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Stability analysis of transportation networks with multiscale driver decisions

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***Abstract*— Stability of Wardrop equilibria is analyzed for dynamical transportation networks in which the drivers' route choices are influenced by information at multiple temporal and spatial scales. The considered model involves a continuum of indistinguishable drivers commuting between a common origin/destination pair in an acyclic transportation network. The drivers' route choices are affected by their, relatively infrequent, perturbed best responses to global information about the current network congestion levels, as well as their instantaneous local observation of the immediate surroundings as they transit through the network. A novel model is proposed for the drivers' route choice behavior, exhibiting local consistency with their preference toward globally less congested paths as well as myopic decisions in favor of locally less congested paths. The simultaneous evolution of the traffic congestion on the network and of the aggregate path preference is modeled by a system of coupled ordinary differential equations. The main result shows that, if the frequency of updates of path preferences is sufficiently small as compared to the frequency of the traffic flow dynamics, then the state of the transportation network ultimately approaches a neighborhood of the Wardrop equilibrium. The proposed analysis combines techniques from singular perturbation theory, evolutionary game theory, and cooperative dynamical systems.**

I. Introduction

As transportation demand is dramatically approaching its infrastructure capacity, a rigorous understanding of the relationship between the macroscopic properties of transportation networks and realistic driver route choice behavior is attracting renewed research interest. A particularly relevant issue is the impact of drivers' *en route* responses to *unexpected events* on the overall transportation network dynamics. This issue is particularly significant in a modern real-life transportation network scenario, where recent technological advancements in intelligent traveller information devices have enabled drivers to be much more flexible in selecting their routes to destination even while being *en route*. While there has been a significant research effort to investigate the effect of such technologies on the route choice behavior of drivers, e.g., see [1], [2], the analytical study of the dynamical properties of the whole network under such behavior has attracted very little attention.

This paper is focused on the stability analysis of transportation networks in a setup where the drivers have access to traffic information at multiple temporal and spatial scales and they have the flexibility to switch their route to destination at every intermediate traffic intersection. Specifically, we consider a model in which the drivers choose their routes while having access to relatively infrequent global information about the network congestion state, and real-time local information as they transit through the network. The drivers' route choice behavior is then influenced by relatively slowly evolving path preferences as well as myopic responses to the instantaneous observation of the local congestion levels at the intersections. This setup captures many real-life scenarios where unexpected events observed *en route* might cause drivers to take a temporary detour, but not necessarily to change their path preferences. Such path preferences may instead be updated, e.g., on a daily, weekly, or longer time basis, in response to information about the global congestion state of the different origin-destination paths collected from the drivers' personal experience, their opinion exchanges with their peers, as well as from information media. However, since the traffic dynamics is significantly influenced by the drivers' response to real-time local information, such responses can influence the drivers' path preference thereby modifying their global route choice behavior in the long run.

The proposed driver decision model gives rise to a double feedback dynamics, governed by a finite-dimensional system of coupled ordinary differential equations. We study the long-time behavior of this dynamical system: our main result shows that, in the limit of small update rate of the aggregate path preferences, a state of approximate *Wardrop equilibrium* [3] is approached. The latter is a configuration in which the delay associated to any source-destination path chosen by a nonzero fraction of the drivers does not exceed the delay associated to any other path. Our results contribute to providing a stronger evidence in support of the significance of Wardrop's postulate of equilibrium for a transportation network. They may also be read as a sort of robustness of such equilibrium notion with respect to non-persistent perturbations of the network.

Our work is naturally related to two streams of literature on transportation networks. On the one hand, traffic flows on networks have been widely analyzed with fluid-dynamical and kinetic models: see, e.g., [4], and references therein. As compared to these models (typically described by partial, or integro-differential equations), ours significantly simplifies the evolution of the traffic parameters (treating them as homogeneous quantities on the links, representative of spatial averages), whereas it highlights the role of the drivers' route choice behavior with its double feedback dynamics, which is typically neglected in that literature.

On the other hand, transportation networks have been

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studied from a decision-theoretic perspective within the framework of *congestion games* [5], [6]. Such an approach has been used, for example in [7]. The stability of Wardrop equilibrium in the context of communication networks has been studied in [8]. It is important to note that the two salient features of a typical congestion game setup are that information is available to the drivers at a single temporal and spatial scale, and that the dynamics of traffic parameters are completely neglected by assuming that they are instantaneously equilibrated. In contrast, we study the stability of Wardrop equilibrium in a setting where the dynamics of the traffic parameters are not neglected, and the drivers' route choice decisions are affected by, relatively infrequent global information, as well as their real-time local information as they transit through the network. As a consequence, classic results of evolutionary game theory and population dynamics [9], [10] are not directly applicable to our framework, and novel analytical tools have to be developed, particularly for the analysis of the fast scale dynamics of the traffic parameters. We do not report all the technical details here due to space limitations; we refer to [11] for complete details.

Before proceeding, we establish here some notation to be used throughout the paper. Let $\mathbb{R}$ be the set of reals, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of nonnegative reals. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. Then, $\mathbb{R}^{\mathcal{A}}$ (respectively, $\mathbb{R}_+^{\mathcal{A}}$) will denote the space of real-valued (nonnegative-real-valued) vectors whose components are labeled by elements of $\mathcal{A}$, and $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ the space of matrices whose real entries labeled by pairs of elements in $\mathcal{A}\times\mathcal{B}$. The transpose of a matrix $M \in \mathbb{R}^{\mathcal{A}\times\mathcal{B}}$, will be denoted by $M' \in \mathbb{R}^{\mathcal{B}\times\mathcal{A}}$, while I to be an identity matrix, and $\mathbf{1}$ the all-one vector, whose size will be clear from the context. The simplex of probability vectors over $\mathcal{A}$ will be denoted by $\mathcal{S}(\mathcal{A}) := \{x \in \mathbb{R}_+^{\mathcal{A}} : \sum_{a\in\mathcal{A}} x_a = 1\}$. For $p \in [1,\infty]$, $\|\cdot\|_p$ is the p-norm. By default, let $\|\cdot\| := \|\cdot\|_2$ denote the Euclidean norm. Let $\mathrm{int}(\mathcal{X})$ be the interior of a set $\mathcal{X} \subseteq \mathbb{R}^d$.

II. Model formulation and main result

In this section, we formulate the problem and state the main result. In our formulation, we represent the dynamics of the traffic and the route choice behavior on a transportation network as a system of coupled ordinary differential equations with two time scales representative of route choice behavior influenced by the two levels of information. The key components of our model are: network topology, congestion properties of the links, path preference dynamics, and node-wise route choice decision. We next describe these components in detail.

A. Network characteristics

Let the topology of the transportation network be described by a directed graph (shortly, *di-graph*) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of nodes and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of (directed) links. For every node $v \in \mathcal{V}$, we shall denote by $\mathcal{E}_v^-$, and $\mathcal{E}_v^+$, the sets of its incoming, and, respectively, outgoing links. A length-l (directed) path from $u \in \mathcal{V}$ to $v \in \mathcal{V}$ is an l-tuple of consecutive links $\{(v_{j-1}, v_j) \in \mathcal{E} : 1 \leq j \leq l\}$ with $v_0 = u$, and $v_l = v$. A cycle is path of length $l \geq 1$ from a node v to itself. Throughout this paper, we shall assume that:

Assumption 1: The di-graph $\mathcal{G}$ contains no cycles, has a unique origin (i.e., some $v \in \mathcal{V}$ such that $\mathcal{E}_v^- = \emptyset$), and a unique destination (i.e., $v \in \mathcal{V}$ such that $\mathcal{E}_v^+ = \emptyset$). Moreover, there exists a path to the destination node from every other node in $\mathcal{V}$.

Assumption 1 implies that one can find a (not necessarily unique) topological ordering of the node set $\mathcal{V}$ (see, e.g., [12]). We shall assume to have fixed one such ordering, identifying $\mathcal{V}$ with the integer set $\{0, 1, \ldots, n\}$, where $n := |\mathcal{V}| - 1$, in such a way that $\mathcal{E}_v^- \subseteq \bigcup_{0\leq u<v} \mathcal{E}_u^+$ for all $v = 0, \ldots, n$.

We shall model the traffic parameters as time-varying quantities which are homogeneous over each link of the network. Specifically, for every link $e \in \mathcal{E}$, and time instant $t \geq 0$, we shall denote the current traffic density, and flow, by $\rho_e(t)$, and $f_e(t)$, respectively, while $\rho(t) := \{\rho_e(t) : e \in \mathcal{E}\}$, $f(t) := \{f_e(t) : e \in \mathcal{E}\}$ will stand for the vectors of all traffic densities, and flows, respectively. Current traffic flow and density on each link are related by a functional dependence

$$f_e = \mu_e(\rho_e), \qquad e \in \mathcal{E}. \tag{1}$$

Such functional dependence models the drivers' speed and lane adjustment behavior in response to traffic density on a particular segment of a road. It will be assumed to satisfy the following:

Assumption 2: For every link $e \in \mathcal{E}$, the flow-density function $\mu_e : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable, strictly increasing, strictly concave and is such that $\mu_e(0) = 0$ and $\lim_{\rho_e \downarrow 0} \mathrm{d}\mu_e/\mathrm{d}\rho_e(\rho_e) < +\infty$.

Remark 1: Flow-density functions commonly used in transportation theory typically are not globally increasing, but rather have a $\cap$-shaped graph [4]: $\mu_e(\rho_e)$ increases from $\mu_e(0) = 0$ until achieving a maximum $C_e = \mu_e(\tilde{\rho}_e)$, and then decreases for $\rho_e \geq \tilde{\rho}_e$. Assumption 2 remains a good approximation of this setting, provided that ρ_e stays in the interval $[0, \tilde{\rho}_e)$.

For every link $e \in \mathcal{E}$, let $C_e := \sup\{\mu_e(\rho_e) : \rho_e \geq 0\} = \lim_{\rho_e \to +\infty} \mu_e(\rho_e)$ be its *maximum flow capacity*. Moreover, let $\mathcal{F}_v := \times_{e\in\mathcal{E}_v^+}[0, C_e)$, $\mathcal{F} := \times_{e\in\mathcal{E}}[0, C_e)$ be the sets of local, and, respectively, global admissible flow vectors. Observe that our formulation allows for both the cases of bounded and unbounded maximum flow capacities. As the flow f_e is the product of speed and density, it is natural to introduce the delay function $T : \mathbb{R}_+^{\mathcal{E}} \to [0, +\infty]^{\mathcal{E}}$

$$T_e(f_e) := \begin{cases} +\infty & \text{if} \quad f_e \geq C_e \\ \mu_e^{-1}(f_e)/f_e & \text{if} \quad f_e \in (0, C_e), \\ 1/\frac{\mathrm{d}\mu_e}{\mathrm{d}\rho_e}(0) & \text{if} \quad f_e = 0, \end{cases} \tag{2}$$

whose components measure the flow-dependent time taken to traverse the different links.[1]

[1]Here it has implicitly been assumed, without any loss of generality, that all the links are of unit length.

Example 1: A flow-density function that satisfies Assumption 2 is given by

$$\mu_e(\rho_e) = C_e \left(1 - e^{-\theta_e \rho_e}\right) \quad \forall e \in \mathcal{E}, \tag{3}$$

where $C_e > 0$, and $\theta_e > 0$. The corresponding delay function is

$$T_e(f_e) = \frac{1}{\theta_e f_e} \log \frac{C_e}{C_e - f_e}\,.$$

We shall denote by $\mathcal{P}$ the set of distinct paths in $\mathcal{G}$ from the origin node 0 to the destination node n, and let

$$A \in \mathbb{R}^{\mathcal{E}\times\mathcal{P}}\,, \qquad A_{ep} = \begin{cases} 1 & \text{if} \quad e \in p \\ 0 & \text{if} \quad e \notin p\,, \end{cases}$$

be the link-path incidence matrix of $\mathcal{G}$. The relative appeal of the different paths to the drivers will be modeled by a time-varying probability vector over $\mathcal{P}$, which will be referred to as the current *aggregate path preference*, and denoted by $\pi(t)$. If one assumes, as we shall do throughout this paper, a constant unit incoming flow in the origin node, it is natural to consider the vector $f^\pi := A\pi$ of the flows associated to the current aggregate path preference. Indeed, $f^\pi_e = \sum_p A_{ep}\pi_p$ represents the total traffic flow that a link $e \in \mathcal{E}$ would sustain in a hypothetic equilibrium condition in which the fraction of drivers choosing any path $p \in \mathcal{P}$ is given by π_p. Now, let $\Pi := \{\pi \in \mathcal{S}(\mathcal{P}) : (A\pi)_e < C_e, \forall e \in \mathcal{E}\}$ be the set of *feasible path preferences*. Here, the term 'feasible' refers to the fact that the flow vector f^π associated to any $\pi \in \Pi$ satisfies the capacity constraint $f^\pi_e < C_e$ for every $e \in \mathcal{E}$. Observe that, whenever $C_e > 1$ for every $e \in \mathcal{E}$ (or, in particular, when link capacities are infinite), the set of admissible path preferences Π coincides with the whole simplex $\mathcal{S}(\mathcal{P})$. In contrast, when $C_e \leq 1$ for some $e \in \mathcal{E}$, $\Pi \subset \mathcal{S}(\mathcal{P})$ is a strict inclusion. On the other hand, whether Π is empty or not depends solely on the value of the min-cut capacity of the network [13, Ch. 4]

$$C^* := \min_{\substack{\mathcal{U}\subseteq\mathcal{V}:\\ 0\in\mathcal{U}, n\notin\mathcal{U}}} C_{\mathcal{U}}\,, \qquad C_{\mathcal{U}} := \sum_{\substack{e=(u,v)\in\mathcal{E}:\\ u\in\mathcal{U},\, v\in\mathcal{V}\setminus\mathcal{U}}} C_e\,,$$

as shown in the following, simply established, result.

Proposition 1: The set Π is nonempty if and only if $C^* > 1$.

In the case when $C^* \leq 1$ it is not hard to show that, since the incoming flow exceeds the outgoing flow of the network, the system will grow unstable, i.e., $\rho_e(t)$ is unbounded as t grows large, for some link $e \in \mathcal{E}$. Therefore, throughout this paper we shall confine ourselves to transportation networks satisfying:

Assumption 3: The min-cut capacity satisfies $C^* > 1$.

B. Route choice behavior and traffic dynamics

We now describe the drivers' route choice behavior and traffic dynamics on the network. We envision a continuum of indistinguishable drivers traveling through the network. Drivers enter the network from the origin node 0 at a constant unit rate, travel through it, and leave the network from the destination node n. While inside the network, drivers occupy some link $e \in \mathcal{E}$. The time required by the drivers to traverse link e, and the current flow on such link are governed by its congestion properties, as given by (2), and (1), respectively. When entering the network from the origin node $v = 0$, as well as when reaching the tail node $v \in \{1, 2, \ldots, n-1\}$ of some link $e \notin \mathcal{E}^-_n$, the drivers instantaneously join some link $e \in \mathcal{E}^+_v$. In this paper, we shall model the choice of such new link to depend on infrequently updated perturbed best responses of the drivers to global information about the congestion status of the whole network as well as on their instantaneous observation of the local congestion levels. We next describe these two aspects of the model in detail.

Aggregate path preference dynamics: The drivers' aggregate path preference $\pi(t)$, already introduced in Sect. II-A, models the relative appeal of the different paths to the drivers' population. The aggregate path preference $\pi(t)$ is updated as drivers access global information about the current congestion status of the whole network. This occurs at some rate $\eta > 0$, which will be assumed small with respect to the time-scale of the network flow dynamics. Information about the current status of the network is embodied by the current traffic flow vector $f(t)$. From $f(t)$, drivers can evaluate the vector $A'T(f(t))$, whose p-th component $\sum_{e\in\mathcal{E}} A_{ep}T_e(f_e(t))$ coincides with the total delay one expects to incur on path p assuming that the congestion levels on such path won't change. Drivers' are assumed to react to such global information by a *perturbed best response*

$$F^h(f) := \operatorname*{argmin}_{\omega\in\Pi_h} \left\{\omega' A' T(f) + h(\omega)\right\}, \tag{4}$$

where $h : \Pi_h \to \mathbb{R}$ is an *admissible perturbation*, satisfying the following:

Assumption 4: An admissible perturbation is a function $h : \Pi_h \to \mathbb{R}$ where $\Pi_h \subseteq \Pi$ is a closed convex set, $h(\,\cdot\,)$ is strictly convex, twice differentiable in $\mathrm{int}(\Pi_h)$, and is such that $\lim_{\pi\to\partial\Pi_h} ||\tilde{\nabla} h(\pi)|| = +\infty$, where $\tilde{\nabla} := (I - |\mathcal{P}|^{-1}\mathbf{1}\mathbf{1}')\nabla$ is the projected gradient on $\mathcal{S}(\mathcal{P})$[2].

As a result, the aggregate path preference $\pi(t)$ evolves as

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi = \eta\left(F^h(f) - \pi\right)\,. \tag{5}$$

The perturbed best response function $F^h(f)$ provides an idealized description of the behavior of drivers whose decisions are based on inexact information about the state of the network. In particular, it can be shown that the form of $F^h(f)$ given in (4) is equivalent to the minimization over paths $p \in \mathcal{P}$ of the expected delay $\sum_{e\in\mathcal{E}} A_{ep}T_e(f_e)$ corrupted by some (admissible) stochastic perturbation (see e.g. [14]).

It is easy to establish that the perturbed best response $F^h(f)$ is continuously differentiable on $\mathcal{F}$. Moreover, it is well known [10] that, as $\|h\|_\infty \downarrow 0$, and $\Pi_h \uparrow \overline{\Pi}$, the perturbed best response $F^h(f)$ converges to the set $\mathrm{argmin}\{\omega' A' T(f) : \omega \in \Pi\}$ of best responses.[3]

[2] We shall use the notation $\Phi := I - |\mathcal{P}|^{-1}\mathbf{1}\mathbf{1}' \in \mathbb{R}^{\mathcal{P}\times\mathcal{P}}$ to denote the corresponding projection matrix.

[3] Here, the convergences $\Pi_h \uparrow \overline{\Pi}$, and $\{F^h(f)\} \to \mathrm{argmin}\{\omega' A' T(f) : \omega \in \Pi\}$ are intended to hold in the Hausdorff metric. (see, e.g., [15, Def. 4.4.11])

Example 2: Assume that $C_e > 1$ for all $e \in \mathcal{E}$. Then, an example of perturbed best response satisfying Assumption 4 is the logit function with noise level $\beta > 0$, which is defined as

$$F_p^h(f) = \frac{\exp(-\beta(A'T(f))_p)}{\sum_{q\in\mathcal{P}} \exp(-\beta(A'T(f))_q)}\,, \qquad p \in \mathcal{P}\,. \quad (6)$$

This corresponds to the admissible perturbation function $h(\omega) = -\beta^{-1}\sum_p \omega_p \log \omega_p$. For any fixed $f \in \mathcal{F}$, one has that $\lim_{\beta\to+\infty} F^h(f)$, with $F^h(f)$ as defined in (6), is a uniform distribution over the set $\operatorname{argmin}\{(A'T(f))_p : p \in \mathcal{P}\}$. We refer the reader to [16] for more on the connection between F^h characterized by Assumption 4 and smooth best response functions.

Remark 2: In the evolutionary game theory literature, e.g., see [9], [10], the domain of an admissible perturbation function h, as well as the one of the minimization in the right-hand side of (4), is typically assumed to be the whole simplex $\mathcal{S}(\mathcal{P})$, instead of a closed polytope $\Pi_h \subseteq \Pi \subseteq \mathcal{S}(\mathcal{P})$. Notice that, as already observed in Sect. II-A, when $C_e > 1$ for every $e \in \mathcal{E}$, $\Pi = \mathcal{S}(\mathcal{P})$ is a closed polytope, so that one can choose $\Pi_h = \Pi$. Therefore, in this case, Assumption 4 does not introduce any additional restriction with respect to such theory.

On the other hand, when $C_e \leq 1$ for some $e \in \mathcal{E}$, then the inclusions of $\Pi_h \subset \Pi \subset \mathcal{S}(\mathcal{P})$ are both strict, so that Assumption 4 does introduce additional restrictions on the admissible perturbations. However, it is worth observing that, in a classic evolutionary game theoretic framework, the dynamics of the aggregate path preference would be autonomous rather than coupled to the one of the actual flow. In particular, perturbed best response dynamics in that framework would read as

$$\frac{\mathrm{d}}{\mathrm{d}t}\pi = F^h(f^\pi) - \pi\,, \quad (7)$$

rather than as in (5). For such dynamics, the fact that $T_e((A\pi)_e) = +\infty$ whenever $(A\pi)_e \geq C_e$, can be shown to imply that $\pi(t)$ would reach a compact $\Pi_h \subseteq \Pi$ in some finite time and never leave it. In contrast, in the two time-scale model of coupled dynamics considered in this paper (see (11)), such more restrictive assumption is needed in order to ensure the same property for the trajectories of $\pi(t)$.

Local route decisions: We now describe the local route decisions, characterizing the fraction of drivers choosing each link $e \in \mathcal{E}_v^+$ when traversing a non-destination node v. Such a fraction will be assumed to be a continuously differentiable function $G_e^v(f_{\mathcal{E}_v^+}, \pi)$ of the local traffic flow $f_{\mathcal{E}_v^+} := \{f_e : e \in \mathcal{E}_v^+\}$, as well as of the current aggregate path preference π. We shall refer to

$$G^v : \mathcal{F}_v \times \Pi \to \mathcal{S}(\mathcal{E}_v^+) \quad (8)$$

as the *local decision function* at node $v \in \{0, 1, \ldots, n-1\}$, and assume that it satisfies the following:

Assumption 5: For all $0 \leq v < n$, and $\pi \in \Pi$,

$$\left(\sum\nolimits_{j\in\mathcal{E}_v^+} f_j^\pi\right) G_e^v\left(f_{\mathcal{E}_v^+}^\pi, \pi\right) = f_e^\pi\,, \qquad \forall e \in \mathcal{E}_v^+\,.$$

Assumption 6: For all $0 \leq v < n$, $\pi \in \Pi$, and $f_{\mathcal{E}_v^+} \in \mathcal{F}_v$,

$$\frac{\partial}{\partial f_e} G_j^v(f_{\mathcal{E}_v^+}, \pi) \geq 0\,, \qquad \forall j \neq e \in \mathcal{E}_v^+\,.$$

Assumption 5 is a consistency assumption. It postulates that, when the locally observed flow coincides with the one associated to the aggregate path preference π, drivers choose to join link $e \in \mathcal{E}_v^+$ with frequency equal to the ratio between the flow f_e^π and the total outgoing flow $\sum_{j\in\mathcal{E}_v^+} f_j^\pi$.

Assumption 6 instead models the drivers' myopic behavior in response to variations of the local congestion levels. It postulates that, if the congestion on one link increases while the congestion on the other links outgoing from the same node is kept constant, the frequency with which each of the other outgoing links is chosen does not decrease. It is worth observing that Assumption 6 is reminiscent of Hirsch's notion of *cooperative dynamical system* [17], [18].

Example 3: An example of local decision function G^v satisfying Assumptions 5 and 6 is the i-logit function. The i-logit route choice with sensitivity $\gamma > 0$ is given by

$$G_e^v(f_{\mathcal{E}_v^+}, \pi) = \frac{f_e^\pi \exp(-\gamma(f_e - f_e^\pi))}{\sum_{j\in\mathcal{E}_v^+} f_j^\pi \exp(-\gamma(f_j - f_j^\pi))}\,, \quad (9)$$

for every $e \in \mathcal{E}_v^+$, $0 \leq v < n$.

For every non-destination node $v \in \{0, 1, \ldots, n-1\}$, and outgoing link $e \in \mathcal{E}_v^+$, conservation of mass implies that $\frac{\mathrm{d}}{\mathrm{d}t}\rho_e = H_e(f, \pi)$, where

$$H_e(f,\pi) := \begin{cases} G_e^v(f_{\mathcal{E}_v^+}, \pi) - f_e & \text{if} \quad v = 0 \\ (\sum_{j\in\mathcal{E}_v^-} f_j) G_e^v(f_{\mathcal{E}_v^+}, \pi) - f_e & \text{if} \quad 1 \leq v < n\,. \end{cases} \quad (10)$$

C. Objective of the paper and main result

The objective of this paper is to study the evolution of the coupled dynamics

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\pi = \eta\left(F^h(f) - \pi\right) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\rho = H(f, \pi)\,, \end{cases} \quad (11)$$

where F^h is the perturbed best response function defined in (4), $\eta > 0$ is the rate at which global information becomes available, $H(f, \pi) = \{H_e(f, \pi) : e \in \mathcal{E}\}$, with H_e defined in (10), and f and ρ are related by the functional dependence (1). In particular, our analysis will focus on the double limiting case of small η and small h. We shall prove that, in such limiting regime, the long-time behavior of the system is approximately at Wardrop equilibrium [3], [19]. The latter is a configuration in which the delay is the same on all the paths chosen by a nonzero fraction of the drivers. More formally, one has the following:

Definition 2 (Wardrop Equilibrium): An admissible flow vector $f^W \in \mathcal{F}$ is a *Wardrop equilibrium* if $f^W = A\pi$ for some $\pi \in \Pi$ such that, for all $p \in \mathcal{P}$,

$$\pi_p > 0 \Longrightarrow (A'T(A\pi))_p \leq (A'T(A\pi))_q\,, \forall q \in \mathcal{P}\,. \quad (12)$$

Existence and uniqueness of a Wardrop equilibrium are guaranteed by the following standard result which easily follows from Theorems 2.4 and 2.5 in [19].

Proposition 3: Let Assumptions 1-3 be satisfied. Then, there exists a unique Wardrop equilibrium $f^W \in \mathcal{F}$.

The following is the main result of this paper.

Theorem 4: Let Assumptions 1–6 be satisfied. Then, for every initial condition $\pi(0) \in \mathrm{int}(\mathcal{S}(\mathcal{P}))$, $\rho(0) \in (0, +\infty)^{\mathcal{E}}$, there exists a unique solution of (11). Moreover, there exists a perturbed equilibrium flow $f^h \in \mathcal{F}$ such that, for all $\eta > 0$,

$$\limsup_{t \to +\infty} ||f(t) - f^h|| \leq \delta(\eta)\,, \tag{13}$$

where $\delta(\eta)$ is a nonnegative-real-valued, nondecreasing function of $\eta > 0$, such that $\lim_{\eta \downarrow 0} \delta(\eta) = 0$. Moreover, for every sequence of admissible perturbations $\{h_k : k \in \mathbb{N}\}$ such that $\lim_{kt\ +\infty} ||h_k||_\infty = 0$, and $\lim_{k \to +\infty} \Pi_{h_k} = \overline{\Pi}$, one has

$$\lim_{k \to +\infty} f^{h_k} = f^W\,. \tag{14}$$

Theorem 4 states that, in large time limit, the flow vector $f(t)$ approaches a neighborhood of the Wardrop equilibrium, whose size vanishes as both the time-scale ratio η and the perturbation norm $||h||_\infty$ vanish. While a qualitatively similar result is known to hold [10] in a classic evolutionary game theoretic framework (i.e., neglecting the traffic dynamics, and assuming it is instantaneously equilibrated, as in the ODE system (7)), the significance of the above is to show that an approximate Wardrop equilibrium configuration is expected to emerge also in our more realistic model of two-time scale dynamics. Therefore, our results provide a stronger evidence in support of the significance of Wardrop's postulate of equilibrium for a transportation network. In fact, they may be read as a sort of robustness of such equilibrium notion with respect to non-persistent perturbations.

D. Proof sketch

We provide a brief sketch of the proof of Theorem 4 here; all the details are in [11]. The main idea consists in adopting a singular perturbation approach (e.g., see [**?**]), viewing the traffic density ρ (or, equivalently, the traffic flow f) as a fast transient, and the aggregate path preference π as a slow component. Hence, one first thinks of π as quasi-static (i.e., 'almost a constant') while analyzing the fast-scale dynamics (10), and then assume that f is 'almost equilibrated', i.e. close to f^π, and study the slow-scale dynamics (5) as a perturbation of (7). A first main result shows that f^π is a globally attractive equilibrium of the fast-scale dynamics (10) with frozen π. This is proven by showing that a suitably weighted l_1-distance between ρ and ρ^π (with weight exponentially decreasing with the distance from the origin node) is a Lyapunov function for such dynamics, and that the gradient of this Lyapunov function is non-increasing along the trajectories of the fast scale dynamics. A consequence of this is that the density vector ρ remains bounded in time. These facts are then combined with the property that the slow-scale dynamics with equilibrated flow, i.e., (7), is the gradient flow of the potential

$$\Psi(\pi) = \sum_{e \in \mathcal{E}} \int_0^{f_e^\pi} T_e(s)\, ds - \beta^{-1} \sum_p \pi_p \log \pi_p \tag{15}$$

(e.g., see [10]).

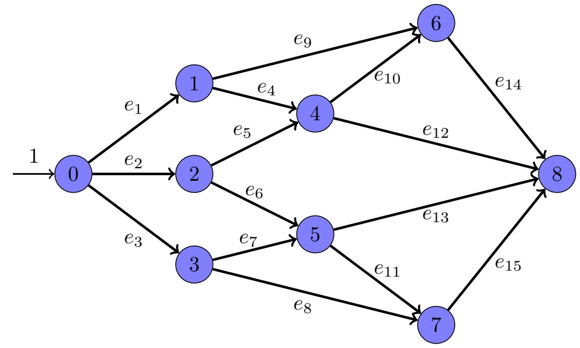

Fig. 1. The graph topology used in simulations.

III. Simulations

In this section, we present results from numerical experiments. We performed several experiments with different graph topologies and for values of η ranging from 0.01 to 100. In all the cases, we found that the trajectories converge exactly to the perturbed Wardrop equilibrium, i.e., $\delta(\eta)$ in Theorem 4 was estimated to be uniformly zero. We suspect that this might be because of the exponential convergence also of the slow scale dynamics. Additionally, we compared the convergence of the trajectories corresponding to local decision function from Example 3 with trajectories corresponding to local decision function of the form

$$G_e^v\left(f_{\mathcal{E}_v^+}, \pi\right) = f_e^\pi \big/ \sum_{j \in \mathcal{E}_v^+} f_j^\pi\,, \forall f_{\mathcal{E}_v^+} \in \mathcal{F}_v,\ \forall e \in \mathcal{E}_v^+\,. \tag{16}$$

The latter corresponds to the case when the drivers do not take into account the local observation on the currently observed flow, and always act in a way that is consistent with their aggregate path preference. We found that the trajectories corresponding to local decision function in (16) converged faster than the trajectories corresponding to the local decision function in Example 3.

We demonstrate these findings through an illustrative example. For this example, the parameters were selected as follows:

- graph topology $\mathcal{G}$ as shown in Figure 1,
- link-wise flow functions as given by (3) with $C_1 = 2$ and $\theta_e = 1$, for all $e \in \mathcal{E}$;
- F^h as in (6) with $\beta = 1$,
- G as in (9) with $\gamma = 1$,
- initial conditions: $\pi_e(0) = 1/15$ for all $e \in \mathcal{E}$, $\rho_{e_1}(0) = \rho_{e_{12}}(0) = 5$, $\rho_{e_2}(0) = \rho_{e_6}(0) = \rho_{e_8}(0) = 7$, $\rho_{e_3}(0) = \rho_{e_7}(0) = 3$, $\rho_{e_4}(0) = 6$, $\rho_{e_5}(0) = 1$, $\rho_{e_9}(0) = 9$, $\rho_{e_{10}}(0) = 10$, $\rho_{e_{13}}(0) = 12$, $\rho_{e_{14}}(0) = 4$, $\rho_{e_{15}}(0) = 8$.
- $\eta = 0.1$.

For these values, $\rho^h := \mu^{-1}(f^h)$ was numerically calculated by implementing a gradient descent algorithm for the corresponding potential function (15). The evolution of the 1-norm distance of ρ from ρ^h is plotted on a log-linear scale in Figure 2 for two cases: (i) local route choice decision function of Example 3, and (ii) local decision function given in (16). Figure 2 also shows that there is no significant difference

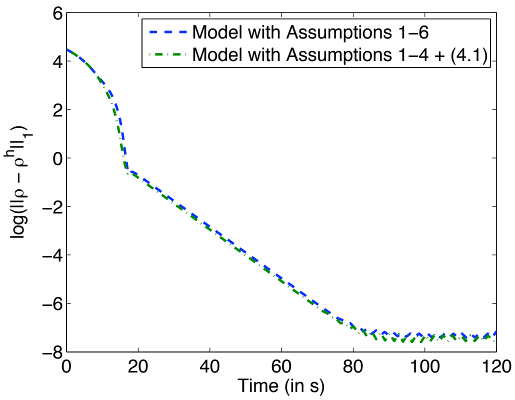

Fig. 2. Log-linear plot for comparison of the evolution of $\|\rho(t)-\rho^h\|_1$ for the local decision function of Example 3 versus the local decision function of (16).

between the convergence of trajectory corresponding to local decision function in (16) and the trajectory corresponding to the local decision function of Example 3. However, as we increase η, we observed that the trajectory corresponding to the local decision function in (16) converge faster than the trajectory corresponding to the local decision function of Example 3.

IV. Conclusion

In this paper, we analyzed the stability of Wardrop equilibria in dynamical transportation networks characterized by dual temporal and spatial scales of the drivers' route choice behavior. We showed that, if the frequency of updates of path preferences is sufficiently small, then the state of the transportation network ultimately approaches a neighborhood of the Wardrop equilibrium. Our results provide a stronger evidence in support of the significance of Wardrop's postulate of equilibrium for a transportation network. They may be read as a sort of robustness of such equilibrium notion with respect to non-persistent perturbations of the network.

There are several possible directions for future work. We plan to formally justify our dynamical model as a macroscopic approximation of the underlying driver level microscopic process. We also plan to extend our analysis to the case with multiple origin-destination pairs and possibly cyclic topologies. We also plan to study the effect of persistent, and possibly adversarial, perturbations on the traffic dynamics under driver behavior model similar to the one considered in this paper, e.g., see [20], [21].

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