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Linear Programming Based Routing Design for a Class of Positive Systems with Piecewise Constant Capacity Constraints

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Abstract— **We present a technique to design routing parameters for positive compartmental conservative systems with capacity constraints. Such systems describe the flow of material through a network of interconnected reservoirs and have become popular, in particular, as models of air traffic flows. The technique presented here is a Linear Programming based method to design time varying routing parameters to satisfy piecewise constant capacity constraints. Under these routing parameters, the resulting system is positive, conservative and exhibits the desired interconnection.**

I. INTRODUCTION

In this work, we focus on control design for positive compartmental systems. Such systems represent the dynamics of the flow of material through an interconnected network of reservoirs. The dynamics are derived from conservation laws and the underlying interconnection of the network [1]. These models have been used to describe a variety of different systems including automobile or aircraft traffic flow, job-balancing in computer clusters [2], or any system of connected reservoirs with natural constraints, such as irrigation networks [3].

We are motivated by the application of the control design techniques presented here to problems in air traffic flow management (ATFM). A popular method of describing air traffic networks is through the use of Eulerian models, first introduced in [4], which describe the aggregate dynamics of groups of aircraft rather than focusing on individual flights. The use of Eulerian models in ATFM problems has become popular in part because these models lend themselves to traditional linear system control design. A survey and comparison of available Eulerian frameworks can be found in [5] and [6]. Although we are motivated by ATFM applications, the control techniques described here can be applied to any positive compartmental system.

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The issue of routing air traffic while satisfying capacity constraints is a fundamental problem of air traffic management research. Weather conditions and other factors can decrease the capacity of a given region of airspace. Several solution methods have been proposed for this problem (for example, [7], [8]). Typically, aggregate ATFM methods do not use routing as a control parameter. Routing parameters are used as a control input in [9] in which a nonlinear control technique based on Max Weight policy is presented. Their model aggregates flights based on destination, which makes it of higher state-space dimension than that of [4], but can be used to address routing in networks with multiple destinations. We addressed routing design in [10] in which we developed Linear Programs (LP) to design static routing parameters for a single destination network. Capacity constraints were considered, but incorporated only as constant capacity constraints.

In this work, we use an aggregate flow model with routing parameters as the control input to satisfy time varying capacity constraints. We derive linear constraints to ensure that the state of the system lies below a piecewise linear capacity bound. Time varying routing parameters which ensure that these constraints are satisfied can be recovered from any feasible point. If a feasible solution cannot be found for a given set of capacity constraints, these constraints can be altered to generate a feasible problem. We give an LP which can be used to adjust these constraints while minimizing the integral of the difference between the given capacity constraints and the altered capacity constraints.

Notation: Matrices are denoted by capital letters, such as A, with entry A_{ij} (i^{th} row, j^{th} column). Vectors are represented by lower case letters, such as x with coordinate x_i. A set of scalar values parameterized by time is denoted by $\beta(t)$. We denote the cone of entry-wise non-negative vectors of dimension n by $\mathbb{R}^n_+$ and write "$x \geq 0$" to mean that vector x belongs to that set, and "$x > 0$" to mean that it belongs to its interior, i.e., that every entry of vector x is strictly positive. Likewise,

$\mathbb{R}_+^{n\times m}$ will denote the set of all $n \times m$ matrices with non-negative entries. A real matrix M is called a *Metzler* matrix if its off-diagonal elements are non-negative, i.e., $M_{ij} \geq 0,\ i \neq j$. For $i = 1, \ldots, n$, e_i is the i^{th} canonical basis vector of $\mathbb{R}^n$.

II. NETWORK DESCRIPTION

The network model used in this work is a continuous time analog of the Eulerian model of air traffic introduced in [4]. Here we focus on a description of the model which is relevant to the development of the current work. For a more thorough justification of the use of this type of model to describe air traffic, see our previous work [11], [12], [10].

We consider positive systems which can be described as a network of sections through which material can travel. Some subset of the sections in the network are "final sections," or sinks. We denote final sections by $\mathcal{S_F}$, that is section i is a final section if $i \in \mathcal{S_F}$.

The state of section i, denoted by x_i, represents the amount of material in that section. Material in each section is assumed to be traveling at a constant speed, corresponding to a section traversal time of $\tau_i > 0$. This leads to an outflow rate of section i of $\frac{x_i(t)}{\tau_i}$. The subset of sections which material in section $i \notin \mathcal{S_F}$ can flow into is denoted $\mathcal{O}_i$. Material can always flow back into the section that it has just exited, therefore $i \in \mathcal{O}_i$ for all $i \notin \mathcal{S_F}$. Note that routing material back into the section which it has just exited effectively reduces the flow rate out of that section. In the physical system, this recirculation can be realized by holding or slowing down the material moving through the section.

Any material in a final section will flow out of the network, therefore $\mathcal{O}_i = \emptyset$ for all $i \in \mathcal{S_F}$. Fractions of the outflow of section i are routed to sections $j \in \mathcal{O}_i$ according to the routing parameter $\beta_{ij}(t)$. To simplify notation in the remainder of this paper, we will refer to these routing parameters collectively as $\beta(t)$.

Material can flow into any section in the network from sources outside of the network. Let S be the number of sources supplying the network. The output of source s is represented by $d_s(t) \geq 0$. The fraction of the output of source s routed to section i at time t is denoted by $b_{si}(t)$, with $0 \leq b_{si}(t) \leq 1$ for all i, s and t, and $\sum_{i=1}^{n} b_{si}(t) = 1$ for all s and t.

Under routing strategy $\beta(t)$, the dynamics of section i is described by

$$\dot{x}_i(t) = -\frac{x_i(t)}{\tau_i} + \sum_{j:i\in\mathcal{O}_j} \beta_{ji}(t)\frac{x_j(t)}{\tau_j} + \sum_{s=1}^{S} b_{si}(t)d_s(t).$$

The dynamics of an n section network can thus be described by the following dynamical system

$$\begin{aligned} \dot{x}(t) &= A(\beta(t))x(t) + B(t)d(t) \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where the i^{th} row and s^{th} column of $B(t)$ is $b_{si}(t)$, the s^{th} row of $d(t)$ is $d_s(t)$, and

$$A(\beta(t)) = A_0 + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\beta_{ij}(t)}{\tau_i} e_j {e_i}^T \tag{2}$$

with $A_0 = \operatorname{diag}\left(-\frac{1}{\tau_1}, \ldots, -\frac{1}{\tau_n}\right)$. The unique solution of (1) under routing strategy $\beta(t)$ is designated by $x^{\beta}(t)$.

III. PROBLEM DESCRIPTION

A. Basic Control Design Objectives

In this work, we design routing parameters to satisfy the performance objectives discussed in Section III-B. In designing these control inputs, we must ensure that the resulting system satisfies the following constraints:

Positivity: System (1) is internally positive, i.e.,

$$x_0 \geq 0 \text{ and } d(t) \geq 0\ \forall t \geq 0 \Rightarrow x(t) \geq 0,\ \forall\ t \geq 0.$$

Conservation: For all $t \geq 0$,

$$\beta_{ij}(t) \geq 0,\ \forall\ i, j, \tag{3}$$

$$\beta_{ij}(t) = 0,\ \text{for } j \notin \mathcal{O}_i, \tag{4}$$

$$\sum_{j\in\mathcal{O}_i} \beta_{ij}(t) = 1,\ \forall\ i \notin \mathcal{S_F}. \tag{5}$$

Positivity ensures that each coordinate of state x, which represents the quantity of material present in a section, is non-negative at all times. Physically, the *Conservation* requirement expresses that material leaving every non-final section must be conserved.

The following sufficient and necessary conditions, which follow directly from Theorem 2 in [13], will be useful to ensure that positivity is satisfied.

Theorem 1: Let $A(\cdot), B(\cdot)$ be continuous matrix-valued maps with $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times S}$ for all $t \geq 0$. Then, the linear time-varying system

$$\dot{x}(t) = A(t)x(t) + B(t)d(t)$$

is internally positive *if and only if*

(i) $B(t) \in \mathbb{R}_+^{n\times S}$ for all $t \geq 0$,

(ii) matrix $\int_0^t A(s)ds$ is Metzler for all $t \geq 0$.

Because matrix $B(t)$ in system (1) is entry-wise non-negative for all t by construction, it follows from Theorem 1 that a sufficient condition for internal positivity of this system is that matrix $A(\beta(t))$ be Metlzer for all $t \geq 0$. Note that a matrix $A(\beta(t))$ of the form given in (2) is Metzler if and only if $\beta_{ij}(t) \geq 0$ for all i, j, which is required by the *Conservation* constraint.

B. Performance Control Design Objective

The control design objective is to generate routing parameters $\beta(t)$ to satisfy a time varying, piecewise constant, capacity constraint. This type of constraint often arises naturally, such as in ATFM problems. For example, weather conditions or other factors can affect airspace capacity. With coarse weather predictions, it is not reasonable to construct an exact time varying capacity constraint. Thus, in this application, available capacity is considered to be constant over some time interval of length ΔT. A similar type of piecewise constant capacity constraint is used, for example, in [8].

Our objective is to find routing parameters to drive the state of system (1) to remain below a given capacity constraint. Our solution to this problem is developed in several stages. First, we give sufficient conditions to ensure that the state of system (1) remains below some time varying capacity bound. We then focus on designing routing parameters to satisfy a linear capacity bound. We give a nonlinear relation that can be used to generate routing parameters $\beta(t)$ between the end points of the linear bound, if routing parameters can be found which satisfy certain constraints at the end points. Finally, we give linear constraints to generate a piecewise linear lower approximation to the given piecewise constant capacity constraint and associated routing parameters to ensure that system (1) remains below the piecewise linear capacity bound.

IV. LP ROUTING FOR TIME VARYING CAPACITY CONSTRAINTS

In [10] we give conditions to ensure that the state of a linear system with no input remains below some constant capacity bound. Here we extend the claim given in [10] to ensure that the state of system (1) remains below a continuous time varying capacity bound.

Claim 1: If $A(t)$ is Metzler, $c(t) > 0$, $A(t)c(t) + B(t)d(t) \leq \dot{c}(t)$, for all $t \geq 0$, and $x_0 \leq c(0)$, then the solution of system (1) satisfies $x(t) \leq c(t)$ for all $t \geq 0$.

Proof: Let $\xi(t) = c(t) - x(t)$. In order for $\xi(t)$ to remain positive for all $t \geq 0$, we must show that

$$\left.\begin{array}{ll} \xi_i(t) & = 0 \\ \xi(t) & \geq 0 \end{array}\right\} \Rightarrow \dot{\xi}_i(t) \geq 0.$$

Differentiating ξ we have

$$\begin{aligned} \dot{\xi}(t) &= \dot{c}(t) - (A(t)x + B(t)d(t)) \\ &\geq A(t)(c(t) - x(t)) \\ &= A(t)\xi(t). \end{aligned}$$

Since $A(t)$ is Metzler, if $\xi_i(t) = 0$ and $\xi(t) \geq 0$ then $[A\xi(t)]_i \geq 0$. In particular, $\dot{\xi}_i(t) \geq [A\xi(t)]_i \geq 0$. Thus, $\xi(t) \geq 0$ and $x(t) \leq c(t)$ for all $t \geq 0$. ∎

A. Linear Capacity Bound

We now concern ourselves with a single linear capacity bound over an interval of length T. We assume constant inflow and constant matrix B over this interval,

$$\begin{aligned} d(t) = d \geq 0 \text{ for all } 0 \leq t \leq T, \\ B(t) = B \in \mathbb{R}^n_+ \text{ for all } 0 \leq t \leq T. \end{aligned}$$

Let us also assume that the capacity constraint varies linearly according to

$$c(t) = b + tm \quad \text{for all } 0 \leq t \leq T, \tag{6}$$

where b and m are constant vectors in $\mathbb{R}^n$.

The following results show how to design routing strategies $\beta(t)$ that satisfy the positivity and conservation conditions and such that $x^{\beta}(t) \leq c(t)$ for all t.

Theorem 2: Let constraint vector $c(t)$ be given as in (6) and $x_0 \leq c(0)$. If there exist $\beta(0)$ and $\beta(T)$ such that constraints (3) - (5) are satisfied and

$$A(\beta(0))c(0) + Bd \leq m, \; A(\beta(T))c(T) + Bd \leq m, \tag{7}$$

then the parameters $\beta(t)$ defined by

$$\beta_{ij}(t) = \frac{\left(1 - \frac{t}{T}\right)\beta_{ij}(0)c_i(0) + \frac{t}{T}\beta_{ij}(T)c_i(T)}{\left(1 - \frac{t}{T}\right)c_i(0) + \frac{t}{T}c_i(T)}, \tag{8}$$

for all i, j and $0 \leq t \leq T$ are such that $A(\beta(t))$ is positive and conservative for all $0 \leq t \leq T$. In addition, $x^{\beta}(t) \leq c(t)$ for all $0 \leq t \leq T$.

Proof: First, note that since each $\beta_{ij}(t)$ is a convex combination of $\beta_{ij}(0)$ and $\beta_{ij}(T)$, it satisfies (3) - (5) whenever $\beta_{ij}(0)$ and $\beta_{ij}(T)$ do, thus the resulting system is conservative. This also implies that $A(\beta(t))$ is Metzler for all $0 \leq t \leq T$, thus the resulting system is positive. With $\beta(t)$ given by (8), let us define $G(t)$ as

$$G(t) = A(\beta(t))c(t) + Bd,$$

From (8) and the fact that $c(t)$ is linear, we find that

$$\begin{aligned} \frac{\beta_{ij}(t)c_i(t)}{\tau_i}e_j &= \frac{\beta_{ij}(t)\left[\left(1 - \frac{t}{T}\right)c_i(0) + \frac{t}{T}c_i(T)\right]e_j}{\tau_i} \\ &= \frac{\left(1 - \frac{t}{T}\right)\beta_{ij}(0)c_i(0) + \frac{t}{T}\beta_{ij}(T)c_i(T)}{\tau_i}e_j. \end{aligned}$$

Summing both sides over i, j and adding $A_0c(t) + Bd$ to both sides results in

$$G(t) = \left(1 - \frac{t}{T}\right)G(0) + \frac{t}{T}G(T).$$

In turn, $G(t) \leq \left(1 - \frac{t}{T}\right)m + \frac{t}{T}m = m$, which, according to Proposition 1, implies that $x^{\beta}(t) \leq c(t)$ for all $0 \leq t \leq T$. ∎

B. Piecewise Constant Capacity Bounds

With Theorem 2 in hand, we are now ready to design routing parameters to satisfy piecewise constant capacity bounds. However, some new notation must be introduced before we can proceed.

Let T be the length of the time interval of interest. In order to allow for flexibility in the solution, T will be divided into time intervals of two sizes, Δt and ΔT where $\Delta t \leq \Delta T$, ΔT is an integer multiple of Δt and T is an integer multiple of both Δt and ΔT. Define $K = \frac{T}{\Delta t}$, $t_k = k\Delta t$ for $k = 0, \ldots, K$. Using time steps of length Δt, T can be divided into intervals of the form $i_k = [t_k, t_{k+1})$ with $\cup_{k=0}^{K-1} i_k = [0, T)$. Similarly, $L = \frac{T}{\Delta T}$, $T_l = l\Delta T$ for $l = 0, \ldots, L$. Using time steps of length ΔT, T can be divided into intervals of the form $I_l = [T_l, T_{l+1})$ with $\cup_{l=0}^{L-1} I_l = [0, T)$.

We will be dealing with discontinuous functions, therefore for any function g we define

$$g(t_k^+) = \lim_{\substack{t \to t_k \\ t \geq t_k}} g(t) \quad \text{and} \quad g(t_k^-) = \lim_{\substack{t \to t_k \\ t \leq t_k}} g(t).$$

Recall that our goal is to find, when possible, a time-varying routing strategy $\beta(t)$ such that $A(\beta(t))$ is positive and conservative for all t and

$$x^\beta(t) \leq \bar{c}(t) \text{ for all } 0 \leq t \leq T. \tag{9}$$

The given constraint $\bar{c}$ is assumed to be constant over intervals I_l for $l = 0, \ldots, L$.

Note that neither Proposition 1 nor Theorem 2 can be used directly in this case, because function $\bar{c}$ is discontinuous from the left at T_l for every l. In particular, it is possible that

$$x^\beta(t) \leq \bar{c}(t) \text{ and } A(\beta(t))\bar{c}(t) + Bd(t) \leq 0 \text{ for all } t \in I_l$$

but that $x^\beta(T_{l+1}) > \bar{c}(T_{l+1}^+)$.

In order to design routing strategies $\beta(t)$ such that constraint (9) is satisfied, and guarantee that the inequality is enforced at points of discontinuity of $\bar{c}$, we proceed in two steps. First, we introduce a *continuous*, positive, piecewise linear function c such that

$$c(t) \leq \bar{c}(t) \text{ for all } 0 \leq t \leq T. \tag{10}$$

We parametrize this function as

$$c(t) = c(t_k) + (t - t_k)m(t_k^+)$$

for all $0 \leq t \leq T$ where $k = \lfloor \frac{t}{\Delta t} \rfloor$ and $m(t)$ is constant over intervals i_k for $k = 0, \ldots, K$. Condition (10) and the positivity requirement can be formulated as

$$0 \leq c(t_k) \leq \min\{\bar{c}(t_k^+), \bar{c}(t_k^-)\}.$$

Note that $\bar{c}(t_k^+) = \bar{c}(t_k^-)$ for all k, unless $t_k = T_l$ for some l. By defining $c(t)$ in this way and making the assumption that B is constant and $d(t)$ is constant over intervals i_k for $k = 0, \ldots, K$, the problem of satisfying piecewise constant capacity bounds becomes a series of problems of the form discussed in section IV-A.

Second, treating this piecewise linear under-approximation c as a free variable, we apply Theorem 2 over every interval $[t_k, t_{k+1}]$ ($k = 0, ..., K-1$) to design routing strategies such that $x^\beta(t) \leq c(t)$ for all $t_k \leq t \leq t_{k+1}$. Constraints of the form (7) involve terms which are nonlinear in free variables $\beta(t)$ and $c(t)$, namely $\beta_{ij}(t_k)c_i(t_k)$. These nonlinearities are removed by making the substitutions $z_{ij}(t_k) = \beta_{ij}(t_k)c_i(t_k)$ for all i, j, k. These parameters are computed by

1) Finding a feasible point for the following set of linear constraints, denoted by $\phi(\bar{c})$:

$$\begin{array}{rcl}
c(t_k) & \leq & \min\{\bar{c}(t_k^-), \bar{c}(t_k^+)\}, \forall\ k, \\
c(t_k) & \geq & 0, \forall\ k, \\
c(t_{k+1}) & = & c(t_k) + \Delta t m(t_k^+), \\
 & & k = 0, \ldots, K-1, \\
z_{ij}(t_k) & \geq & 0,\ \forall\ i, j, k, \\
z_{ij}(t_k) & \leq & c_i(t_k),\ \forall\ i, j, k, \\
z_{ij}(t_k) & = & 0,\ \forall\ i,\ \forall\ j \notin \mathcal{O}_i, \forall\ k, \\
\sum_{j=1}^n z_{ij}(t_k) & = & c_i(t_k), \forall\ i \notin \mathcal{S}_\mathcal{F}, \forall\ k, \\
Bd(t_k^+) & + & A_0 c(t_k) + \sum_{i=1}^n \sum_{j=1}^n \frac{z_{ij}(t_k)}{\tau_i} e_j \\
 & \leq & m(t_k^+), \forall\ k, \\
Bd(t_{k+1}^-) & + & A_0 c(t_{k+1}) + \sum_{i=1}^n \sum_{j=1}^n \frac{z_{ij}(t_{k+1})}{\tau_i} e_j \\
 & \leq & m(t_k^+), k = 0, \ldots, K-1.
\end{array}$$

2) Recovering $\beta(t_k)$ according to

$$\beta_{ij}(t_k) = \frac{z_{ij}(t_k)}{c_i(t_k)}, \forall\ i, j, k.$$

3) Interpolating non-linearly between $\beta_{ij}(t_k)$ and $\beta_{ij}(t_{k+1})$ according to

$$\beta_{ij}(t) = \frac{\left(1 - \frac{t-t_k}{\Delta t}\right)\beta_{ij}(t_k)c_i(t_k) + \frac{t-t_k}{\Delta t}\beta_{ij}(t_{k+1})c_i(t_{k+1})}{\left(1 - \frac{t-t_k}{\Delta t}\right)c_i(t_k) + \frac{t-t_k}{\Delta t}c_i(t_{k+1})},$$

where $k = \lfloor \frac{t}{\Delta t} \rfloor$.

The resulting routing strategy is continuous over $[0, T)$ since function c is continuous by assumption.

If linear constraints $\phi(\bar{c})$ are infeasible, it is natural to try to alter the desired capacity $\bar{c}$ so as to find a feasible solution, while ensuring that the resulting bounds are close to $\bar{c}$ in some sense. In such a case, we allow $\bar{c}$ to be increased to $\hat{c}$ over intervals of length Δt in order to achieve feasible piecewise linear bounds below $\hat{c}$. The problem of finding capacity constraints $\hat{c}$ and piecewise

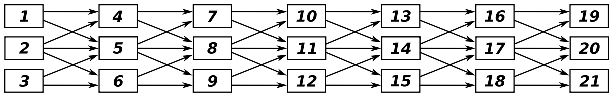

Fig. 1. Network of interconnected sections. Material flows into the network at sections 1, 2 and 3, material exits the network from sections 19, 20, 21, i.e., $\mathcal{S}_{\mathcal{F}} = \{19, 20, 21\}$.

linear bounds c can then be written as the following LP:

$$\begin{array}{rl} \min & \sum_{k=0}^{K} \sum_{i=1}^{N} \left(\hat{c}_i(t_k^+) - \bar{c}_i(t_k^+) \right) \Delta t \\ \text{subject to} & \phi(\hat{c}(t)) \\ & \hat{c}(t_k^+) \geq \bar{c}(t_k^+), \ k = 0, \ldots, K \end{array} \tag{11}$$

V. APPLICATION

Here we give the details and results of simulated problems making use of the proposed control design method. We then discuss some issues that arise in the application of this control method to an ATFM problem.

A. Application Example

In order to illustrate the proposed control method, we applied this routing design technique to the compartmental system depicted in Figure 1. We chose a traversal time $\tau_i = 0.4$ hours for every $i = 1, \ldots, 21$, to agree with typical orders of magnitude encountered in the air traffic management literature [4]. The connectivity of the network can be inferred from the diagram.

The inflow rate of sections 1 and 3 is set to 25 material units per hour, while the inflow rate of section 2 is equal to 30 material units per hour. The initial conditions were set to 10 material units for all sections in the top and bottom rows and 12 for all sections in the middle row. With this inflow and initial conditions, the state of every section remains constant when flows are routed along the rows of the network (i.e., when $\beta_{1,4} = \beta_{4,7} = \beta_{7,10} = \beta_{10,13} = \beta_{13,16} = \beta_{16,19} = 1$ and similar equalities hold for the second and third rows).

Each section except 14 has a constant capacity of fifteen, i.e., $\bar{c}_i(t) = 15$ material units for all $i \neq 14$ and all $t \geq 0$. Section 14, on the other hand, has the piecewise constant capacity profile pictured in Figure 2, where each base interval has length $\Delta t = 15$ min.

Based on this profile, the linear constraints $\phi(\bar{c})$ formulated using this value of Δt were found it to be feasible. The corresponding routing parameters for selected sections are plotted in Figure 3. Notice that in sections 1, 2 and 3, well upstream of the capacity constrained section, the majority of the section outflow is routed to the upper and lower sections of the graph. Closer to the capacity constrained section, in sections 10, 11 and 12, a larger portion of the section outflow is routed to the upper and lower sections of the graph.

In a second example, we imposed the capacity profile pictured in Figure 4 on section 14. In this case, we found

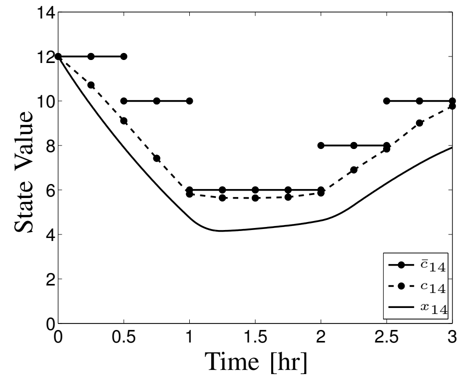

Fig. 2. Piecewise constant capacity constraint $\bar{c}_{14}$, piecewise linear capacity bound c_{14} and state x_{14}.

that linear constraints $\phi(\bar{c})$ are not feasible. Thus, LP (11) was used to find constraints $\hat{c}$ for which a feasible solution could be found. The integral of the difference between $\hat{c}$ and $\bar{c}$ is 0.84 (material units) $\times$ hour. That is, using this solution, the actual section count will be above the constraint $\bar{c}$ by no more than an average of 0.84 material units over a one hour time period.

Notice that in both Figures 2 and 4 the state $x_{14}(t)$ does not match the capacity bound $c_{14}(t)$ for most of the simulation. This is because the use of Claim 1 to ensure that $x(t) \leq c(t)$ implicitly assumes that the state of the system is at capacity at all times, which is conservative, resulting in the gap between the $x_{14}(t)$ and $c_{14}(t)$.

B. Use of Aggregate Models for ATFM Applications

Although the objective of the present paper is not specifically to solve ATFM problems, the methods we present are applicable to classes of such problems provided the points addressed below are taken into account.

An underlying assumption in the development of aggregate models is that the state of each section is infinitely divisible. In an ATFM application, the state represents the number of aircraft in each section, which is a discrete quantity. However, these models have been shown to accurately describe the flow of aircraft in dense traffic [5], [6]. The implementation of an aggregate model and control method for ATFM problems requires the use of some disaggregation method to translate the aggregate control input to a flight-by-flight control input. As suggested in [14], rounding heuristics can be used to generate integer values of control inputs when needed.

The identity of each aircraft is lost in an aggregate model. Thus, when routing is used as a control parameter, all aircraft involved in the problem must have the same destination airport. Such a problem is proposed, for example, in [15]. If multiple destinations are required, traffic flow can be aggregated based on destination.

In current operations, human air traffic controllers

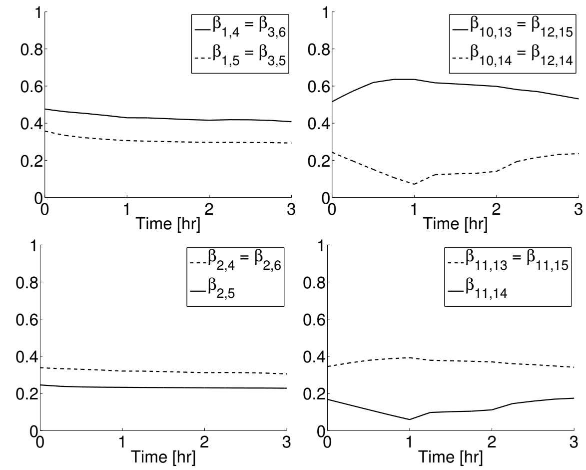

Fig. 3. Routing parameters associated with results plotted in Figure 2. Note that, due to symmetry of the problem, several of the routing parameters have identical profiles. Also recall that $\beta_{ij}(t)$ must sum to 1 for each section, and thus recirculation accounts for the remainder of the flow routing (i.e. $\beta_{1,1}(t) = 1 - \beta_{1,4}(t) - \beta_{1,5}(t)$, etc.).

direct individual aircraft. In order to achieve the control design objective, air traffic controllers can use the routing parameters generated by the proposed method as a guideline when determining air traffic control commands for individual aircraft.

VI. CONCLUSIONS AND FUTURE WORK

We presented sufficient conditions which can be used to design time varying routing parameters to ensure that the state of a positive conservative system remains below a specified piecewise linear capacity bound. We also give an LP which can be used to adjust capacity constraints in order to find a feasible routing solution. Resulting routing parameters satisfy the specified capacity constraints (or adjusted constraints) while ensuring that the resulting system is positive, conservative and exhibits the desired interconnection of sections.

The proposed solution method makes use of a deterministic capacity forecast. However, it is more realistic to assume that an uncertain forecast is available, with more accurate updates available as time progresses. For example, in an ATFM application airspace capacity is related to uncertain weather forecasts. In future work we will incorporate probabilistic capacity forecasts. We will adapt this method into a receding horizon method in which capacity constraints are revealed over time.

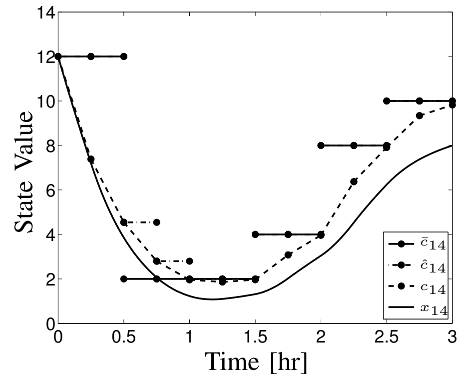

Fig. 4. Piecewise constant capacity constraint $\bar{c}_{14}$, adjusted capacity constraint $\hat{c}_{14}$, piecewise linear capacity bound c_{14} and state x_{14}

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