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Synergistic Lyapunov functions and backstepping hybrid feedbacks

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Abstract— **The notion of synergistic potential functions has been introduced recently in the literature and has been used as the basis for the design of hybrid feedback laws that achieve global asymptotic stabilization of a point on a compact manifold (without boundary) such as $\mathbb{S}^1$, $\mathbb{S}^2$, and $\mathsf{SO}(3)$. Here, synergistic potential functions are generalized—to synergistic Lyapunov functions—and are shown to be amenable to backstepping. In particular, if an affine control system admits a (weak) synergistic Lyapunov function and feedback pair then the system with an integrator added at the input also admits a synergistic Lyapunov and feedback pair. This fact enables "smoothing" hybrid feedbacks, or implementing them through a chain of integrators. In this way, hybrid control designed at a kinematic level can be redesigned for control through forces, torques, or even the derivative of these quantities. We demonstrate the backstepping procedure for attitude stabilization of a rigid body using a quaternion parametrization.**

I. Introduction

Hybrid feedback is a powerful tool for achieving robust global asymptotic stabilization in situations where topological constraints preclude achieving this goal with classical feedback. Such situations include point stabilization for systems having states evolving on a compact boundaryless manifold [1], or, more generally, point stabilization for systems whose state space is not contractible [2] or diffeomorphic to some Euclidean space [3], [4]. Also included in this list of situations is stabilization to a disconnected set of points in a connected state space [5], which arises naturally when considering point stabilization of rigid-body attitude with a unit quaternion parametrization [6], [7]. These topological obstructions to global asymptotic stability have been emphasized recently in a series of papers [8]–[11] where the notion of a family of synergistic potential functions has been introduced and used to achieve global asymptotic stability of a point by hybrid feedback for systems whose state space is not diffeomorphic to any Euclidean space.

Roughly speaking, a family of potential functions is synergistic in the sense of [9] if, at each point where the gradient of one of the potential functions vanishes (other than at the point being stabilized), there is another potential function in the family whose value is strictly less than the value of the given potential function. A synergistic family of potential functions gives rise to a simple hybrid controller based on hysteretically choosing the minimum potential function and its corresponding feedback control law for global asymptotic stability. This "min-switch" hybrid control paradigm has appeared in the literature in various contexts over the past two decades. In particular, an early application of this idea for implementing hysteresis in adaptive control was presented in [12], which was later made scale independent in [13]. Later, [14] proposed this method (without hysteresis) for multi-controller systems where it has been applied for the problem of stabilizing a pendulum on a cart in [15] and for control of a double-tank system in [16] (which suggests a similar form for the hysteresis used in this paper); see also [17]. Ideas related to synergistic potential functions also appear in [18]–[20] where multiple Lyapunov functions are proposed for analysis and control design.

In this paper, we extend the notion of synergistic potential functions to a larger class of functions, which contains synergistic potential functions as a special case. We call these functions synergistic Lyapunov functions. We show that if an affine control system admits a family of (weak) synergistic Lyapunov functions, then the system with an integrator added at the input also admits a family of synergistic Lyapunov functions. In turn, since synergistic Lyapunov functions admit global hybrid stabilizers, this result shows that hybrid feedback can be smoothed or implemented through multiple integrators. This observation is significant for extending hybrid feedback designs from a kinematic level to a dynamic level or further through multiple integrators in an effort to avoid exciting unmodeled dynamics that might be sensitive to jump discontinuities in the control variable.

The backstepping feature of synergistic Lyapunov functions has its antecedent in the nonlinear control literature of the late 1980s and early 1990s. A summary of the important references in integrator backstepping can be found in the notes and references of [21, Chapter 2]. Our result on passing from a family of weak synergistic Lyapunov functions and feedbacks for an affine control system to a family of synergistic Lyapunov functions and feedbacks for the system extended with an integrator at the input parallels the integrator backstepping idea summarized in [21, Lemma 2.8(ii)]. See also [22, Theorem 5.3]. Similar backstepping results for switched systems have appeared in [23]; however, the crucial notion of synergism ensuring global asymptotic stability does not appear in [23].

Our paper is organized as follows. In the next section, we give some preliminaries including a description of the hybrid systems framework we use. In Section III we de-

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fine synergistic Lyapunov function and feedback pairs. In Section IV we show how to build a globally asymptotically stabilizing hybrid feedback from synergistic Lyapunov function and feedback pairs. In Section V, we define *weak* synergistic Lyapunov function and feedback pairs, while in Section VI we show that if an affine control system admits a family of weak synergistic Lyapunov function and feedback pairs then the system with an integrator added at the input admits a family of (non-weak) synergistic Lyapunov function and feedback pairs. In Section VII, we apply the method to the problem of rigid-body attitude stabilization using a unit-quaternion parameterization. Finally, we provide some concluding remarks in Section VIII.

II. Preliminaries

A. Notation

In this paper, $\mathbb{R}$ denotes the real numbers, $\mathbb{R}_{\geq 0}$ the nonnegative real numbers, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, and $\mathbb{N}$ denotes the natural numbers including 0. Given a vector $x \in \mathbb{R}^n$, $|x|$ denotes the Euclidean vector norm. Given a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, $|x|_S$ denotes the distance from x to S, i.e., $|x|_S := \inf_{y \in S} |x - y|$. For a closed set $X \subset \mathbb{R}^n \times Q$, where $Q \subset \mathbb{R}$ is a finite set, and a smooth function $V : X \to \mathbb{R}$, we use $\nabla V(z, q)$ to denote gradient of V relative to z, with q considered to be constant. Given a smooth function $\kappa : X \to \mathbb{R}^m$, we use $\mathcal{D}\kappa(q, z)$ to denote the Jacobian matrix of κ relative to z, i.e., $\mathcal{D}\kappa(z, q)$ is an $\mathbb{R}^{m \times n}$ matrix with ij-th entry given as $\frac{\partial \kappa_i(z,q)}{\partial z_j}$.

B. Hybrid Systems

Hybrid systems are dynamical systems with both continuous and discrete dynamics. For the purposes of this paper we consider the framework used in [24]. Here, a hybrid system $\mathcal{H}$ is defined by the following objects:

- A set $C \subset \mathbb{R}^n$ called the *flow set*.
- A set $D \subset \mathbb{R}^n$ called the *jump set*.
- A map $f : C \to \mathbb{R}^n$ called the *flow map*.
- A set-valued map $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ called the *jump map*.

The flow map f defines the continuous dynamics on the flow set C, while the jump map G defines the jump dynamics on the jump set D. A hybrid system $\mathcal{H}$ is written compactly as

$$\mathcal{H} : \quad x \in \mathbb{R}^n \quad \begin{cases} \dot{x} = f(x) & x \in C \\ x^+ \in G(x) & x \in D \ . \end{cases} \tag{1}$$

Solutions are given on extended time domains by functions that satisfy the conditions suggested by (1). More precisely:

Definition 1 (hybrid time domain)**.** A set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *compact hybrid time domain* if

$$E = \bigcup_{j=0}^{J-1} ([t_j, t_{j+1}], j)$$

for some finite sequence of times $0 = t_0 \leq t_1 \leq t_2 \ldots \leq t_J$. It is a *hybrid time domain* if for all $(T, J) \in E$, $E \cap ([0, T] \times \{0, 1, \ldots J\})$ is a compact hybrid time domain.

Definition 2 (hybrid arc)**.** A function $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *hybrid arc* if $\operatorname{dom} x$ is a hybrid time domain and, for each $j \in \mathbb{N}$, $t \mapsto x(t, j)$ is locally absolutely continuous.

Definition 3 (solution to $\mathcal{H}$)**.** A hybrid arc $x : \operatorname{dom} x \mapsto \mathbb{R}^n$ is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$;

(S1) $\forall j \in \mathbb{N}$ such that $I_j := \{t \ : \ (t, j) \in \operatorname{dom} x\}$ has nonempty interior

$$x(t, j) \in C \qquad \text{for all } t \in [\min I_j, \sup I_j)$$
$$\dot{x}(t, j) = f(x(t, j)) \qquad \text{for almost all } t \in I_j;$$

(S2) $\forall (t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$,

$$x(t, j) \in D, \quad x(t, j + 1) \in G(x(t, j)) \ .$$

Hybrid arcs, and solutions to $\mathcal{H}$ in particular, are parametrized by pairs (t, j), where t is the ordinary time component and j is the number of jumps accrued. A solution x is said to be *nontrivial* if $\operatorname{dom} x$ contains at least one point different from $(0, 0)$, *maximal* if there does not exist another solution x' such that x is a truncation of x' to some proper subset of $\operatorname{dom} x'$, *complete* if $\operatorname{dom} x$ is unbounded, and *Zeno* if it is complete but the projection of $\operatorname{dom} x$ onto $\mathbb{R}_{\geq 0}$ is bounded. Maximal solutions to $\mathcal{H}$ may not be unique, not only due to the jump dynamics being set-valued map, but also because when $C \cap D \neq \emptyset$, solutions from $C \cap D$ jump and, depending on the flow map, may be able to flow as well.

The stability definitions below are generalizations of the standard stability concepts to the setting where completeness or even existence of solutions is not required. It is a natural stability notion for hybrid systems since, often, local existence of solutions is not guaranteed because the set $C \cup D$ does not cover $\mathbb{R}^n$. For the hybrid control problem studied here, the flow and jump sets will be subsets of the form $M \times Q$, where M is a closed subset of an Euclidean space and Q is a finite set, hence, not covering $\mathbb{R}^n$ for some n. Nonetheless, in our applications, local existence of solutions will hold.

Definition 4 (asymptotic stability)**.** Consider a hybrid system $\mathcal{H}$. Let $\mathcal{A} \subset \mathbb{R}^n$ be compact. Then:

- The compact set $\mathcal{A}$ is *stable* for $\mathcal{H}$ if for each $\varepsilon > 0$ there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ satisfies $|x(t, j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom} x$.
- The compact set $\mathcal{A}$ is *attractive* for $\mathcal{H}$ if there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0, 0)|_{\mathcal{A}} \leq \delta$ is bounded and if it is complete then $x(t, j) \to \mathcal{A}$ as $t + j \to \infty$.
- The compact set $\mathcal{A}$ is *asymptotically stable* if it is both stable and attractive.

The set from which all solutions are bounded and the complete ones converge to $\mathcal{A}$ is called the *basin of attraction* of $\mathcal{A}$. The compact set $\mathcal{A}$ is *globally asymptotically stable* when the basin of attraction is equal to $\mathbb{R}^n$.

By definition, the basin of attraction contains a neighborhood of $\mathcal{A}$. Points in $\mathbb{R}^n \setminus (C \cup D)$ always belong to the basin of attraction since there are no solutions starting at such points.

Definition 5 (weak invariance)**.** For a hybrid system $\mathcal{H}$ in $\mathbb{R}^n$, the set $S \subset \mathbb{R}^n$ is said to be

(a) *weakly forward invariant* if for each $x(0,0) \in S$, there exists at least one complete solution x to $\mathcal{H}$ starting from $x(0,0)$ with $x(t,j) \in S$ for all $(t,j) \in \operatorname{dom} x$;

(b) *weakly backward invariant* if for each $q \in S$, $N > 0$, there exist $x(0,0) \in S$ and at least one solution x to $\mathcal{H}$ starting from $x(0,0)$ such that for some $(t^*, j^*) \in \operatorname{dom} x$, $t^* + j^* \geq N$, we have $x(t^*, j^*) = q$ and $x(t,j) \in S$ for all $(t,j) \preceq (t^*, j^*)$, $(t,j) \in \operatorname{dom} x$;

(c) *weakly invariant* if it is both weakly forward invariant and weakly backward invariant.

III. Synergistic Lyapunov function and feedback

In this section, a synergistic Lyapunov function and feedback pair is defined for the affine control system

$$\left.\begin{aligned} \dot{z} &= \phi(z,q) + \psi(z,q)\omega \\ \dot{q} &= 0 \end{aligned}\right\} \quad (z,q) \in M \times Q \tag{2}$$

where the functions ϕ and ψ are smooth[1], $\omega \in \mathbb{R}^m$ is the control, the set $M \subset \mathbb{R}^n$ is closed, and the set Q is discrete. Smooth functions $V : M \times Q \to \mathbb{R}_{\geq 0}$ and $\kappa : M \times Q \to \mathbb{R}^m$ form a *synergistic Lyapunov function and feedback pair candidate* relative to the compact set $\mathcal{A} \subset M \times Q$ if

- $\forall r \geq 0$, $\{(z,q) \in M \times Q : V(z,q) \leq r\}$ is compact;
- V is positive definite with respect to $\mathcal{A}$;
- For all $(z,q) \in M \times Q$,

$$\langle \nabla V(z,q), \phi(z,q) + \psi(z,q)\kappa(z,q) \rangle \leq 0 \ . \tag{3}$$

Given a synergistic Lyapunov function and feedback pair candidate (V, κ), define

$$\begin{aligned} \mathcal{E} := \{&(z,q) \in M \times Q : \\ &\langle \nabla V(z,q), \phi(z,q) + \psi(z,q)\kappa(z,q) \rangle = 0\} \end{aligned} \tag{4}$$

and let $\Psi \subset \mathcal{E}$ denote the largest weakly invariant set for the system

$$\left.\begin{aligned} \dot{z} &= \phi(z,q) + \psi(z,q)\kappa(z,q) \\ \dot{q} &= 0 \end{aligned}\right\} \quad (z,q) \in \mathcal{E} \ . \tag{5}$$

Let

$$\rho_V(z) = \min_{q \in Q} V(z,q) \tag{6}$$

and define

$$\mu(V,\kappa) := \inf_{(z,q) \in \Psi \setminus \mathcal{A}} V(z,q) - \rho_V(z), \tag{7}$$

using the convention that $\mu(V,\kappa) = \infty$ when $\Psi \setminus \mathcal{A}$ is empty. The pair (V,κ) is called a *synergistic Lyapunov function and feedback pair* if $\mu(V,\kappa) > 0$, in which case $\mu(V,\kappa)$ is called the *synergy gap*. When $\mu(V,\kappa) > \delta > 0$ we say that the synergy gap exceeds δ.

[1]Here and in the rest of the paper, "smooth" means continuously differentiable enough times so that all used derivatives are well defined and continuous. For k steps of backstepping, it is enough for ϕ and ψ to be C^{k-1}.

Remark 6. In the setting of [9], Ψ is a set corresponding to the critical values of the potential function, which is finite under some mild conditions.

IV. Hybrid control using a synergistic Lyapunov function and feedback

In this section, we develop a hybrid feedback for the control system (2) using a synergistic Lyapunov function and feedback pair relative to the compact set $\mathcal{A}$ that globally asymptotically stabilizes $\mathcal{A}$. Let (V,κ) be a synergistic Lyapunov function and feedback pair with gap exceeding $\delta > 0$. We propose the hybrid controller

$$\begin{aligned} C &= \{(z,q) \in M \times Q : V(z,q) - \rho_V(z) \leq \delta\} \\ \omega &= \kappa(z,q) \\ D &= \{(z,q) \in M \times Q : V(z,q) - \rho_V(z) \geq \delta\} \\ G(z) &= \{q \in Q : V(z,q) = \rho_V(z)\}, \end{aligned} \tag{8}$$

where $C, D \subset M \times Q$, resulting in the closed-loop system

$$\underbrace{\begin{aligned} \dot{z} &= \phi(z,q) + \psi(z,q)\kappa(z,q) \\ \dot{q} &= 0 \end{aligned}}_{(z,q) \in C} \qquad \underbrace{\begin{aligned} z^+ &= z \\ q^+ &\in G(z) \end{aligned}}_{(z,q) \in D.} \tag{9}$$

Theorem 7. *Suppose that (V,κ) is a synergistic Lyapunov function and feedback pair relative to the compact set $\mathcal{A}$ with synergy gap exceeding δ for the system* (2)*. Then, the compact set $\mathcal{A}$ is globally asymptotically stable for the closed-loop system (9).*

Proof of Theorem 7: Consider the synergistic Lyapunov function V and feedback κ and note that (3) holds for all $(z,q) \in M \times Q$. In particular, (3) holds for all $(z,q) \in C$. Also, by the construction of D and G in (8), for all $(z,q) \in D$ and $g \in G(z)$, we have $V(z,q) - V(z,g) \geq \delta > 0$. In particular, V is nonincreasing along flows of (9) and strictly decreasing over jumps of (9). Using the properties of V, it follows that the set $\mathcal{A}$ is stable and all solutions are bounded. It remains to establish that all complete solutions converge to $\mathcal{A}$. By the invariance principle in [25], since $\{(z,q) \in M \times Q : V(z,q) - V(z,g) = 0, g \in G(z)\} \cap D = \emptyset$, all complete solutions to (9) converge to the largest weakly invariant set contained in the set $\mathcal{E} \cap C$, where $\mathcal{E}$ was defined in (4). From the definition of the closed-loop system (9), computing such a set amounts to finding the largest weakly invariant set of

$$\left.\begin{aligned} \dot{z} &= \phi(z,q) + \psi(z,q)\kappa(z,q) \\ \dot{q} &= 0 \end{aligned}\right\} \quad (z,q) \in \mathcal{E} \cap C. \tag{10}$$

According to the definition of Ψ, this weakly invariant set must be contained in $\Psi \cap C$. Since V is positive definite with respect to $\mathcal{A}$, $V(z,q) - \rho_V(z) = 0 \leq \delta$ for all $(z,q) \in \mathcal{A}$ which implies that $\mathcal{A} \subset C$. Then, it follows that $\Psi \cap C \subset ((\Psi \setminus \mathcal{A}) \cup \mathcal{A}) \cap C = ((\Psi \setminus \mathcal{A}) \cap C) \cap \mathcal{A}$. But then, since $\mu(V,\kappa) > \delta > 0$, it follows that $(\Psi \setminus \mathcal{A}) \cap C = \emptyset$, so that all complete solutions to (9) converge to $\mathcal{A}$. □

The next corollary follows from Theorem 7 together with the fact that $C \cup D = M \times Q$.

Corollary 8. *Under the conditions of Theorem 7, if for each $(z,q) \in M \times Q$, $\phi(z,q) + \psi(z,q)\kappa(z,q)$ belongs to the tangent cone of M at z then each maximal solution is complete.*

V. Weak synergistic Lyapunov function and feedback

In this section, we introduce the notion of a *weak* synergistic Lyapunov function and feedback for the system (2). Given a synergistic Lyapunov function and feedback pair candidate, define

$$\mathcal{W} := \left\{(z,q) \in M \times Q : \psi(z,q)^\top \nabla V(z,q) = 0\right\}. \quad (11)$$

Recall the definition of $\mathcal{E}$ in Section III, and let $\Omega \subset \mathcal{E} \cap \mathcal{W}$ denote the largest weakly invariant set for the system

$$\left.\begin{aligned} \dot{z} &= \phi(z,q) + \psi(z,q)\kappa(z,q) \\ \dot{q} &= 0 \end{aligned}\right\} \quad (z,q) \in \mathcal{E} \cap \mathcal{W}. \quad (12)$$

Define

$$\mu_{\mathcal{W}}(V,\kappa) := \inf_{(z,q) \in \Omega \setminus \mathcal{A}} V(z,q) - \rho_V(z), \quad (13)$$

using the convention that $\mu_{\mathcal{W}}(V,\kappa) = \infty$ when $\Omega \setminus \mathcal{A}$ is empty. The pair (V,κ) is called a *weak synergistic Lyapunov function and feedback pair* relative to the compact set $\mathcal{A} \subset M \times Q$ if $\mu_{\mathcal{W}}(V,\kappa) > 0$, in which case $\mu_{\mathcal{W}}(V,\kappa)$ is called the *weak synergy gap*. When $\mu_{\mathcal{W}}(V,\kappa) > \delta > 0$, we say that the weak synergy gap exceeds δ.

Lemma 9. *If (V,κ) is a synergistic Lyapunov function and feedback pair with synergy gap exceeding δ then it is also a weak synergistic Lyapunov function and feedback pair with weak synergy gap exceeding δ.*

Proof. By definition, the set Ω of this section is contained in the set Ψ in Section III. Thus, $\mu_{\mathcal{W}}(V,\kappa) \geq \mu(V,\kappa)$. Hence, if $\mu(V,\kappa) > \delta$ then $\mu_{\mathcal{W}}(V,\kappa) > \delta$. ☐

VI. Backstepping

Consider the control system

$$\left.\begin{aligned} \dot{\zeta} &= \phi_1(\zeta,q) + \psi_1(\zeta,q)u \\ \dot{q} &= 0 \end{aligned}\right\} \quad (\zeta,q) \in M_1 \times Q \quad (14)$$

with controls $u \in \mathbb{R}^m$, where $\zeta = (z,\omega,p)$ is the state and

$$\phi_1(\zeta,q) = \begin{bmatrix} \phi_0(z,q) + \psi_0(z,q)\omega \\ 0 \\ v(z,p,q) \end{bmatrix} \quad \psi_1(\zeta,q) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \quad (15)$$

For an appropriate choice for the function v we construct a (non-weak) synergistic Lyapunov function and feedback pair with synergy gap exceeding $\delta > 0$ for (14) by supposing we have a weak synergistic Lyapunov function and feedback pair with weak synergy gap exceeding δ for the reduced system

$$\left.\begin{aligned} \dot{z} &= \phi_0(z,q) + \psi_0(z,q)\omega \\ \dot{q} &= 0 \end{aligned}\right\} \quad (z,q) \in M_0 \times Q \quad (16)$$

with controls $\omega \in \mathbb{R}^m$.

Let (V_0,κ_0) be a weak synergistic Lyapunov function and feedback pair relative to the compact set $\mathcal{A}_0 \subset M_0 \times Q$, where $M_0 \subset \mathbb{R}^n$ is closed and Q is a discrete set, with weak synergy gap exceeding $\delta > 0$, for the system (16). We suppose that $\kappa_0 : M_0 \times Q \to \mathbb{R}^m$ can be written as linear in some function of the variable q. In particular, we assume that there exists a smooth function $\vartheta : M_0 \to \mathbb{R}^{m \times L}$ and some function $\sigma : Q \to \mathbb{R}^L$, where $L \geq 1$, such that

$$\kappa_0(z,q) = \vartheta(z)\sigma(q). \quad (17)$$

Remark 10. We note that κ_0 can always be decomposed as in (17). Assuming, without loss of generality, that $Q = \{1,\ldots,N\}$, let $\sigma(q) = \mathbf{e}_q$, where $\mathbf{e}_i \in \mathbb{R}^L$ denotes the ith unit vector, and let $\vartheta(z) = \begin{bmatrix}\kappa_0(z,1) & \cdots & \kappa_0(z,N)\end{bmatrix}$. Then, (17) holds.

Define

$$\begin{aligned} \mathcal{A}_1 := \{(\zeta,q) \in M_1 \times Q : (z,q) \in \mathcal{A}_0,& \\ p = \sigma(q),\ \omega = \kappa_0(z,q)\}.& \end{aligned} \quad (18)$$

For a vector $\xi \in \mathbb{R}^r$ and a symmetric, positive definite matrix $\Gamma \in \mathbb{R}^{r \times r}$, let $\lambda_{\max}(\Gamma)$ denote the largest eigenvalue of Γ and define $|\xi|_\Gamma^2 := \xi^T \Gamma \xi$. Consider the function $V_1 : M_1 \times Q \to \mathbb{R}_{\geq 0}$, $M_1 = M_0 \times \mathbb{R}^m \times \mathbb{R}^L$, defined, for each $(\zeta,q) \in M_1 \times Q$, as

$$V_1(\zeta,q) := V_0(z,q) + \frac{1}{2}|p - \sigma(q)|_{\Gamma_1}^2 + \frac{1}{2}|\omega - \vartheta(z)p|_{\Gamma_2}^2, \quad (19)$$

where $\Gamma_1 \in \mathbb{R}^{L \times L}$ and $\Gamma_2 \in \mathbb{R}^{m \times m}$ are symmetric positive definite matrices such that

$$\mu_{\mathcal{W}}(V_0,\kappa_0) - \tfrac{1}{2}\lambda_{\max}(\Gamma_1) \max_{s,q \in Q} |\sigma(s) - \sigma(q)|^2 > \delta, \quad (20)$$

which is possible since the weak synergistic Lyapunov function and feedback pair (V_0,κ_0) has a weak synergy gap exceeding δ and Q is a finite set.

Let $\theta_1,\theta_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be continuous, positive definite functions, and let the smooth functions $\Theta_1 : \mathbb{R}^L \to \mathbb{R}^L$ and $\Theta_2 : \mathbb{R}^m \to \mathbb{R}^m$ satisfy

$$v^T \Gamma_i \Theta_i(v) + \Theta_i(v)^T \Gamma_i v \leq -\theta_i(|v|) \quad \forall i \in \{1,2\} \quad (21)$$

where the inequality should hold for all $v \in \mathbb{R}^L$ for $i = 1$ and for all $v \in \mathbb{R}^m$ for $i = 2$. Let $\vartheta_i(z) = \vartheta(z)\mathbf{e}_i$. Define

$$\begin{aligned} \kappa_1(\zeta,q) &= \Theta_2(\omega - \vartheta(z)p) \\ &\quad - \Gamma_2^{-1}\psi_0(z,q)^\top \nabla_z V_0(z,q) \\ &\quad + \sum_{i=1}^{L} \mathbf{e}_i^\top p \mathcal{D}\vartheta_i(z)(\phi_0(z,q) + \psi_0(z,q)\omega) \\ &\quad + \vartheta(z)v(z,p,q) \\ v(z,p,q) &= \Theta_1(p - \sigma(q)) \\ &\quad - \Gamma_1^{-1}\vartheta(z)^\top \psi_0(z,q)^\top \nabla_z V_0(z,q). \end{aligned} \quad (22)$$

The following theorem establishes that (V_1,κ_1) is a synergistic Lyapunov function and feedback pair with synergy gap exceeding δ.

Theorem 11. *Let the compact set $\mathcal{A}_1$ be defined as in (18) and let the pair (V_1, κ_1) and the function v be defined by (19), (22). If, for the system (16), the pair (V_0, κ_0) is a weak synergistic Lyapunov function and feedback pair relative to the compact set $\mathcal{A}_0$ with weak synergy gap exceeding δ then, for the system (14)-(15), the pair (V_1, κ_1) is a (non-weak) synergistic Lyapunov function and feedback pair relative to $\mathcal{A}_1$ and with (non-weak) synergy gap exceeding δ.*

Proof. For all $(\zeta, q) \in M_1 \times Q$,

$$\begin{aligned}\langle \nabla_\zeta V_1(\zeta,q), \phi_1(\zeta,q) + \psi_1(\zeta,q)\kappa_1(\zeta,q)\rangle = \\ \langle \nabla_z V_0(z,q), \phi_0(z,q) + \psi_0(z,q)\omega\rangle \\ - \tfrac{1}{2}\theta_1(|p-\sigma(q)|) - \tfrac{1}{2}\theta_2(|\omega - \vartheta(z)p|) \\ - \langle \nabla_z V_0(z,q), \psi_0(z,q)\vartheta(z)(p-\sigma(q))\rangle \\ - \langle \nabla_z V_0(z,q), \psi_0(z,q)(\omega - \vartheta(z)p)\rangle \\ = \langle \nabla_z V_0(z,q), \phi_0(z,q) + \psi_0(z,q)\vartheta(z)\sigma(q)\rangle \\ - \tfrac{1}{2}\theta_1(|p-\sigma(q)|) - \tfrac{1}{2}\theta_2(|\omega - \vartheta(z)p|) \\ \leq 0.\end{aligned} \tag{23}$$

Define

$$\begin{aligned}\mathcal{E}_1 = \{(z,q) \in M_1 \times Q : \\ \langle \nabla_\zeta V_1(\zeta,q), \phi_1(z,q) + \psi_1(z,q)\kappa_1(\zeta,q)\rangle = 0\}, \\ \mathcal{W}_1 = \left\{(z,q) \in M_1 \times Q : \psi_1(z,q)^\top \nabla_\zeta V_1(\zeta,q) = 0\right\}.\end{aligned} \tag{24}$$

Let $\mathcal{E}_0$, $\mathcal{W}_0$, and Ω_0 come from the definitions in Section V for the weak synergistic Lyapunov function and feedback pair (V_0, κ_0) for the system (16). It follows from (23), the properties of θ_i, the definition of ψ_1 in (15), and the definition of V_1 in (19) that

$$\mathcal{E}_1 = \{(z,q) \in \mathcal{E}_0,\ \omega = \vartheta(z)p,\ p = \sigma(q)\} \subset \mathcal{W}_1. \tag{25}$$

Let $\Psi_1 \subset M_1 \times Q$ denote the largest weakly invariant set for the system

$$\left.\begin{aligned}\dot\zeta &= \phi_1(z,q) + \psi_1(z,q)\kappa_1(\zeta,q) \\ \dot q &= 0\end{aligned}\right\} \quad (\zeta,q) \in \mathcal{E}_1\ . \tag{26}$$

It follows from the definition of u in (22), the fact that $\dot\omega = \kappa_1(\zeta, q)$ and the characterization of $\mathcal{E}_1$ in (25) that

$$\begin{aligned}\Psi_1 = \{(\zeta,q) \in M_1 \times Q : (z,q) \in \Omega_0, \\ \omega = \vartheta(z)p,\ p = \sigma(q)\}.\end{aligned} \tag{27}$$

Then, it follows from (19) that

$$\begin{aligned}\mu(V_1,\kappa_1) &= \inf_{(\zeta,q)\in\Psi_1\setminus\mathcal{A}_1} V_1(\zeta,q) - \rho_{V_1}(\zeta) \\ &\geq \mu_{\mathcal{W}}(V_0,\kappa_0) - \tfrac{1}{2}\max_{q,s\in Q}|\sigma(q)-\sigma(s)|^2_{\Gamma_1} \\ &\geq \mu_{\mathcal{W}}(V_0,\kappa_0) - \tfrac{1}{2}\lambda_{\max}(\Gamma_1)\max_{q,s\in Q}|\sigma(q)-\sigma(s)|^2 \\ &> \delta\ .\end{aligned} \tag{28}$$

Thus, the pair (V_1, κ_1) is a synergistic Lyapunov function and feedback pair with gap exceeding $\delta > 0$. □

Remark 12. It follows by combining Theorems 7 and 11 that we can use synergistic Lyapunov functions to build hybrid stabilizers with an arbitrarily number of integrators between the ideal system and the control variables. At each level of backstepping, we add L additional states, corresponding to the state p in (14)-(15).

Remark 13. If $\kappa(z,q) = \vartheta(z)\sigma(q)$ is independent of q, i.e., all of the columns of $\vartheta(z)$ are the same, then the variable p can be removed from the control scheme, by replacing p by $\sigma(q)$ in the Lyapunov function in (19).

Remark 14. If the goal is just to make the control ω continuously differentiable without insisting on controlling through an integrator, the state ω can be removed from the control scheme, by replacing ω by $\vartheta(z)p$ in the Lyapunov function in (19). However, in this case, one must begin with a non-weak synergistic Lyapunov function and feedback pair.

VII. Hybrid control of rigid-body attitude

The attitude of a rigid body is represented by a 3×3 rotation matrix $R \in \mathsf{SO}(3) = \{R \in \mathbb{R}^{3\times 3} : RR^\top = R^\top R = I,\ \det R = 1\}$. Consider the kinematic equations of a rigid body in a quaternion parametrization given by

$$z = \begin{bmatrix}\eta \\ \epsilon\end{bmatrix} \in \mathbb{S}^3 \qquad \dot z = \frac{1}{2}\begin{bmatrix} -\epsilon^\top \\ \eta I + [\epsilon]_\times \omega, \end{bmatrix} \tag{29}$$

where $\mathbb{S}^3 = \{(\eta,\epsilon) \in \mathbb{R}\times\mathbb{R}^3 : \eta^2 + \epsilon^\top\epsilon = 1\}$ is the unit 3-sphere embedded in $\mathbb{R}^4$, $z \in \mathbb{S}^3$ is the unit quaternion representing the attitude, $\omega \in \mathbb{R}^3$ is the angular velocity, and

$$[v]_\times = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0\end{bmatrix}.$$

A quaternion $z = (\eta, \epsilon)$ is related to a rigid-body attitude through the Rodrigues formula, $\mathcal{R} : \mathbb{S}^3 \to \mathsf{SO}(3)$, defined as

$$\mathcal{R}(z) = I + 2\eta\,[\epsilon]_\times + 2\,[\epsilon]^2_\times\ .$$

We note that for each $R \in \mathsf{SO}(3)$ there exist exactly two antipodal unit quaternions satisfying $\mathcal{R}(\pm z) = R$. Furthermore, since $\mathcal{R}(z) = I$ if and only if $z = \pm\mathbf{e}_1 = (\pm 1, 0) \in \mathbb{S}^3$, we wish to globally asymptotically stabilize the disconnected set $z = \pm\mathbf{e}_1$ for the system (29).

Let $Q = \{-1, 1\}$, $\mathcal{A}_0 = \{(z,q) \in \mathbb{S}^3 \times Q : z = q\mathbf{e}_1\}$, $V_0(z,q) = 2k(1 - q\eta) = 2k(1 - \langle z, q\mathbf{e}_1\rangle)$, and $\kappa_0(z,q) = 0$. Since V_0 is continuous and $\mathbb{S}^3$ is compact, its sub-level sets are compact and furthermore, it is positive definite with respect to $\mathcal{A}_0$. Since $\kappa_0(z,q) = 0$, it follows that V_0 satisfies (3), $\mathcal{E}_0 = \mathbb{S}^3 \times Q$, and $\mathcal{W} = \Omega_0 = \{(z,q) : z = \pm\mathbf{e}_1\}$ so that $\Omega_0 \setminus \mathcal{A}_0 = \{(z,q) \in \mathbb{S}^3 \times Q : z = -q\mathbf{e}_1\}$. Finally we see that

$$\begin{aligned}\mu_{\mathcal{W}}(V_0,\kappa_0) &= \inf_{z=-q\mathbf{e}_1} V_0(z,q) - \rho_{V_0}(z) \\ &= V_0(-q\mathbf{e}_1, q) - \rho_{V_0}(-q\mathbf{e}_1) \\ &= 4k > 0,\end{aligned}$$

so that (V_0, κ_0) is a weak synergistic Lyapunov function and feedback pair for (29) relative to $\mathcal{A}_0$ with gap exceeding any $\delta \in (0, 4k)$.

Consider the angular velocity dynamics

$$J\dot{\omega} = [J\omega]_\times \omega + \tau, \tag{30}$$

where $\tau \in \mathbb{R}^3$ is a control torque. We let $\tau = -[J\omega]_\times \omega + Ju$ so that $\dot{\omega} = u$ and now apply the backstepping procedure with $\Gamma_2 = J$ and, since κ_0 does not depend on q, $p = \sigma(q) = 0$, to obtain

$$V_1(z,\omega,q) = 2k(1-q\eta) + \tfrac{1}{2}\omega^\top J\omega$$
$$\kappa_1(q,\omega,q) = \Theta_2(\omega) - qJ^{-1}k\epsilon,$$

where Θ_2 satisfies (21). A possible choice for Θ_2 is $\Theta_2(\omega) = J^{-1}([J\omega]_\times \omega - \Phi(\omega))$, where $\Phi(0) = 0$ and $\omega^\top\Phi(\omega) \geq \theta(|\omega|)$ for some positive definite $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Then, it follows that, with $u = \kappa_1$,

$$\tau(z,\omega,q) = -qk\epsilon - \Phi(\omega) \tag{31}$$

and finally, (V_1, τ) is a (non-weak) synergistic Lyapunov function and feedback pair for (29), (30) relative to $\mathcal{A}_1 = \{(z,\omega,q) : z = q\mathbf{e}_1,\ \omega = 0\}$ with gap exceeding any $\delta \in (0, 4k)$. Applying the hybrid controller (8) recovers the tracking controller of [6] when applied to point stabilization, which globally asymptotically stabilizes $\mathcal{A}_1$ for the closed-loop hybrid system.

To smooth the torque feedback (31), we can replace $q \in Q$ by $p \in \mathbb{R}$ in (31) and apply the backstepping procedure without controlling τ through an integrator. We form the Lyapunov function

$$V_2(z,\omega,p,q) = V_1(z,\omega,q) + \gamma\tfrac{1}{2}(p-q)^2,$$

where $\gamma > 0$ and $4k - \gamma > 0$ and obtain, through the backstepping procedure, the dynamics for $p \in \mathbb{R}$ as

$$\dot{p} = v(z,\omega,p,q) = \frac{k}{\gamma}\omega^\top\epsilon - k_p(p-q), \tag{32}$$

where $k_p > 0$. By defining $\kappa_2(z,\omega,p,q) = \tau(z,\omega,p)$, it follows that (V_2, κ_2) is a synergistic Lyapunov function and feedback pair for the system (29), (30), (32), relative to $\mathcal{A}_2 = \{(z,\omega,p,q) : z = q\mathbf{e}_1,\ \omega = 0,\ p = q\}$ with gap exceeding any $\delta \in (0, 4k - \gamma)$.

VIII. Conclusion

We have defined synergistic Lyapunov function and feedback pairs, in both weak and non-weak versions. Our main result has been to show how to pass from weak synergistic Lyapunov function and feedback pairs to non-weak synergistic Lyapunov function and feedback pairs through backstepping. In turn, this result permits constructing hybrid feedback control laws through a chain of integrators. The latter result is useful in the case where unmodeled dynamics would be sensitive to abrupt changes in the control signal. This construction allowed us to recover the hybrid feedback of [6] for rigid-body attitude stabilization in a quaternion setting and smooth it through backstepping. In a similar fashion, this methodology can recover the control laws proposed in [10], [11] and furthermore, allow those control laws to be smoothed.

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