda^* m_\nu(t) - \lambda^* G_t\widetilde{W}_{t+1}, \tag{42}$$

where $\lambda^* m_\nu(t) - \beta_t = \lambda^*\big[m_\nu(t) - \nu_t\big]$.
Now introduce a process

$$\eta_t = m_\nu(t) - \nu_t, \quad \text{with} \quad \eta_0 = 0. \tag{43}$$

It follows from (43) and Kalman filter equations (38) that

$$\eta_{t+1} - \eta_t = m_\nu(t+1) - m_\nu(t) - (u_t - u_{t+1}) = Q_t\widetilde{W}_{t+1}, \tag{44}$$

where $Q_t$ is a matrix consisting of the last two rows of matrix
$a\gamma_t A_1^*(t)D_t^+$ of (38). Therefore,

$$\eta_t = \sum_{s\le t} Q_{s-1}\widetilde{W}_s \sim \mathcal{N}\big(0, \gamma_\eta(t)\big),$$

where $\gamma_\eta(t) \in R^{2\times 2}$ is a covariance matrix calculated through the elements of the matrices $Q_s$,

$s = 0, 1, 2, \ldots, t-1$. Define a scalar process

$$\begin{aligned} \varepsilon_{t+1} &= \lambda^* \big(\eta_t + G_t \widetilde{W}_{t+1}\big) = \\ &= \lambda^* \textstyle\sum_{0<s\le t+1} Q_{s-1} \widetilde{W}_s \sim \mathcal{N}\big(0, \varrho_t^2\big), \end{aligned} \tag{45}$$

where $\varrho_t^2$ is a covariance calculated through the elements of the matrices $Q_s$, $s = 0, 1, 2, \ldots, t$, and vector $\lambda$. Here we set formally $Q_t = G_t$.

It follows from (42)–(45) that

$$y_{t+1} = y_t + \alpha_t - \varepsilon_{t+1}.$$

This equation coincides with (21). Consequently, the calculations of Step 6 of the procedure in Section 4 remain valid for the present case as well, and the optimal control law takes the identical form, i.e. is given by (25).

## VI. Conclusion

The strobing optimization procedure carried out in the present work could be potentially extended to encompass heavy tail and other non-Gaussian but symmetric distributions, and address tracking with target freely moving in the 2-dimensional and 3-dimensional space. The procedure could be also extended to the multi-sensor/multi-target setting. Finally, observation control in the form of the strobe gates selection could be attempted.

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