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# Finite-time Consensus of Multi-agent Networks with Inherent Nonlinear Dynamics Under an Undirected Interaction Graph

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*Abstract*— **This paper studies finite-time consensus of multi-agent networks with inherent nonlinear dynamics where each agent is driven by a nonlinear term based on its state under an undirected interaction graph. We propose two distributed nonlinear algorithms to guarantee finite-time consensus. To facilitate the stability analysis of the closed-loop system using the proposed nonlinear algorithms, we present a general comparison lemma. The general comparison lemma provides an important tool in the stability analysis of linear/nonlinear closed-loop systems by making use of known results in linear/nonlinear systems. With the aid of the general comparison lemma, the two nonlinear algorithms are shown to guarantee finite-time consensus by comparing the original closed-loop systems with one or more predesigned closed-loop systems that can guarantee finite-time consensus.**

## I. Introduction

The past decade has witnessed an increasing research interest in the study of distributed control of multi-agent networks. The main objective of multi-agent networks is to design proper local controllers for a team of networked agents such that a desired group behavior can be accomplished. As one of the fundamental research topics in distributed control of multi-agent networks, consensus over multi-agent networks has been studied extensively. The main objective of consensus is to design distributed algorithms such that a group of agents reach an agreement on some state of interest.

The study of consensus and its extensions can be roughly categorized as leaderless consensus, consensus tracking, and containment control based on the number of leaders involved in a team of networked agents. Leaderless consensus refers to the agreement of a group of agents on some state which is not specified beforehand. Leaderless consensus has been investigated under different scenarios, including deterministic interaction setting [1]–[3], stochastic interaction setting [4]–[6], and finite-time convergence [7]–[9]. Consensus tracking refers to the case where a group of agents tracks a time-varying leader's state. Consensus tracking has also been investigated under different scenarios, including continuous-time setting [10], [11], discrete-time setting [12], and finite-time convergence [13], [14]. Containment control refers to the convergence of some agents, designated as followers, into the minimal geometric space formed by the other agents, designated as leaders. Containment control is mainly studied for single-integrator kinematics [15], [16] and double-integrator dynamics [17]. In the aforementioned papers, consensus is studied either when there is no leader or when the leaders' states are some known external signals.

When each agent has its own reference state driven by an external signal, the authors in [18] study the dynamic average consensus problem where all agents reach an agreement on the average of all the reference states. When the reference states have steady-state values, a dynamic average consensus algorithm is proposed and analyzed by using a frequency-domain approach. In [19], two dynamic average consensus algorithms, *i.e.*, proportional-like (P-like) and proportional-and-integral-like (PI-like) algorithms, are proposed. In particular, the PI-like algorithm is efficient when the reference states are constant. However, when the reference states are varying, accurate average dynamic consensus cannot be guaranteed. In [20], the authors study second-order consensus of multi-agent systems with inherent nonlinear dynamics under a directed fixed interaction graph. Although the nonlinear dynamics can be considered external signals, the nonlinear dynamics in [20] are not necessarily bounded while the external signals in [18], [19] are assumed to be bounded.

In this paper, we study the finite-time consensus problem for multi-agent networks with inherent nonlinear dynamics where each agent is driven by a nonlinear term based on its state under an undirected interaction graph. Although finite-time consensus and consensus with inherent nonlinear dynamics have been studied previously, finite-time consensus with inherent nonlinear dynamics has not been investigated yet. First, we propose two nonlinear algorithms to guarantee finite-time consensus with inherent nonlinear dynamics. We then present a general comparison lemma which serves as the main tool used in the following stability analysis. The general comparison lemma provides an important tool in the stability analysis of linear/nonlinear closed-loop systems by making use of known results in linear/nonlinear systems. With the general comparison lemma, the two nonlinear algorithms are shown to guarantee finite-time consensus by comparing the original closed-loop systems with one or more predesigned closed-loop systems that can guarantee finite-time consensus.

## II. Preliminaries

### A. Graph Theory Notions

For a team of $n$ agents, we model the interaction among the $n$ agents by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the agent set and the edge set. Each edge denoted as $(v_i, v_j)$ means that agents $i$ and $j$ can obtain information from each other. That is, $v_i$ and $v_j$ are neighbors of each other. A path is a sequence of edges of the form $(v_1, v_2), (v_2, v_3), \cdots,$

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where $v_i \in \mathcal{V}$. An undirected graph is connected if there is an undirected path between every pair of distinct agents.

Two matrices are frequently used to represent the interaction graph: the adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ with $a_{ij} > 0$ if $(v_j, v_i) \in \mathcal{W}$ and $a_{ij} = 0$ otherwise, and the Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ with $\ell_{ii} = \sum_{j=1,j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}$, $i \neq j$. In particular, we let that $a_{ii} = 0$, $i = 1, \cdots, n$, (*i.e.*, agent $i$ is not a neighbor of itself) and $a_{ij} = a_{ji}$ (*i.e.*, $\mathcal{A}$ and $\mathcal{L}$ are symmetric). It is straightforward to verify that $\mathcal{L}$ is symmetric semi-definite and $\mathcal{L}$ has at least one zero eigenvalue with a corresponding left eigenvector $\mathbf{1}_n^T$ and a corresponding right eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is an $n \times 1$ all-one column vector.

### B. Notations

We use $\mathbb{R}$ to denote the set of real number. $\mathbf{0}_n \in \mathbb{R}^n$ is used to denote the $n \times 1$ all-zero column vector. $I_n \in \mathbb{R}^{n\times n}$ is used to denote the identity matrix. We use $\text{sgn}(\cdot)$ to denote the signum function. Define $\text{sig}(x)^\alpha = \text{sgn}(x)\,|x|^\alpha$. Let $f : [0,\infty) \mapsto J \subseteq \mathbb{R}^n$ be a continuous function. The upper right-hand derivative of $f(t)$ is given by $D^+ f(t) = \limsup_{h\to 0^+} \frac{1}{h}[f(t+h) - f(t)]$.

## III. Finite-time Consensus with Inherent Nonlinear Dynamics

Consider a group of $n$ agents with dynamics given by

$$\dot{r}_i = f(t, r_i) + u_i, \quad i = 1, \cdots, n, \tag{1}$$

where $r_i \in \mathbb{R}$ is the state of the $i$th agent, $f(t, r_i) \in \mathbb{R}$ is the inherent nonlinear dynamics for the $i$th agent, and $u_i \in \mathbb{R}$ is the control input for the $i$th agent. Note that each agent is driven by a nonlinear term $f(t, r_i)$. Here we assume that $|f(t, r_i) - f(t, r_j)| \leq \gamma\,|r_i - r_j|$, where $\gamma$ is a known positive constant.

The objective here is to design $u_i$ such that $\left|r_i(t) - \frac{1}{n}\sum_{j=1}^{n} r_j(t)\right| \to 0$ in finite time. That is, all agents' states converge to the average of all $r_j(t)$ in finite time. We propose the following two nonlinear finite-time consensus algorithms for (1) as

$$u_i = -\beta \sum_{j=1}^{n} a_{ij}(t)\text{sgn}(r_i - r_j), \tag{2}$$

$$u_i = -\beta \sum_{j=1}^{n} a_{ij}(t)\text{sig}(r_i - r_j)^{\alpha(|r_i - r_j|)}, \tag{3}$$

where $\beta$ is a positive constant, $a_{ij}(t)$ is the $(i,j)$th entry of the adjacency matrix $\mathcal{A}(t)$ characterizing the interaction among the $n$ agents at time $t$, and $\alpha(|r_i - r_j|)$ satisfies that

$$\alpha(|r_i - r_j|) \begin{cases} = \alpha^\star \in (0,1), & 0 \leq |r_i - r_j| < 1, \\ = 1, & |r_i - r_j| \geq 1, \end{cases}$$

where $\alpha^\star$ is a positive constant. Note that the existence of the solution to (1) using (2) (respectively, (3)) can be guaranteed by Proposition 3 in [21].

We assume that the adjacency matrix $\mathcal{A}(t)$ is constant for $t \in [t_i, t_{i+1})$ and switches at time $t_{i+1}$, $i = 0, 1, \cdots$. Let $\mathcal{G}_i$, $\mathcal{A}_i$, and $\mathcal{L}_i$ denote, respectively, the directed graph, the adjacency matrix, and the Laplacian matrix associated with the $n$ agents for $t \in [t_i, t_{i+1})$. We assume that $t_{i+1} - t_i \geq t_L$, where $t_L$ is a positive constant. We also assume that each nonzero and hence positive entry of $\mathcal{A}_i$ has a lower bound $\underline{a}$ and an upper bound $\overline{a}$, where $\underline{a}$ and $\overline{a}$ are positive constants.

### A. Stability Analysis for Algorithm (2)

In this subsection, we analyze the stability of (1) when using (2). Before moving on, we need the following lemmas.

*Lemma 3.1:* [22, Theorem 3.5] Let $f(t, x, \lambda)$ be continuous in $(t, x, \lambda)$ and locally Lipschitz in $x$ (uniformly in $t$ and $\lambda$) on $[t_0, t_1] \times J \times \{\|\lambda - \lambda_0\| \leq c\}$, where $J \subset \mathbb{R}^n$ is an open connected set. Let $y(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ with $y(t_0, \lambda_0) = y_0 \in J$. Suppose that $y(t, \lambda_0)$ is defined and belongs to $J$ for all $t \in [t_0, t_1]$. Then, given $\epsilon > 0$, there is $\delta > 0$ such that if $\|z_0 - y_0\| < \delta$ and $\|\lambda - \lambda_0\| < \delta$, then there is a unique solution $z(t, \lambda)$ of $\dot{x} = f(t, x, \lambda)$ defined on $[t_0, t_1]$, with $z(t_0, \lambda) = z_0$, and $z(t, \lambda)$ satisfies $\|z(t, \lambda) - y(t, \lambda_0)\| < \epsilon$ for all $t \in [t_0, t_1]$.

With Lemma 3.1, we next present the comparison lemmas for vector differential equations.

*Lemma 3.2:* Consider the following vector differential equation

$$\dot{z} = f(t, z), \quad z(t_0) = \mu_0,$$

where $z = [z_1, \cdots, z_p]^T \in \mathbb{R}^p$, $f(t, z) = [f_1(t, z), \cdots, f_p(t, z)]^T$ is defined such that $f_i(t, z)$, $i = 1, \cdots, p$, is continuous in $t$ and locally Lipschitz in $z_i$, $i = 1, \cdots, p$, for all $t > 0$ and all $z \in J \subset \mathbb{R}^p$. Let $[t_0, T)$ ($T$ could be infinity) be the maximal interval of existence of the solution $z$, and suppose that $z \in J$ for all $t \in [t_0, T)$. Let $\omega \in \mathbb{R}^p$ be a continuous function whose upper right-hand derivative $D^+\omega$ satisfies the inequality

$$D^+\omega \leq f(t, \omega), \quad \omega(t_0) \leq \mu_0,$$

where $\omega \in J$ for all $t \in [t_0, T)$. Then $\omega(t) \leq z(t)$ for all $t \in [t_0, T)$.

*Proof:* The proof is motivated by that of Lemma 3.4 in [22]. Consider the following vector differential equation

$$\dot{x} = f(t, x) + \lambda, \quad x(t_0) = z(t_0) \tag{4}$$

for $i = 1, \cdots, p$, where $x \in \mathbb{R}^p$ and $\lambda = [\lambda_1, \cdots, \lambda_p]^T$ is a positive constant vector. For $t \in [t_0, t_1]$, where $t_1 > t_0$, it follows from Lemma 3.1 that for any $\epsilon > 0$, there is $\delta > 0$ such that if $\|\lambda\| < \delta$, (4) has a unique solution $\xi(t, \lambda)$ defined on $[t_0, t_1]$ and $\|\xi(t, \lambda) - z(t)\| < \epsilon, \forall t \in [t_0, t_1]$. Therefore, we have that

$$\|\xi_i(t, \lambda) - z_i(t)\| < \epsilon, \forall t \in [t_0, t_1]. \tag{5}$$

Claim 1: $\omega_i(t) \leq \xi_i(t, \lambda)$ for all $t \in [t_0, t_1]$. We prove this by contradiction. Assume that there exist times $a, b \in (t_0, t_1]$ such that $\omega_i(a) = \xi_i(a, \lambda)$ and $\omega_i(t) > \xi_i(t, \lambda)$ for $a < t \leq b$. Accordingly, we have that $\omega_i(t) - \omega_i(a) > \xi_i(t, \lambda) - \xi_i(a, \lambda), \forall t \in (a, b]$, which implies that $D^+\omega_i(a) \geq D^+\xi_i(a, \lambda) = \dot{\xi}_i(a, \xi) = f_i(a, \xi) + \lambda_i > f_i(a, \xi)$. This contradicts the inequality $D^+\omega \leq f(t, \omega)$.

Claim 2: $\omega_i(t) \leq z_i(t)$ for all $t \in [t_0, t_1]$. Again, we prove this by contradiction. Assume that there exists $a \in (t_0, t_1]$ such that $\omega_i(a) > z_i(a)$. Letting $\epsilon = \frac{\omega_i(a) - z_i(a)}{2}$ and using (5), we obtain that

$$\begin{aligned}\omega_i(a) - \xi_i(a, \lambda) =& \omega_i(a) - z_i(a) + z_i(a) - \xi_i(a, \lambda)\\ =& 2\epsilon + z_i(a) - \xi_i(a, \lambda) \geq \epsilon,\end{aligned}$$

which contradicts the statement of Claim 1. ■

*Lemma 3.3:* Suppose that $F(t, z) : [t_0, T) \times J \subseteq \mathbb{R}^p \mapsto \mathbb{R}^q$, is a continuous function satisfying that $D^+F = f(t, z)$, where $z \in \mathbb{R}^p$, and $f(t, z)$ is piecewise continuous in $t$ and is locally Lipschitz in $z$ when $f(t, z)$ is continuous at $t$. Let $G(t, \omega) : [t_0, T) \times J \subseteq \mathbb{R}^p \mapsto \mathbb{R}^q$ be a continuous function whose upper right-hand derivative $D^+G$ satisfies the differential inequality

$$D^+G \leq f(t, \omega), \quad G[t_0, \omega(t_0)] \leq F[t_0, z(t_0)].$$

Then $G(t) \leq F(t)$ for all $t \in [t_0, T)$.

*Proof:* The proof of the lemma can be divided into three cases:

Case 1: $q = p$. Without loss of generality, assume that $f(t, z)$ is continuous in $t$ for $t \in [t_i, t_{i+1}),\ i = 0, 1, \ldots$. For $t \in [t_0, t_1)$, consider a new vector differential equation given by

$$\dot{x} = f(t, x), \quad x(t_0) = F[t_0, z(t_0)]. \tag{6}$$

Because $D^+F = f(t, z) \leq f(t, z)$ and $F(t_0) = x(t_0) \leq x(t_0)$ are trivially satisfied, it follows from Lemma 3.2 that $F(t) \leq x(t)$ for all $t \in [t_0, t_1)$. Noting also that $D^+(-F) = -f(t, z) \leq -f(t, z)$ and $-F(t_0) = -x(t_0) \leq -x(t_0)$ are trivially satisfied, it follows from Lemma 3.2 that $-F(t) \leq -x(t)$ for all $t \in [t_0, t_1)$. Combining the two arguments shows that $F(t) = x(t)$ for all $t \in [t_0, t_1)$. Note that $D^+G \leq f(t, \omega)$ and $G[t_0, z(t_0)] \leq F[t_0, \omega(t_0)] = x(t_0)$. It thus follows from Lemma 3.2 that $G(t) \leq x(t)$ for all $t \in [t_0, t_1)$. Because $F(t) = x(t)$ for all $t \in [t_0, t_1)$, it follows that $G(t) \leq F(t)$ for all $t \in [t_0, t_1)$. Because $F(t)$ is a continuous function, by employing a similar analysis for $t \in [t_i, t_{i+1}),\ i = 1, \ldots,$ it can be shown that $G(t) \leq F(t)$ for all $t \in [t_i, t_{i+1}),\ i = 1, \ldots$. Therefore $G(t) \leq F(t)$ for all $t \in [t_0, T)$.

Case 2: $1 \leq q < p$. Define $\widetilde{F} \stackrel{\triangle}{=} [F^T, \mathbf{1}_{p-q}^T]^T$ and $\widetilde{G} \stackrel{\triangle}{=} [G^T, \mathbf{1}_{p-q}^T]^T$. By letting $\widetilde{F}$ and $\widetilde{G}$ play the role of, respectively, $F$ and $G$, it follows from a similar analysis to that of the case when $q = p$ that $\widetilde{G}(t) \leq \widetilde{F}(t)$ for all $t \in [t_0, T)$, which implies that $G(t) \leq F(t)$ for all $t \in [t_0, T)$.

Case 3: $q > p$. Note that there exists a positive integer $m$ such that $q = mp + q_d$, where $0 \leq q_d < p$ is a nonnegative integer. When $q_d = 0$, $G$ and $F$ can be written as $G = [G_1^T, \ldots, G_m^T]^T$ and $F = [F_1^T, \ldots, F_m^T]^T$, where $G_i \in \mathbb{R}^p$ and $F_i \in \mathbb{R}^p$ for all $i = 1, \ldots, m$. By applying the result of Case 1 repeatedly, we have $G_i(t) \leq F_i(t), \forall i = 1, \ldots, m$, for all $t \in [t_0, T)$. This implies $G(t) \leq F(t)$ for all $t \in [t_0, T)$. When $q_d \neq 0$, $G$ and $F$ can be written as $G = [G_1^T, \ldots, G_m^T, G_{m+1}^T]^T$ and $F = [F_1^T, \ldots, F_m^T, F_{m+1}^T]^T$, where $G_i \in \mathbb{R}^p$ and $F_i \in \mathbb{R}^p$ for all $i = 1, \ldots, m$, and $G_{m+1} \in \mathbb{R}^{q_d}$ and $F_{m+1} \in \mathbb{R}^{q_d}$. By applying the result of Case 1 repeatedly, we also have $G_i(t) \leq F_i(t), \forall i = 1, \ldots, m$, for all $t \in [t_0, T)$. By applying the result of Case 2, we have $G_{m+1}(t) \leq F_{m+1}(t)$ for all $t \in [t_0, T)$. This implies $G(t) \leq F(t)$ for all $t \in [t_0, T)$.

Combining the previous cases completes the proof. ■

*Theorem 3.1:* Assume that the interaction graph $\mathcal{G}_i, i = 0, 1, \ldots,$ is undirected and connected. Using (2) for (1), $|r_i(t) - r_j(t)| \to 0, \forall i, j = 1, \ldots, n$, in finite time if $\beta > \frac{2(n-1)\gamma \max_i |r_i(0) - \overline{r}(0)|}{n\underline{a}}$, where

$$\overline{r}(t) \stackrel{\triangle}{=} \frac{1}{n} \sum_{i=1}^{n} r_i(t). \tag{7}$$

*Proof:* Define $\delta_i \stackrel{\triangle}{=} r_i - \overline{r}$. We can get

$$\begin{aligned}\dot{\delta}_i &= \dot{r}_i - \dot{\overline{r}}\\ &= -\beta \sum_{j=1}^{n} a_{ij}(t) \text{sgn}(\delta_i - \delta_j)\\ &\quad - \frac{1}{n} \sum_{j=1, j\neq i}^{n} [f(t, \delta_j + \overline{r}) - f(t, \delta_i + \overline{r})],\end{aligned}$$

where we have used the fact that $\sum_{j=1}^{n} u_j = 0$ due to the symmetry of $\mathcal{A}(t)$ to derive the last equality. Define $\delta \stackrel{\triangle}{=} [\delta_1, \ldots, \delta_n]^T$. Consider the nonnegative function $G(t, \delta) = \max_i |\delta_i|$. Then the upper right-hand derivative of $G(t, \delta)$ can be derived as

$$\begin{aligned}D^+G(t, \delta) &= \limsup_{h \to 0^+} \frac{1}{h} \{G[t + h, \delta(t + h)] - G[t, \delta(t)]\}\\ &= \limsup_{h \to 0^+} \frac{1}{h} \left[\max_i |\delta_i(t + h)| - \max_i |\delta_i(t)|\right].\end{aligned} \tag{8}$$

We next study $D^+G(t, \delta)$ in the following three cases:

Case 1: $\max_i |\delta_i(t)| = \max_i \delta_i(t) > 0$ (*i.e.*, there exists at least one agent $j$ such that $\delta_j(t) > 0$ and $\max_i |\delta_i(t)| = \delta_j(t)$, and $|\min_i \delta_i(t)| < \max_i |\delta_i(t)|$) for some time interval $t \in [\underline{t}, \overline{t}]$, where $\underline{t} < \overline{t}$. In this case, it follows from (8) that

$$\begin{aligned}D^+G(t, \delta) &= \max_{i \in \arg\max_\ell \delta_\ell} D^+\delta_i\\ &\leq \max_{i \in \arg\max_\ell \delta_\ell} \left\{ -\beta \sum_{j=1}^{n} a_{ij}(t) \text{sgn}(r_i - r_j) \right.\\ &\qquad \left. - \frac{1}{n}\gamma \sum_{j=1, j\neq i}^{n} (r_j - r_i) \right\}\\ &= \max_{i \in \arg\max_\ell \delta_\ell} \left\{ -\beta \sum_{j=1}^{n} a_{ij}(t) \text{sgn}(\delta_i - \delta_j) \right.\\ &\qquad \left. - \frac{1}{n}\gamma \sum_{j=1, j\neq i}^{n} (\delta_j - \delta_i) \right\},\end{aligned} \tag{9}$$

Define $\varsigma_i \stackrel{\triangle}{=} -\beta \sum_{j=1}^{n} a_{ij}(t) \text{sgn}(\delta_i - \delta_j) - \frac{1}{n}\gamma \sum_{j=1, j\neq i}^{n} (\delta_j - \delta_i)$, $i \in \arg\max_\ell \delta_\ell$. For $i \in \arg\max_\ell \delta_\ell$ and $t \in [\underline{t}, \overline{t}]$, when $\beta > \frac{2(n-1)\gamma \max_i |r_i(t) - \overline{r}(t)|}{n\underline{a}}$

$\geq \left|-\frac{1}{n}\gamma\sum_{j=1,j\neq i}^{n}(\delta_j-\delta_i)\right|$, we have the following two statements:

1. $D^+G(t,\delta)\leq 0$ if there exists $j\in\mathcal{N}_i(t)$ such that $\delta_j(t)\neq\delta_i(t)$;
2. $D^+G(t,\delta)\leq 0$ if $\delta_j(t)=\delta_i(t),\forall j\in\mathcal{N}_i(t)$.

Note that Statement 1 holds apparently because $D^+\delta_i\leq\varsigma_i$ from (9), $\delta_j\neq\delta_i$ implies that $\delta_j<\delta_i$, and $\delta_j<\delta_i$ implies that $\varsigma_i<0$. We next briefly prove Statement 2. We prove this statement by contradiction. Assume that $D^+G(t,\delta)\leq 0$ does not hold for $t\in[\underline{t},\overline{t}]$ (*i.e.*, $\delta_i(t)$ increases for some interval $t\in[t_1,t_2]$, where $t_1$ and $t_2$ are positive constants satisfying $\underline{t}\leq t_1<t_2\leq\overline{t}$). Because $\delta_j(t)=\delta_i(t),\ \forall j\in\mathcal{N}_i(t)$, for $t\in[t_1,t_2]$, it follows from Statement 1 and the assumption that $\delta_i(t)$ increases for some interval $t\in[t_1,t_2]$ that $\delta_k(t)=\delta_j(t),\forall k\in\mathcal{N}_j(t)$ for $t\in[t_1,t_2]$. Repeating the analysis shows that $\delta_\ell(t)=\delta_i(t),\ell=1,\ldots,n$, for $t\in[t_1,t_2]$ because the interaction graph $\mathcal{G}_i,i=0,1,\ldots,$ is undirected and connected. Note from (9) that $D^+\delta_i\leq\varsigma_i$ and that $\varsigma_i=0$ when $\delta_\ell(t)=\delta_i(t),\ell=1,\ldots,n$. It follows that $D^+\delta_i\leq 0$, which implies that $\delta_i(t)$ is nonincreasing for $t\in[t_1,t_2]$. This contradicts the assumption that $\delta_i(t)$ increases for some interval $t\in[t_1,t_2]$. Note also that $\delta_i(t)$ is continuous with respect to $t$. Therefore, Statement 2 holds. Combining Statements 1 and 2 shows that $D^+G(t,\delta)\leq 0$ for $t\in[\underline{t},\overline{t}]$, which implies that $D^+G(t,\delta)\leq 0$ for $t\in[\underline{t},\overline{t}]$ when $\beta>\frac{2(n-1)\gamma\max_i|r_i(t)-\overline{r}(t)|}{n\underline{a}}$.

Case 2: $\max_i|\delta_i(t)|=-\min_i\delta_i(t)>0$ (*i.e.*, there exists at least one agent $h$ such that $\delta_h(t)<0$ and $\max_i|\delta_i(t)|=-\delta_h(t)$, and $|\max_i\delta_i(t)|<\max_i|\delta_i(t)|$) for some time interval $t\in[\underline{t},\overline{t}]$, where $\underline{t}<\overline{t}$. In this case, it follows from (8) that

$$\begin{aligned}D^+G(t,\delta)&=\max_{i\in\arg\min_\ell\delta_\ell}-D^+\delta_i\\&\leq\max_{i\in\arg\min_\ell\delta_\ell}\left\{\beta\sum_{j=1}^{n}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)\right.\\&\qquad\left.+\frac{1}{n}\gamma\sum_{j=1,j\neq i}^{n}(\delta_j-\delta_i)\right\}.\end{aligned}$$

By following a similar analysis to that of Case 1, we can get that $D^+G(t,\delta)\leq 0$ for $t\in[\underline{t},\overline{t}]$ when $\beta>\frac{2(n-1)\gamma\max_i|r_i(t)-\overline{r}(t)|}{n\underline{a}}$.

Case 3: $\max_i|\delta_i(t)|=\max_i\delta_i(t)=-\min_i\delta_i(t)>0$ (*i.e.*, there exist at least one agent, labeled as $j$, such that $\delta_j(t)>0$ and $\max_i|\delta_i(t)|=\delta_j(t)$ and at least one agent, labeled as $h$, such that $\delta_h(t)<0$ and $\max_i|\delta_i(t)|=-\delta_h(t)$) for some time interval $t\in[\underline{t},\overline{t}]$, where $\underline{t}<\overline{t}$. In this case, it can be computed that

$$D^+G(t,\delta)=\max_{i\in\arg\max_\ell\delta_\ell,j\in\arg\min_j\delta_j}\{D^+\delta_i,-D^+\delta_j\}.$$

By following the analysis in Cases 1 and 2, it follows that $D^+G(t,\delta)\leq 0$ for $t\in[\underline{t},\overline{t}]$ when $\beta>\frac{2(n-1)\gamma\max_i|r_i(t)-\overline{r}(t)|}{n\underline{a}}$.

Combining the three cases shows that $D^+G(t,\delta)\leq 0$ if $\beta>\frac{2(n-1)\gamma\max_i|r_i(t)-\overline{r}(t)|}{n\underline{a}}$. Note that $D^+G(t,\delta)\leq 0$ implies that $\frac{2(n-1)\gamma\max_i|r_i(t)-\overline{r}(t)|}{n\underline{a}}\leq\frac{2(n-1)\gamma\max_i|r_i(0)-\overline{r}(0)|}{n\underline{a}}$. Therefore, $D^+G(t,\delta)\leq 0$ if $\beta>\frac{2(n-1)\gamma\max_i|r_i(0)-\overline{r}(0)|}{n\underline{a}}$. Define $k_1\triangleq\frac{2(n-1)\gamma\max_i|r_i(0)-\overline{r}(0)|}{n\underline{a}}$. We next study the relationship between $\sum_{j=1}^{n}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)$ and $\mathrm{sgn}\left[\sum_{j=1}^{n}a_{ij}(t)(\delta_i-\delta_j)\right]$ and then rewrite $D^+G(t,\delta)$.

① $\max_i|\delta_i|=\max_i\delta_i$. When $\delta_i=\max_i|\delta_i|$ and $\delta_j=\delta_i,\forall j\in\mathcal{N}_i$, $\sum_{j=1}^{n}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)=0=\mathrm{sgn}\left[\sum_{j=1}^{n}a_{ij}(t)(\delta_i-\delta_j)\right]$. When $\delta_i=\max_i|\delta_i|$ and there exists at least one $j\in\mathcal{N}_i$ such that $\delta_j<\delta_i$, it follows that

$$\begin{aligned}\sum_{j=1}^{n}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)&=\sum_{j\in\mathcal{N}_i}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)\\&\geq\min_{\{j|\delta_j<\delta_i,j\in\mathcal{N}_i\}}a_{ij}(t)\\&\geq\underline{a}\mathrm{sgn}\left[\sum_{j=1}^{n}a_{ij}(t)(\delta_i-\delta_j)\right],\end{aligned}\tag{10}$$

where we have used the fact that $\min_{\{j|\delta_j<\delta_i,j\in\mathcal{N}_i\}}a_{ij}(t)\geq\underline{a}$. It can be further computed from (9) that

$$\begin{aligned}D^+G(t,\delta)&\leq\max_{i\in\arg\max_\ell\delta_\ell}\left\{k_1-\beta\sum_{j=1}^{n}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)\right\}\\&\leq\max_{i\in\arg\max_\ell\delta_\ell}\left\{k_1-\beta\underline{a}\mathrm{sgn}\left[\sum_{j=1}^{n}a_{ij}(t)(\delta_i-\delta_j)\right]\right\}.\end{aligned}\tag{11}$$

② $\max_i|\delta_i|=-\min_i\delta_i$. When $\delta_i=-\max_i|\delta_i|$, by following a similar analysis to that of Case 1, it can be derived that

$$\sum_{j=1}^{n}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)\leq\underline{a}\mathrm{sgn}\left[\sum_{j=1}^{n}a_{ij}(t)(\delta_i-\delta_j)\right].\tag{12}$$

It can also be computed that

$$\begin{aligned}D^+G(t,\delta)&\leq\max_{i\in\arg\min_\ell\delta_\ell}\left\{\beta\sum_{j=1}^{n}a_{ij}(t)\mathrm{sgn}(\delta_i-\delta_j)-k_1\right\}\\&\leq\max_{i\in\arg\min_\ell\delta_\ell}\left\{\beta\underline{a}\mathrm{sgn}\left[\sum_{j=1}^{n}a_{ij}(t)(\delta_i-\delta_j)\right]-k_1\right\}.\end{aligned}\tag{13}$$

③ $\max_i|\delta_i|=\max_i\delta_i=-\min_i\delta_i$. $D^+G(t,\delta)$ satisfies both (11) and (13).

Consider the closed-loop dynamics given by

$$\dot{\xi}_i=-\beta\underline{a}\mathrm{sgn}\left[\sum_{j=1}^{n}a_{ij}(t)(\xi_i-\xi_j)\right]+\nu,i=1,\ldots,n,\tag{14}$$

where $\xi_i\in\mathbb{R}$, $\xi_i(0)=\delta_i(0)$, and $\nu=k_1$. Define $F(t,\xi)\triangleq\max_i|\xi_i|$, where $\xi\triangleq[\xi_1,\ldots,\xi_n]^T$. We also study $D^+F(t,\xi)$

in three cases:

① $\max_i |\xi_i| = \max_i \xi_i$. In this case, it can be computed that

$$D^+F(t,\xi) = \max_{i\in\arg\max_\ell \xi_\ell} D^+\xi_i$$
$$= \max_{i\in\arg\max_\ell \xi_\ell} \left\{ -\beta\underline{a}\mathrm{sgn}\left[\sum_{j=1}^n a_{ij}(t)(\xi_i - \xi_j)\right] + k_1 \right\}. \tag{15}$$

② $\max_i |\xi_i| = -\min_i \xi_i$. In this case, it can be computed that

$$D^+F(t,\xi) = \max_{i\in\arg\min_\ell \xi_\ell} -D^+\xi_i$$
$$= \max_{i\in\arg\min_\ell \xi_\ell} \left\{ \beta\underline{a}\mathrm{sgn}\left[\sum_{j=1}^n a_{ij}(t)(\xi_i - \xi_j)\right] - k_1 \right\}. \tag{16}$$

③ $\max_i |\xi_i| = \max_i \xi_i = -\min_i \xi_i$. In this case, it can be computed that $D^+F(t,\delta)$ satisfies (15) and (16).

Note that $D^+F(t,\xi)$ is piecewise continuous in $t$ and is locally Lipschitz in $\xi_i$ when $D^+F(t,\xi)$ is continuous at $t$. Note also that $D^+G(t,\delta) \le D^+F(t,\delta)$. Because $\delta(0) = \xi(0)$ (*i.e.*, $G[0,\delta(0)] = F[0,\delta(0)]$), it follows from Lemma 3.3 that $G(t) \le F(t)$ for all $t \ge 0$. Given (14), if $\beta\underline{a} > k_1$ and the directed graph $\mathcal{G}_i, i = 0,1,\ldots,$ has a directed spanning tree, then it follows from Theorem 3.1 in [13] that $F(t) \to 0$ in finite time. Note from the definition of $G(t)$ that $G(t) \ge 0$ for all $t \ge 0$. It then follows from the fact that $G(t) \le F(t)$ that $G(t) \to 0$ in finite time. Combining with the definition of $G(t)$ implies that $\delta_i(t) \to 0$ in finite time. That is, $\|r_i(t) - r_j(t)\| \to 0, \forall i = 1,\ldots,n,$ in finite time. ■

*Corollary 3.2:* Let $f(t,r_i)$ in (2) be given by $f_i(t)$ and $|f_i(t)| < \gamma$. Assume that the interaction graph $\mathcal{G}_i, i = 0,1,\ldots,$ is undirected and connected. Using (2) for (1), $|r_i(t) - r_j(t)| \to 0$ in finite time if $\beta > \frac{2(n-1)\gamma}{n\underline{a}}$.

*Proof:* When $|f_i(t)| < \gamma$, it follows that $\left|\frac{1}{n}\sum_{j=1,j\ne i}^n [f_i(t) - f_j(t)]\right| \le \frac{2(n-1)\gamma}{n}$. The proof follows that of Theorem 3.1 by letting $k_1 = \frac{2(n-1)\gamma}{n}$. ■

### *B. Stability Analysis for Algorithm* (3)

In this subsection, we analyze the stability of (1) when using (3). Before stating the main result, we need the following lemma.

*Lemma 3.4:* [9] For the closed-loop system given by

$$\dot{x}_i = -\epsilon \sum_{j=1}^n a_{ij}(t)\mathrm{sig}(x_i - x_j)^\alpha, \tag{17}$$

where $x_i \in \mathbb{R}$, $\epsilon$ is any positive constant, $0 < \alpha < 1$, and $a_{ij}(t)$ is the $(i,j)$th entry of the adjacency matrix $\mathcal{A}(t)$ at time $t$. Assume that the interaction graph $\mathcal{G}_i, i = 0,1,\cdots,$ is undirected and connected. Then $x_i(t) - x_j(t) \to 0$ in finite time.

We next present the main result for (1) using (3).

*Theorem 3.3:* Assume that the interaction graph $\mathcal{G}_i, i = 0,1,\ldots,$ is undirected and connected. Using (3) for (1), $|r_i(t) - r_j(t)| \to 0$ in finite time if $\beta \ge \frac{\gamma}{\underline{\lambda}} + \epsilon$, where $\underline{\lambda} = \min\inf_{i=0,1,\ldots}\{\lambda(\mathcal{L}_i)|\lambda(\mathcal{L}_i) \ne 0\}$ and $\epsilon$ is any positive constant.

*Proof:* Define $\delta_i \triangleq r_i - \overline{r}$, where $\overline{r}$ is defined in (7). We have

$$\begin{aligned}\dot{\delta}_i &= f(t,r_i) + u_i - \dot{r}^d \\ &= f(t,r_i) - \beta\sum_{j=1}^n a_{ij}(t)\mathrm{sig}(r_i - r_j)^\alpha(|r_i - r_j|) \\ &\quad - \frac{1}{n}\sum_{j=1}^n f(t,r_j) \\ &= -\beta\sum_{j=1}^n a_{ij}(t)\mathrm{sig}(\delta_i - \delta_j)^{\alpha(|\delta_i - \delta_j|)} \\ &\quad - \frac{1}{n}\sum_{j=1,j\ne i}^n [f(t,\delta_j + \overline{r}) - f(t,\delta_i + \overline{r})].\end{aligned}$$

Note that $|r_i(t) - r_j(t)| \to 0$ in finite time if and only if $\max_i |\delta_i| \to 0$ in finite time. In order to show that $\max_i |\delta_i| \to 0$ in finite time, we construct the first closed-loop system given by

$$\begin{aligned}\dot{\xi}_i = &-\epsilon\sum_{j=1}^n a_{ij}(t)\mathrm{sig}(\xi_i - \xi_j)^{\alpha(|\xi_i - \xi_j|)} \\ &- \underline{\beta}\sum_{j=1}^n a_{ij}(t)(\xi_i - \xi_j) - \frac{1}{n}\sum_{j=1,j\ne i}^n \gamma(\xi_j - \xi_i), \end{aligned} \tag{18}$$

where $\xi_i \in \mathbb{R}$, $\xi_i(0) = \delta_i(0)$, and $\underline{\beta} \triangleq \frac{\gamma}{\underline{\lambda}}$. Define $G_1(t,\delta) \triangleq \max_i |\delta_i|$ and $F_1(t,\xi) \triangleq \max_i |\xi_i|$, where $\delta \triangleq [\delta_1,\ldots,\delta_n]^T$ and $\xi \triangleq [\xi_1,\ldots,\xi_n]^T$. It follows a similar analysis in the proof of Theorem 3.1 that $G_1(t) \le F_1(t)$ for all $t \ge 0$ under the condition of the theorem. Therefore, to show that $G_1(t) \to 0$ in finite time, it is sufficient to show that $F_1(t) \to 0$ in finite time.

Note that $F_1(t) \to 0$ in finite time if and only if $\xi^T\xi \to 0$ in finite time. To show that $\xi^T\xi \to 0$ in finite time, we construct the second closed-loop system given by

$$\dot{x}_i = -\epsilon\sum_{j=1}^n a_{ij}(t)\mathrm{sig}(x_i - x_j)^{\alpha^\star}, \tag{19}$$

where $x_i \in \mathbb{R}$ and $x_i(0) = \xi_i(0)$. Define $x \triangleq [x_1,\ldots,x_n]^T$. Consider the nonnegative function $F_2(t,x) \triangleq \frac{1}{2}x^Tx$. The upper right-hand derivative of $F_2(t,x)$ is given by

$$\begin{aligned}D^+F_2(t,x) &= x^T\dot{x} \\ &= -\epsilon\sum_{i=1}^n\sum_{j=1}^n a_{ij}(t)\left|x_i - x_j\right|^{\alpha^\star+1}.\end{aligned}$$

Consider the nonnegative function $G_2(t,\xi) \triangleq \frac{1}{2}\xi^T\xi$. Then the upper right-hand derivative of $G_2(t,\xi)$ is given by

$$\begin{aligned}D^+G_2(t,\xi) = &-\epsilon\sum_{i=1}^n\sum_{j=1}^n a_{ij}(t)\left|\xi_i - \xi_j\right|^{\alpha(|\xi_i - \xi_j|)+1} \\ &- \underline{\beta}\xi^T\mathcal{L}(t)\xi + \frac{\gamma}{n}\xi^T(nI_n - \mathbf{1}_n\mathbf{1}_n^T)\xi.\end{aligned}$$

Let $\xi = \xi^{\parallel} + \xi^{\perp}$, where $\xi^{\parallel}$ is the projection of $\xi$ along the vector $\mathbf{1}_n$ and $\xi^{\perp}$ is the projection of $\xi$ in the plane that is perpendicular to the vector $\mathbf{1}_n$. Note that $\mathcal{L}(t)\mathbf{1}_n = \mathbf{0}_n$ and $\mathbf{1}_n^T\mathcal{L}(t) = \mathbf{0}_n^T$. It follows that

$$\begin{aligned}
&-\underline{\beta}\xi^T\mathcal{L}(t)\xi + \frac{\gamma}{n}\xi^T(nI_n - \mathbf{1}_n\mathbf{1}_n^T)\xi \\
=&-\underline{\beta}(\xi^{\parallel}+\xi^{\perp})^T\mathcal{L}(t)(\xi^{\parallel}+\xi^{\perp}) \\
&+\frac{\gamma}{n}(\xi^{\parallel}+\xi^{\perp})^T(nI_n - \mathbf{1}_n\mathbf{1}_n^T)(\xi^{\parallel}+\xi^{\perp}) \\
=&-\underline{\beta}(\xi^{\perp})^T\mathcal{L}(t)\xi^{\perp} + \frac{\gamma}{n}(\xi^{\perp})^T(nI_n - \mathbf{1}_n\mathbf{1}_n^T)\xi^{\perp} \\
\leq&-\underline{\beta}\underline{\lambda}(\xi^{\perp})^T\xi^{\perp} + \lambda_{\max}(nI_n - \mathbf{1}_n\mathbf{1}_n^T)\frac{\gamma}{n}(\xi^{\perp})^T\xi^{\perp} \\
=&-\underline{\beta}\underline{\lambda}(\xi^{\perp})^T\xi^{\perp} + \gamma(\xi^{\perp})^T\xi^{\perp} = 0,
\end{aligned}$$

where we have used Theorem 4.2.2 in [23] to derive the inequality. Therefore, we have

$$\begin{aligned}
D^+G_2(t,\xi) = &-\epsilon\sum_{i=1}^n\sum_{j=1}^n a_{ij}(t)\left|\xi_i-\xi_j\right|^{\alpha(|\xi_i-\xi_j|)+1} \\
&-\underline{\beta}\xi^T\mathcal{L}(t)\xi + \frac{\gamma}{n}\xi^T(nI_n - \mathbf{1}_n\mathbf{1}_n^T)\xi \\
\leq &-\epsilon\sum_{i=1}^n\sum_{j=1}^n a_{ij}(t)\left|\xi_i-\xi_j\right|^{\alpha(|\xi_i-\xi_j|)+1}.
\end{aligned}$$

It follows from the definition of $\alpha(\cdot)$ that $D^+G_2(t,\xi) \leq D^+F_2(t,\xi)$. Because $G_2(0) = F_2(0)$, it follows from Lemma 3.3 that $G_2(t) \leq F_2(t)$ for all $t \geq 0$. Therefore, to show that $G_2(t) \to 0$ in finite time, it is sufficient to show that $F_2(t) \to 0$ in finite time. From Lemma 3.4, because $0 < \alpha^\star < 1$, $y_i - y_j \to 0$ in finite time. Therefore, $F_2(t) \to 0$ in finite time. Combining with the fact that $G_2(t) \leq F_2(t)$ shows that $G_2(t) \to 0$ in finite time. For (18), $G_2(t) \to 0$ in finite time implies that $F_1(t) \to 0$ in finite time. Combining with the fact $G_1(t) \leq F_1(t)$ shows that $G_1(t) \to 0$ in finite time. That is, consensus is reached in finite time. ■

*Remark 3.4:* Although both (2) and (3) can be used to solve finite-time consensus for (1), there are fundamental differences between the two algorithms. On one hand, (2) can be applied to the case when the inherent nonlinear dynamics is bounded (see Corollary 3.2) while (3), in general, requires that the inherent nonlinear dynamics is locally Lipschitz. On the other hand, the control gain in (2) depends on the initial states of the agents while the control gain in (3) does not relies on the initial states of the agents. Therefore, both algorithms have unique features and thus deserve investigation in this paper.

## IV. Conclusion

In this paper, we studied finite-time consensus of multi-agent networks with inherent nonlinear dynamics where each agent is driven by a nonlinear term based on its state. We proposed two distributed nonlinear algorithms to solve finite-time consensus with nonlinear dynamics. We then proposed a general comparison lemma which was used as the main tool in the stability analysis. With the general comparison lemma, we analyzed the stability of the closed-loop system using the two distributed nonlinear algorithms by comparing the original closed-loop systems with one or more predesigned closed-loop systems that can guarantee finite-time consensus.

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