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On the Extremum Seeking of Model Reference Adaptive Control in Higher-Dimensional Systems

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***Abstract*— We develop a method for the model reference adaptive control (MRAC) of first order systems via extremum seeking. We show, in special cases in which there exists a partial knowledge of parameter values, proofs of global and exponential convergence of both the tracking and parameter tracking errors. We then extend the proposed method from first order systems to linear systems of any higher dimensions. We prove that our method has similar properties as MRAC. Results are partially illustrated through simulations of a simplified roll rate model of a fixed wing aircraft.**

I. INTRODUCTION

Extremum seeking control is a non-model based control for systems with an extremum in their reference-to-ouput maps [1]. While the mainstream methods of adaptive control [2], [3], [4], [5] are only applicable for regulation to *known* reference trajectories, extremum control enables the designer to keep the output at the extremum value of the reference-to-output map. In order to resolve the issues resulting from uncertainties in the reference-to-output map, extremum control provides an adaptation scheme to find the set point which extremizes the output.

Adaptive control schemes [5], [6], [7] provide exponential stability of the homogenous error system under conditions of persistency of excitation. But the size of the exponents depends upon the initial conditions of parameter estimation error, and hence *a priori* exponential bounds on convergence are unavailable, though several characterizations and bounds of transient performance exist [8], [5]. Similarly, estimates of stability margins also exist [9], [10] though these are different than those available from exponentially stable systems. While extremum seeking [1] provides predictable performance, it only adapts the set point of a control system. In the special case of set point adaptation, extremum seeking permits predictable parameter convergence by design. This is because persistency of excitation requirements are met by the sinusoidal perturbation that is part of the basic control design.

The idea behind the extremum seeking control can be employed in conjunction with adaptive control by defining a convex optimization cost function, which leads to a novel approach for dealing with uncertainties. This was done in our previous work [11], where we introduced Extremum Seeking Model Reference Adaptive Control (ES-MRAC), a scheme for adapting a model reference control law via extremum seeking. The ES-MRAC introduces nonlinearities in the system through a convex cost function. The minimization of this cost function provides the necessary adaptation to account for plant and control uncertainties. In this work, we generalize these results with rigorous analysis and design theorems. In particular, we shall examine the stability proofs for linear first order systems with plant and control uncertainties. We supply simulations that show parameter convergence of the adaptation conforming to predictions from the theory of extremum seeking. The results in this paper point to the possibility of controlling the convergence rates of the parameters in adaptive control with a time varying adaptive controller.

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A. Outline of Paper

Section I introduced the basic idea of extremum seeking adaptive control and the current literature. Section II discusses a rigorous analysis and proof for the stability of first order systems under the presented control scheme. Section III presents a general theory and design criteria for the use of ES-MRAC in general linear systems of higher dimensions. Finally, section IV demonstrates the effectiveness of the proposed scheme through simulations. The paper ends with some of the ideas to be addressed in the future in section V.

II. FIRST ORDER SYSTEMS: ANALYSIS AND DESIGN

We begin by introducing the ES-MRAC method for a first order system - first proposed in [11] - followed up by a rigorous analysis of the proposed scheme, in which we apply the method of averaging in order to convert the nonautonomous equations to autonomous counterparts. Finally, we shall conduct a stability analysis on the averaged system, using Lyapunov functions.

A. General setting

Consider a first order system, governed by the following dynamics

$$\dot{x} = ax + bu, \tag{1}$$

where the parameters a and b are assumed to be constant but unknown to the designer. The objective is to desing a control law

$$u = k_x x + k_r r, \tag{2}$$

such that the state x will follow the dynamics of a model reference, despite of the uncertainty in a and b. The model reference is given by

$$\dot{x}_m = a_m x_m + b_m r. \tag{3}$$

One can easily verify that the substitution of the ideal values

$$k_x^* = \frac{a_m - a}{b} \quad k_r^* = \frac{b_m}{b} \tag{4}$$

into (1) will yield the same dynamics as the model reference, but the uncertainty in the parameters a and b requires some sort of adaptation. This adaptation is carried out in extremum seeking, by perturbing the input signal to a proper convex cost function. The proposed scheme is shown in the block diagram in Fig. 1. We shall prove that the ES-MRAC method as presented by this block diagram, will lead to tracking of the model reference, despite the uncertainties in a and b. To examine the behavior of the system under the control algorithm shown in Fig. 1, we begin by deriving the differential equations governing the error dynamics. To this end, the state tracking error and the parameter tracking error are defined as follows

$$e \triangleq x - x_m \tag{5}$$
$$\tilde{k}_x \triangleq k_x^* - k_x + k_{x0} \tag{6}$$
$$\tilde{k}_r \triangleq k_r^* - k_r + k_{r0}, \tag{7}$$

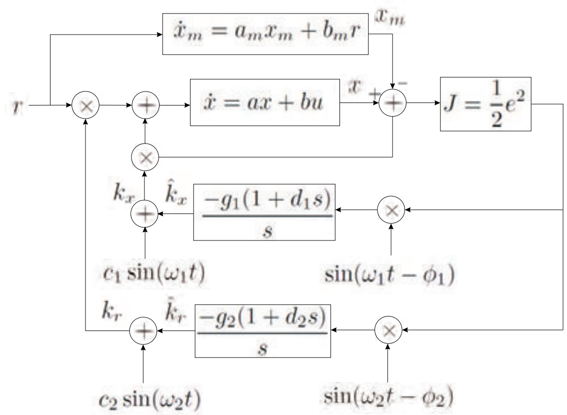

Fig. 1. ES-MRAC control algorithm.

where $k_{x0} = c_1 \sin \omega_1 t$ and $k_{r0} = c_2 \sin \omega_2 t$ are the perturbation signals as shown in Fig. 1. The perturbation frequencies must be sufficiently large (in a qualitative sense) to permit the application of the averaging method. The perturbation amplitude must be chosen so as to produce a measurable change in the plant output.

Substitution of (6) and (7) into (1) and (2), and employing the results in (5) will yield the governing equation for the tracking error, which after simplification becomes

$$\dot{e} = \left(a_m + b(k_{x0} - \tilde{k}_x)\right) e + b\left(k_{x0} - \tilde{k}_x\right) x_m + b\left(k_{r0} - \tilde{k}_r\right) r \tag{8}$$

In order to analyze the system, we also need the equations governing the parameter estimations. One can see from Fig. 1 that

$$\hat{k}_x = -g_1 \frac{(1 + d_1 s)}{s} [\sin(\omega_1 t - \phi_1) J] \tag{9}$$

$$\hat{k}_r = -g_2 \frac{(1 + d_2 s)}{s} [\sin(\omega_2 t - \phi_2) J] \tag{10}$$

where J is a convex cost function to be defined. Since the ideal control gain, k^*, is constant we have $\dot{\tilde{k}} = -\dot{\hat{k}}$. Thus, the last two equations will simplify to

$$\dot{\tilde{k}}_x = g_1 \left[d_1 \left(\omega_1 \cos(\omega_1 t - \phi_1) J + \sin(\omega_1 t - \phi_1) \dot{J} \right) + \sin(\omega_1 t - \phi_1) J \right] \tag{11}$$

$$\dot{\tilde{k}}_r = g_2 \left[d_2 \left(\omega_2 \cos(\omega_2 t - \phi_2) J + \sin(\omega_2 t - \phi_2) \dot{J} \right) + \sin(\omega_2 t - \phi_2) J \right], \tag{12}$$

which constitute the equations governing parameter tracking errors.

Obviously, the governing equations are nonautonomous. Thus, we shall use the method of averaging to find an averaged equivalent system which is autonomous. Once we find the averaged system, we can use standard methods for time invariant systems to analyze the stability of equilibria.

B. The method of averaging

The basic idea behind the method of averaging is that one integrates a set of nonautonomous equations over a time period, which is called the averaging period, and then uses the resulting autonomous equations instead; thereby, a simplification in the analysis occurs [12]. Certain requirements must hold for the procedure to be valid, which we shall discuss in detail as we provide the analysis.

In order to be able to apply the method of averaging to our system, we must first perform a scaling of time. Let ω be the greatest common factor of ω_1 and ω_2. Thus we can write $\omega_1 = p\omega$ and $\omega_2 = q\omega$, where p and q are integers. Let $\tau = \omega_1 t$, where $\omega_1 \gg 1$. Using this time scale, the governing equations can be written as

$$\frac{dx_m}{d\tau} = \varepsilon[a_m x_m + b_m r] \tag{13}$$

$$\frac{de}{d\tau} = \varepsilon \left[\left(a_m + b(c_1 \sin \tau - \tilde{k}_x) \right) e + b\left(c_2 \sin \frac{q}{p}\tau - \tilde{k}_r \right) r \right] + b\left(c_1 \sin \tau - \tilde{k}_x \right) x_m$$

$$\frac{d\tilde{k}_x}{d\tau} = \varepsilon g_1 \left[d_1 \omega_1 \left(\cos(\tau - \phi_1) J + \sin(\tau - \phi_1) \frac{dJ}{d\tau} \right) + \sin(\tau - \phi_1) J \right]$$

$$\frac{d\tilde{k}_r}{d\tau} = \varepsilon g_2 \left[d_2 \omega_1 \left(\frac{q}{p} \cos(\frac{q}{p}\tau - \phi_2) J + \sin(\frac{q}{p}\tau - \phi_2) \frac{dJ}{d\tau} \right) + \sin(\frac{q}{p}\tau - \phi_2) J \right] \tag{14}$$

where we have defined ε to be the reciprocal of ω_1. We shall further assume that $O(\omega_1 d_1) = O(\omega_2 d_2) = 1$, where O denotes the order, and that the reference input r is of constant value. Given these assumptions, (13) to (14) can be summarized in the form

$$\frac{dx_i}{d\tau} = \varepsilon f_i(\tau, x, \varepsilon) \quad i = 1, \ldots, 4 \tag{15}$$

where f_i and their partial derivatives with respect to (x, ε) up to the second order are continuous and bounded for all $t \in [0, \infty)$, and for all $x \in D_0$, where D_0 is any compact set $D_0 \subset \mathbb{R}^4$. The averaged autonomous system associated with (15) is given by $\dot{x} = \varepsilon f_{\text{av}}(x)$ where

$$f_{\text{av}}(x) = \frac{1}{T} \int_0^T f(s, x, \varepsilon) ds \tag{16}$$

Using $T = 2\pi p$ as the period of averaging, and $J = \frac{1}{2}e^2$ as the cost function, we can find the averaged equations after a long sequence of calculations as follows

$$\left(\frac{d\tilde{k}_x}{d\tau}\right)_{\text{av}} = \frac{1}{2}\varepsilon \left[g_1 d_1 b c_1 \cos \phi_1 e(e + x_m) \right] \tag{17}$$

$$\left(\frac{d\tilde{k}_r}{d\tau}\right)_{\text{av}} = \frac{1}{2}\varepsilon \left[g_2 d_2 b c_2 \cos \phi_2 e r \right] \tag{18}$$

$$\left(\frac{de}{d\tau}\right)_{\text{av}} = \varepsilon \left[(a_m - b\tilde{k}_x) e - b\tilde{k}_x x_m - b\tilde{k}_r r \right] \tag{19}$$

$$\left(\frac{dx_m}{d\tau}\right)_{\text{av}} = \varepsilon \left[a_m x_m + b_m r \right] \tag{20}$$

We shall now study the stability of the system via the averaged set of equations, to obtain more insight into the behavior of the system under the proposed control scheme.

One can see that the only equilibrium point of (17) to (20) is the point

$$(e, x_m, \tilde{k}_x, \tilde{k}_r)_{\text{eq}} = (0, -\frac{b_m}{a_m} r, k, k\frac{b_m}{a_m}) \tag{21}$$

where $k \in \mathbb{R}$ can be any real number. This equilibrium point pertains to zero state tracking error which is desirable to us. One can show that the eigenvalues of the Jacobi matrix are given by

$$\lambda_1 = 0 \tag{22}$$

$$\lambda_2 = a_m \tag{23}$$

$$\lambda_{3,4} = \left(\frac{a_m - bk}{2}\right) \pm \left[\left(\frac{a_m - bk}{2}\right)^2 - \frac{g_2 d_2 b^2 c_2 \cos \phi_2 r^2}{2} \right]^{\frac{1}{2}} \tag{24}$$

In order for the system to be stable, we need $\lambda_i < 0$. Note that

$$\lambda_3\lambda_4 = \frac{1}{2} g_2 d_2 b^2 c_2 \cos\phi_2 r^2 > 0.$$

Thus, it is sufficient to only check the sign of either λ_3 or λ_4, for they have the same sign. This analysis is useful in the sense that it provides the rate of convergence of tracking error, and also a criterion on how to choose the design parameters so as to make $\lambda_{3,4}$ negative. Since λ_1 cannot become negative for any values of design parameters, the question remains whether trajectories will converge to this equilibrium and will zero tracking error be achieved. Therefore in the next section, we provide a stability analysis using Lyapunov functions, to study stability in more detail.

C. Analysis using Lyapunov function

Consider the following Lyapunov function

$$V = \frac{1}{2}e^2 + \frac{1}{2}\tilde{k}^T P \tilde{k}, \tag{25}$$

where $P \in \mathbb{R}^{2\times 2}$ is a symmetric positive definite matrix to be defined. In scalar form

$$V = \frac{1}{2}e^2 + \frac{1}{2}P_{11}\tilde{k}_x^2 + P_{12}\tilde{k}_x\tilde{k}_r + \frac{1}{2}P_{22}\tilde{k}_r^2 \tag{26}$$

For simplicity, let $P_{12} = 0$. Since ε appears in all of the equations, without loss of generality, we assume that $\varepsilon = 1$. Taking the time derivative of V (with respect to τ), and substituting the averaged equations from (17) to (20) would yield

$$V' = (a_m - b\tilde{k}_x)e^2 - b\tilde{k}_x x_m e - b\tilde{k}_r r e \tag{27}$$

$$+\frac{1}{2}P_{11} g_1 d_1 b c_1 \cos\phi_1 \tilde{k}_x (e^2 + e x_m) \tag{28}$$

$$+\frac{1}{2}P_{22} g_2 d_2 b c_2 \cos\phi_2 \tilde{k}_r e r \tag{29}$$

where we have used the prime sign to denote differentiation with respect to τ to avoid ambiguity. One can see that by setting

$$P_{11} = \frac{2}{g_1 d_1 c_1 \cos\phi_1}, \quad P_{22} = \frac{2}{g_2 d_2 c_2 \cos\phi_2}, \tag{30}$$

the matrix P will be positive definite, and

$$V' = a_m e^2 \tag{31}$$

will be negative semi-definite since $a_m < 0$ by assumption. This means that the system is stable in the sense of Lyapunov, but will the tracking error converge to zero? We shall now apply the following form of Barbalat's lemma from [13], to examine the convergence of the tracking error.

Lemma 2.1: If a scalar function $V(x,t)$ is lower bounded, $\dot{V}(x,t)$ is negative semi-definite, and $\dot{V}(x,t)$ is uniformly continuous in time, then $\dot{V}(x,t) \to 0$ as $t \to \infty$.

Since the Lyapunov function as given in (26) is positive definite, it is lower bounded, and as shown above, V' is negative semi-definite. Therefore, $V(t) \le V(0), \forall t$. In other words, e, $\tilde{k}_x$ and $\tilde{k}_r$ are bounded. Thus we only have to check the uniform continuity of V. A sufficient condition for uniform continuity of a function is that its derivative be bounded. The derivative of V is given by

$$V'' = 2a_m e\left[(a_m - b\tilde{k}_x)e - b\tilde{k}_x x_m - b\tilde{k}_r r\right] \tag{32}$$

This shows that V'' is bounded since e, $\tilde{k}_x$ and $\tilde{k}_r$ were shown to be bounded. Hence V' is uniformly continuous. Application of Lemma 2.1 will yield $e \to 0$.

A keen observation is that although the state tracking error converges to zero for proper design values, the parameter tracking errors can have non-zero values, i.e. the unknown parameters in the system may converge to values, other than their ideal value while the state tracking error still converges to zero. Simulation results attest to this observation as seen in section IV.

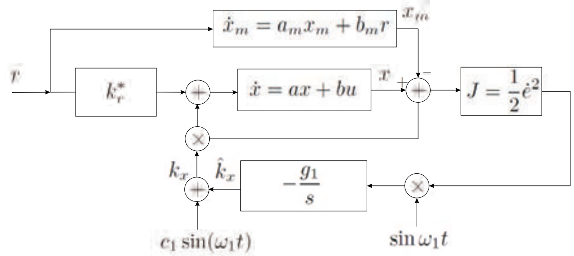

Fig. 2. ES-MRAC control algorithm for the case where b is known.

In what follows we will study two special cases, where b is known and only a is unknown, and vice versa. We shall see that the proof of parameter convergence exists for these special cases.

D. Special Cases

1) b known, a unknown: For this particular case of interest, it can be proved that the Lyapunov function becomes negative definite, and we can estimate the rate of convergence using comparison lemma (see [14]). The result is stated in the following proposition.

Proposition 2.1: For the ES-MRAC system in Fig. 2, the model reference error e and the parameter tracking error $\tilde{k}_x = k_x^* - k_x + c_1 \sin\omega_1 t$ converge globally and exponentially to an $O(1/\omega_1)$ neighborhood of the origin provided the probing frequency ω_1 is sufficiently large. Moreover, the exponent of convergence is at least as fast as the pole in the reference model a_m.

We will show the validity of the above proposition as follows: Since b is known, we have

$$k_r = k_r^* = \frac{b_m}{b} \tag{33}$$

As a result, we don't need to estimate k_r in our extremum seeking block, and the control input simply becomes $u = k_x x + \frac{b_m}{b} r$. Thus, the error dynamics simplifies to the following

$$\dot{e} = \dot{x} - \dot{x}_m \tag{34}$$

$$= [(a + bk_x)x + b_m r] - [a_m x_m + b_m r] \tag{35}$$

which after substituting the value of k_x and simplification becomes

$$\dot{e} = a_m e + b(c_1 \sin\omega_1 t - \tilde{k}_x)x \tag{36}$$

Using $\tau = \omega_1 t$ as the time scale, the averaged error dynamics becomes

$$\left(\frac{de}{d\tau}\right)_{\text{av}} = \frac{1}{\omega_1}\left[(a_m - b\tilde{k}_x)e - b\tilde{k}_x x_m\right] \tag{37}$$

As shown in Fig. 2 the estimation dynamics will be given by

$$\hat{k}_x = -\frac{g_1}{s}\left[\sin\omega_1 t \ J\right] \tag{38}$$

Using $J = q_1 \dot{e}^2$ as the cost function, we get

$$\dot{\tilde{k}}_x = g_1 q_1 \sin\tau \dot{e}^2 \tag{39}$$

Substituting from the error dynamics and performing averaging we find

$$\left(\frac{d\tilde{k}_x}{d\tau}\right)_{\text{av}} = \frac{1}{\omega_1} b c_1 g_1 q_1 \Big[e^2(a_m - b\tilde{k}_x) - b\tilde{k}_x x_m^2 + e x_m(a_m - 2b\tilde{k}_x)\Big] \tag{40}$$

Obviously, Eqs. (37) and (40) have the equilibrium point at the origin, which is desirable. We shall now, perform a Lyapunov analysis to examine the stability properties of this equilibrium. Consider the following Lyapunov function

$$V = \frac{\omega_1}{2}\left(e^2 + \gamma \tilde{k}_x^2\right) \tag{41}$$

Taking the derivative of V with respect to τ we get

$$\frac{dV}{d\tau} = \omega_1 e \frac{de}{d\tau} + \omega_1 \gamma \tilde{k}_x \frac{d\tilde{k}_x}{d\tau} \tag{42}$$

which after substitution of the governing dynamics and grouping similar terms becomes

$$\begin{aligned}\frac{dV}{d\tau} &= a_m e^2 + \overbrace{[\gamma b c_1 g_1 q_1 a_m - b]}^{\text{can set to 0}}(\tilde{k}_x e^2 + \tilde{k}_x e x_m) \\ &\quad - \gamma b^2 c_1 g_1 q_1 \left[(e\tilde{k}_x)^2 + (\tilde{k}_x x_m)^2 + 2\tilde{k}_x^2\right]\end{aligned} \tag{43}$$

Thus, by proper choice of design parameters to satisfy the condition

$$\gamma b c_1 g_1 q_1 a_m - b = 0 \tag{44}$$

we get $\frac{dV}{d\tau} < 0$, which shows that $e \to 0$ and $\tilde{k}_x \to 0$. In other words, not only does the tracking error goes to zero, but also the parameters will reach their actual values.

Next, we will estimate the exponential rate of convergence, using comparison lemma. To this end, we perform the following calculations

$$\begin{aligned}\frac{dV}{d\tau} &= a_m e^2 - \gamma b^2 c_1 g_1 q_1 [e^2 + x_m^2 + 2]\tilde{k}_x^2 \\ &\le a_m e^2 - 2\gamma b^2 c_1 g_1 q_1 \tilde{k}_x^2\end{aligned} \tag{45}$$

The last inequality can be written in terms of V as follows

$$\frac{dV}{d\tau} \le \left(\frac{2a_m}{\omega_1}\right) V - \tilde{k}_x^2 [2b^2 c_1 g_1 q_1 + a_m \gamma] \tag{46}$$

Thus, if we choose γ such that $2b^2 c_1 g_1 q_1 + a_m \gamma > 0$, then

$$\frac{dV}{d\tau} \le \left(\frac{2a_m}{\omega_1}\right) V \tag{47}$$

Hence $V(\tau)$ satisfies the differential inequality (47), with the initial condition $V(0) = \omega_1/2e(0)^2 + \gamma/2\tilde{k}_x^2$. Thus by comparison lemma

$$V(t) \le \left[\frac{\omega_1}{2} e(0)^2 + \frac{\omega_1 \gamma}{2}\tilde{k}_x^2\right] e^{2\frac{a_m}{\omega_1}t}, \tag{48}$$

and our proof is complete.

2) a known, b unknown: This is a very interesting case that can happen in many applications, e.g. degradation of actuator. We shall formulate the problem for any arbitrary value of a, and then narrow the results to the cases where $a = a_m$.

Since the value of b is unknown, we don't have the value for k_x^*, and thus the analysis is not as straight forward as the previous case. Instead, we propose the following. We define $\Delta \triangleq \frac{k_x^*}{k_r^*} = \frac{a_m - a}{b_m}$ and supply the following result:

Proposition 2.2: For the ES-MRAC system in Figure 3, the model reference error e *and the parameter tracking error* $\tilde{k}_r = k_r^* - k_r + c_1 \sin \omega_1 t$ *converge globally and exponentially to an neighborhood of the origin provided the probing frequency is sufficiently large. Moreover, the exponent of convergence is at least as fast as the pole in the reference model* a_m.

This result is arrived at in a similar fashion to Proposition 2.1. Thus, we shall not discuss further to avoid redundancies.

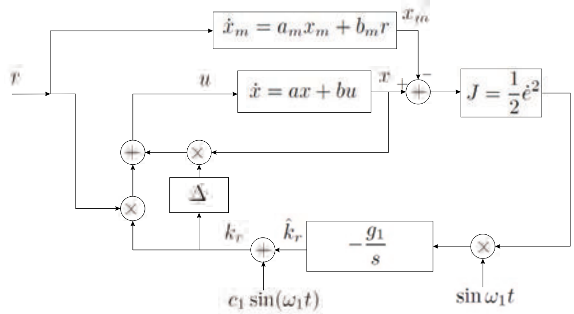

Fig. 3. ES-MRAC control algorithm for the case where a is known.

III. Generalization to Higher Order Systems

In this section, we generalize the results to a single input multi output (SIMO) dynamic system of order n. We shall show that with a little bit of modification in the adaptation procedure, a linear system of an arbitrary order n can be stabilized, and perfect tracking is achieved (in the average sense) for all of the states. Without loss of generality, we assume that the state equations are written in controllable canonical form, as given by

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y = u, \tag{49}$$

where $y^{(i)} \in \mathbb{R}$ are measurable states. The result is summarized as follows.

Proposition 3.1: For a linear system of order n *given by (49), the ES-MRAC system as shown in Fig. 4, will guaranty global and exponential convergence of the tracking error vector* $\left[e, \dot{e}, \ldots, e^{(n-1)}\right]$, *to a neighborhood of the origin, provided that the probing frequency is sufficiently large.*

We shall now explain this proposition in more detail. Assume that all a_i's are unknown. Note that the usual MRAC restricts the sign of a_n to be known, where as this methodology puts no restriction on the sign of a_n; hence less conservative.

Let the greatest common factor of $\omega_i, i = 0, \ldots, n$ be denoted by ω. In other words, $\omega_i = n_i \omega$, where $n_i \in \mathbb{N}$, $i = 0, \ldots, n$. Furthermore, assume that $n_i \neq n_j$ if $i \neq j$. Let the design parameters be chosen such that

$$\omega_0 \gg 1 \tag{50}$$

$$O(d_i \omega_i) = 1, \quad i = 0, \ldots, n \tag{51}$$

$$O(g_i) = 1, \quad i = 0, \ldots, n \tag{52}$$

Suppose that the objective is to design a control law to track the model reference

$$a_{mn} y_m^{(n)} + a_{m(n-1)} y_m^{(n-1)} + \ldots + a_{m0} y_m = r(t), \tag{53}$$

in the presence of parameter uncertainties. Define a signal $z(t)$ as follows

$$z(t) \triangleq y_m^{(n)} - \beta_{n-1} e^{(n-1)} - \ldots - \beta_0 e \tag{54}$$

with the design paramters β_i chosen such that the polynomial $p^n + \beta_{n-1} p^{(n-1)} + \ldots + \beta_0$ is Hurwitz. Let the control input be as shown in Fig. 4. This control input can be written as

$$u = \breve{a}_n z + \breve{a}_{n-1} y^{(n-1)} + \ldots + \breve{a}_0 y \tag{55}$$

where $\breve{a}_i$ denotes the estimation of the parameter a_i after being perturbed by the sinusoidal signal (see Fig. 4). Substituting this control input into the governing dynamics, one can show that the error dynamics is given by

$$\begin{aligned}& a_n[e^{(n)} + \beta_{n-1} e^{(n-1)} + \ldots + \beta_0 e] \\ &\quad = \sum_{k=0}^{n-1} (c_k \sin \omega_k t - \tilde{a}_k) y^{(k)} + (c_n \sin \omega_n t - \tilde{a}_n) z(t)\end{aligned}$$

Defining the state variables as $x_i = e^{(i-1)}$, $i = 1, \ldots, n$, one can write the error dynamics as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \frac{1}{a_n}\mathbf{b}\mathbf{v}^T[\mathcal{S} - \tilde{\mathbf{a}}] \tag{56}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\beta_0 & -\beta_1 & -\beta_2 & \ldots & -\beta_{n-1} \end{bmatrix}, \tag{57}$$

$\mathbf{b} = [0, 0, \ldots, 0, 1]^T$, $\tilde{\mathbf{a}} = [\tilde{a}_0, \ldots, \tilde{a}_n]$, $\tilde{a}_i = a_i - \hat{a}_i$, and $\mathcal{S}$ denotes the vector of perturbation signals given by $\mathcal{S} = [c_0 \sin \omega_0 t, \ldots, c_n \sin \omega_n t]^T$. In addition, the vector $\mathbf{v}$ is the regression vector given by $\mathbf{v} = [y, \dot{y}, \ldots, y^{(n-1)}, z]^T$. Next, we shall use a scaling of time, in order to make the equations suitable for averaging.

Suppose that $\tau = \omega_0 t$, and define $\varepsilon = 1/\omega_0$. One can show that the averaged equation for error dynamics is given by

$$\left(\frac{d\mathbf{x}}{d\tau}\right)_{av} = \varepsilon \left[\mathbf{A}\mathbf{x} - \frac{1}{a_n}\mathbf{b}\mathbf{v}^T\tilde{\mathbf{a}}\right] \tag{58}$$

As seen in Fig. 4, the parameter estimation error is governed by

$$\dot{\tilde{a}}_i = g_i(1 + d_i s)[\sin(\omega_i t - \phi_i)J], \quad i = 0, \ldots, n \tag{59}$$

We shall employ the cost function

$$J = \frac{1}{2}\left[\mathbf{q}^T\mathbf{x}\right]^2 = \frac{1}{2}\left[\sum_{k=1}^{n} q_i x_i\right]^2 \tag{60}$$

via the extremum seeking block as shown in Fig. 4, where $\mathbf{q} = [q_1, \ldots, q_n]^T$ are weighting factors for the cost function.

One can show that the averaged equations governing the parameter estimation errors is given by

$$\left(\frac{d\tilde{a}_i}{d\tau}\right)_{av} = \varepsilon g_i c_i d_i \cos\phi_i v_{i+1} \frac{q_n}{2a_n}\mathbf{q}^T\mathbf{x}, \quad i = 0, \ldots, n$$

where v_{i+1} is the $(i+1)$-th component of the vector $\mathbf{v}$. It is more useful to write this equation in vector form, as given by

$$\left(\frac{d\tilde{a}_i}{d\tau}\right)_{av} = \varepsilon \frac{q_n}{2a_n}\mathcal{C}\mathbf{v}\mathbf{q}^T\mathbf{x}, \tag{61}$$

where $\mathcal{C} = \mathrm{diag}(g_i d_i c_i \cos\phi_i) \in \mathbb{R}^{(n+1)\times(n+1)}$, $i = 0, \ldots, n$.

Note that the perturbation amplitude c_i is chosen so as to produce a measurable variation in the plant output.

Now, we are ready to use the Lyapunov stability analysis, to prove the convergence of tracking error (and all its derivatives) to zero (in the averaged sense). Consider the following Lyapunov function

$$V = \mathbf{x}^T\mathbf{P}\mathbf{x} + 2\tilde{\mathbf{a}}^T\mathbf{\Gamma}\tilde{\mathbf{a}}, \tag{62}$$

where $\mathbf{P}$, and $\mathbf{\Gamma}$ are positive definite matrices. Without loss of generality, we assume that $\varepsilon = 1$, and conduct the stability analysis. We shall use the prime to denote differentiation with respect to τ. Taking the derivative of V with respect to τ and noting that $\mathbf{P}$ and $\mathbf{\Gamma}$ are symmetric matrices, we can write

$$V' = \mathbf{x}'^T\mathbf{P}\mathbf{x} + \mathbf{x}^T P\mathbf{x}' + 4\tilde{\mathbf{a}}^T\mathbf{\Gamma}\tilde{\mathbf{a}}' \tag{63}$$

Substituting (58) and simplifying, we get

$$V' = \mathbf{x}^T(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})\mathbf{x} + \frac{2}{a_n}\tilde{\mathbf{a}}^T\left[q_n\mathbf{\Gamma}\mathcal{C}\mathbf{v}\mathbf{q}^T - \mathbf{v}\mathbf{b}^T\mathbf{P}\right]\mathbf{x}$$

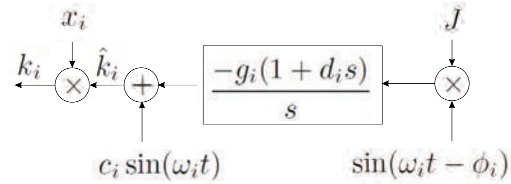

Fig. 4. ES block for updating parameters.

Using the Kalman-Yakubovich-Popov lemma [13], one can write

$$V' = -\mathbf{x}^T Q\mathbf{x} + \frac{2}{a_n}\tilde{\mathbf{a}}^T\left[q_n\mathbf{\Gamma}\mathcal{C}\mathbf{v}\mathbf{q}^T - \mathbf{v}\mathbf{b}^T\mathbf{P}\right]\mathbf{x} \tag{64}$$

where $Q = -(\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A})$ is a negative definite matrix. By setting the right term on the right hand side of (64), we get

$$V' = -\mathbf{x}^T Q\mathbf{x}, \tag{65}$$

which shows that V' is negative semi-definite. A simple application of Barbalat's lemma yields $\mathbf{x} \to 0$, i.e. tracking error and all its derivatives converge to zero. Therefore, the design challenge is summarized into choosing $\mathbf{P}$, $\mathbf{Q}$, $\mathcal{C}$, $\mathbf{q}$, and $\mathbf{\Gamma}$ such that

$$q_n\mathbf{\Gamma}\mathcal{C}\mathbf{v}\mathbf{q}^T - \mathbf{v}\mathbf{b}^T\mathbf{P} = 0 \tag{66}$$

holds. Examples of higher order systems will be presented in forthcoming papers.

IV. SIMULATIONS

In this section, we provide the simulation results for implementing the proposed control scheme to a first order system, and leave the results of the second order system to a future publication.

Suppose that the model reference is given by $\dot{x}_m + 3x_m = 2r$ with zero initial condition. We shall implement the ES-MRAC method as explained here, to a system $\dot{x} = ax + bu$, $x(0) = 0$, where it is assumed that a and b are not known, but their true values are $a = -4$ and $b = 1$. The control law is given by $u = k_x x + k_r r$, where k_x and k_r are updated according to Fig. 1. It is assumed that there is no *a priori* knowledge of the ideal values of parameters. Figs. 5 to 7 show the performance of the system when the design parameters are chosen as follows: perturbation amplitudes $c_1 = c_2 = 0.1$, perturbation frequencies $\omega_1 = 8$ rad/sec, $\omega_2 = 11$ rad/sec, damping coefficients $d_1 = d_2 = 0.1$, and gains $g_1 = g_2 = 150$. To have a better understanding of the performance of the system, results are compared with a usual MRAC method applied to the same system, for which we have used the gradient update laws $\dot{\hat{k}}_x = -\mathrm{sgn}(b)ex$, and $\dot{\hat{k}}_r = -\mathrm{sgn}(b)er$, and a constant reference input $r = 1$.

It can be seen that the ES-MRAC method as proposed here, successfully tracks the model reference in the average value. Moreover, while the usual MRAC method is incapable of reaching the ideal values of parameters, the ES-MRAC has achieved the ideal values. This is due to the fact the convergence of parameters in MRAC depends on persistency of excitation of tracking reference, whereas in the ES-MRAC method the persistency of excitation condition is inherently met, due to sinusoidal perturbations.

V. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

We generalized the method of ES-MRAC to systems of higher dimensions, and provided rigorous analysis and design theorems. It was shown via method of averaging, that there exist design parameters for which the control system becomes stable. Barbalat's lemma was further employed to show that the tracking error converges to zero. Moreover, special cases where some knowlege of parameter values exists, were studied for frist order systems,

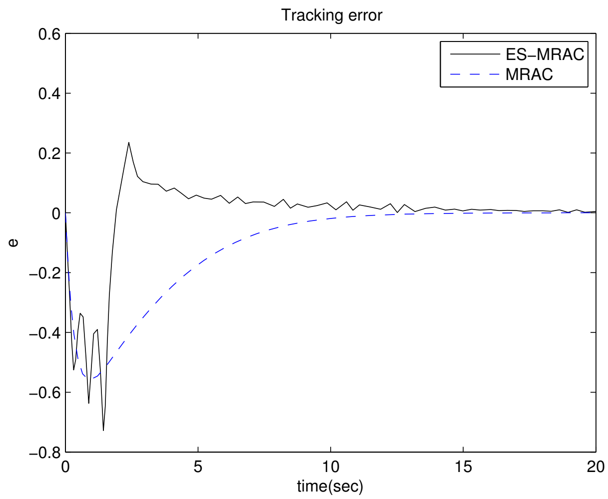

Fig. 5. Convergence of tracking error to zero for ES-MRAC vs. MRAC.

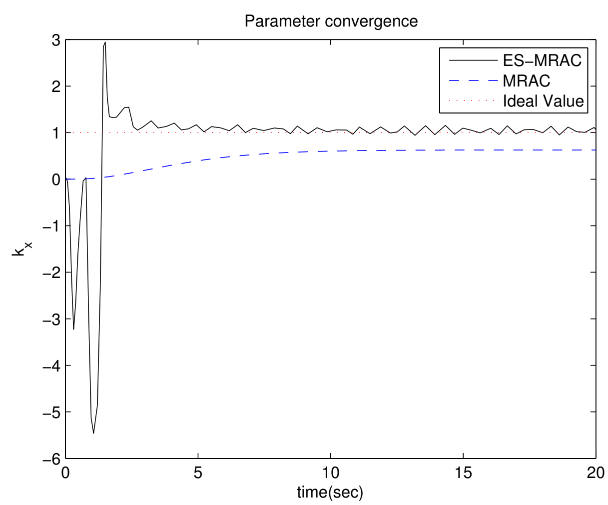

Fig. 6. Convergence of parameter k_x for ES-MRAC vs. MRAC.

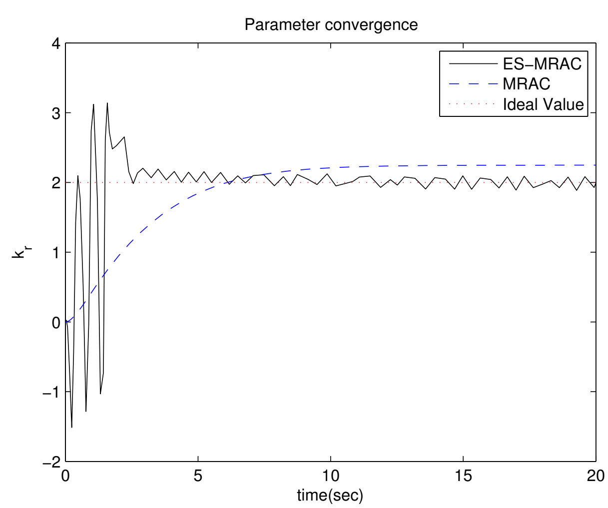

Fig. 7. Convergence of parameter k_r for ES-MRAC vs. MRAC.

and it was shown that convergence to ideal values of parameters is achievable in these cases, and the rate of convergence was estimated using comparison lemma. Finally, simulation results were provided to verify the analysis.

B. Future Works

While the method presented in this paper, accounts for the convergence of state tracking error to zero, it cannot guaranty the convergence of parameters to their true values for higher order systems. Study continues on possible modifications to this control algorithm where convergence of parameters to their ideal values is also possible. In addition, simulations show that the convergence of tracking error to zero is achieved, even for small values of ω_1. This poses the question whether it is possible to utilize the nonlinear control theories of nonautonmous systems to prove stability in a less conservative configuration.

REFERENCES

[1] K. B. Ariyur and M. Krstic, *Real-time optimization by extremum-seeking control*. Hoboken, New Jersey: John Wiley & Sons, 2003.
[2] K. J. Astrom and B. Wittenmark, *Adaptive Control*, 2nd ed. Reading, MA: Addison Wesley, 1995.
[3] G. C. Goodwin and K. S. Sin, *Adaptive Filtering Prediction and Control*. Englewood Cliffs, NJ: Prentice-Hall, 1984.
[4] I. Mareels and J. W. Polderman, *Adaptive Systems: An Introduction*. Berlin: Springer Verlag, 1996.
[5] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: John Wiley & Sons, 1995.
[6] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. New Jersey: Prentice Hall, 1995.
[7] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. New York: Dover Publications, 2005.
[8] C. Cao and N. Hovakimyan, “Design and analysis of a novel l1 adaptive control architecture with guaranteed transient performance,” *IEEE Transactions on Automatic Control*, vol. 53, no. 2, pp. 586–591, 2008.
[9] E. Lavretsky and A. Annaswamy, “On the computation of stability margins for adaptive controllers using linear system tools,” in *Proc. of the AIAA Guidance, Navigation, and Control Conference and Exhibit*, Chicago, Illinois, August 2009.
[10] J. Jang, M. Matsutani, L. G. Crespo, and A. M. Annaswamy, “Stability margins of adaptively controlled systems with time-delays,” in *Proc. of the AIAA Guidance, Navigation, and Control Conference and Exhibit*, Chicago, Illinois, August 2009.
[11] K. B. Ariyur, S. Ganguli, and D. F. Enns, “Extremum seeking for model reference adaptive control,” in *Proc. of the AIAA Guidance, Navigation, and Control Conference*, Chicago, Illinois, August 2009.
[12] L. Meirovitch, *Methods of Analytical Dynamics*. New York, NY: McGraw-Hill, 1970.
[13] J. J. E. Slotine and W. Li, *Applied Nonlinear Control*. New Jersey: Prentice Hall, 1991.
[14] H. K. Khalil, *Nonlinear Systems*, 3rd ed. New Jersey: Prentice Hall, 2002.