2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

Statistical Characterization of the GLR Based Fault Detection

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***Abstract*— This paper is mainly focused on providing the probabilistic performance criteria for generalized likelihood ratio (GLR) test based fault detection schemes. Analytical expressions regarding probability distribution of the detection delay and time between false alarms are presented, which are validated by simulation. The results can be applied to important industrial application, such as abnormal signal monitoring (magnitude or energy), targeting and navigation, as well as the design of fault tolerant control systems (FTCS).**

I. Introduction

Nowadays, data-based fault diagnosis (FD) and integrated FD (combining data-based and model-based approaches) have been more and more highlighted in the recent research work, as technical systems/processes normally consist of abundant signal measurements due to the development of sensor technologies, while the dynamics of many components, e.g. motors, engines, servo valves, and pumps, etc., are well understood and represented by mathematical models. Concerning the online data-driven FD, the distribution of (or the mean of, [1],[2]) the detection delay and the time between false alarms are the most characteristic and visual performance indices. The specific fault detection algorithm includes the exponentially weighted moving average (EWMA) algorithm, the cumulative sum (CUSUM) control chart, the generalized-likelihood ratio (GLR), etc. From the perspective of probability and random process, the detection delay can generally be explained as the first hit(ting) time (FHT) at the boundary(-ies) concerning a drifted random walk or Brownian motion generated by the recorded residual signal data. Some research has been developed about the probabilistic properties of FHT [3], [4], [5], [6]. Reference [7] has succeeded in describing these distributions by means of continuous approximation under CUSUM detection, whereas the results for the GLR still has much room for improvement. Regarding the integrated FD, a general type of approaches is the integration of filters (or innovation nodes as in neural network) and detection algorithms: relevant research has been developed concerning CUSUM [8] and GLR [9].

This paper firstly attempts computing the probability distribution of detection delay with the GLR method, concerning the discrete random walk formed by taking summation of the original Gaussian i.i.d. noise samples upon time. As no exact analytical distribution of the detection delay can be achieved with the discrete GLR test due to the extremely complex expression, the discrete random walk is approximated with the corresponding wiener process in the continuous time domain, so that more mature performance analysis tools can be applied. A square root curve bound based on GLR is then used for detecting any drifted mean of the wiener process, while the distributions of detection delay and false alarm rate are deduced.

The research on this topic helps develop probabilistic FD criteria, which can be used to improve the overall performance of the fault detection system. It not only "refines" the FD criteria from the performance indices providing only average rates (e.g. average run length (ARL)) to a specific description of distribution, but also helps with stochastic modeling of the FD process, which corresponds to the switching mechanism of the hybrid fault tolerant control system (FTCS) driven by Markov [10], [11], [12] or semi-Markov chain [7]. On the other hand, the proposed research results are useful for assessment of fault detection alarms, e.g. the alarm management in a large scale system or process, so as to reduce the number of nuisance alarm signals to avoid unnecessary shut-downs and the related costs. It will find vast applications in process control industries, electro-mechanical systems, and power systems.

The remainder of this paper is organized as follows: in Section II, the modeling and the problem are formulated first, followed by the main results, where the probability distributions for detection delay and the time between false alarms are analyzed for the GLR based detection system; the results are validate by simulation in Section III, while the concluding remarks are given in Section IV.

II. Modeling & Main Results

First hitting time (FHT) [13], concerning a continuous Wiener process (w.p.) $W(t)$ or a discrete random walk $W(k)$ ($W(kT_s)$, where T_s is the sampling period), is defined as the time instant T_h when the w.p. (or random walk) crosses the pre-determined bound $b(t)$ (or $b(k), b(kT_s)$) for the first time. As w.p.s (or random walks) are stochastic processes, the FHT is also random and obeys certain distributions [18],[14]. The probability distribution of FHT is a visual and proper measure reflecting its properties, and thus it has been attracting the attention of researchers. Regarding the field of fault diagnosis (FD), the most important measures are the detection probability within a certain time length and the false alarm probability. Initial research concentration was the FHT with a constant value, resulting in the deduction of the probability with simple probabilistic means [15]. Durbin made representative breakthroughs by investigating the FHT with any linear border and successfully summarizing the expressions of relevant probabilities [18].

This work was supported by NSERC (Natural Science and Engineering Research Council, Canada) Canada.

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In this part we mainly focus on carrying out relevant performance analysis of GLR test standard and the corresponding continuous likelihood ratio (CLR) algorithm. The signal prepared for test is either random walk or wiener process, achieved by taking summation or integration of the original Gaussian i.i.d. noise signal. At first, the FHT distribution of discrete random walk signal is analyzed, and an upper bound of the cumulative density function (CDF) of the detection delay is computed in an analytical form. However, as no exact analytical expression of distribution is worked out for the discrete detection delay, we then have turned to a continuous likelihood ratio (CLR) test method, which is the approximation of GLR in continuous time domain. Given the recordable Gaussian i.i.d. noise sequence $y(t)$, the CLR detection consists of the following procedures: the establishment of signal for fault detection, the detection with GLR, and the analysis on performance indices.

A. Discrete Boundary Hitting Detection

The following research is developed with a typical discrete random walk $\{W(k) : k = 0, 1, \dots\}$ ($\{W(kT_s)\}$ with $W(0) = 0$, $E[W(k)] = 0$, $\mathrm{cov}(W(k)) = \sigma^2 kT_s$, where $W(k+1) - W(k) \sim N(0, \sigma^2 T_s)$, the samples of which are formed by taking summation of a recorded, pre-processed discrete gaussian i.i.d. noise data $y(k)$. A discrete boundary $\{b(k)\}$ is also defined upon the desire. Considering the reality of typical FD issues, we assume $a(0) > 0$ and treat the first hitting as the detection of the fault, and the detection delay is thus the FHT. It is now possible to describe the fault detection probability within the time length kT_s, using the Markovian property of random walks:

$$
\begin{aligned}
& P_{\{W(j):1\le j\le k\}}(\exists j : W(j) \ge b(j)) \\
= \; & 1 - P_{\{W(j):1\le j\le k\}}(\forall j : W(j) < b(j)) \\
= \; & 1 - P_{\{W(j):1\le j\le k\}}(b(k), b(k-1), \cdots, b(1)) \\
= \; & 1 - \int_{-\infty}^{b(k)}\int_{-\infty}^{b(k-1)}\cdots\int_{-\infty}^{b(1)} \\
& f_{\{W(j):1\le j\le k\}}(w_k, w_{k-1}, \cdots, w_1)\ dw_1 dw_2\cdots dw_k \\
= \; & 1 - \int_{-\infty}^{b(k)}\int_{-\infty}^{b(k-1)}\cdots\int_{-\infty}^{b(1)} \\
& f_{W(k)|W(k-1),W(k-2),\cdots,W(1)}(w_k|w_{k-1}, \cdots, w_1) \\
& \cdot f_{W(k-1)|W(k-2),\cdots,W(1)}(w_{k-1}|w_{k-2}, \cdots, w_1) \\
& \cdots \\
& \cdot f_{W(2)|W(1)}(w_2|w_1) \cdot f_{W(1)}(w_1)\ dw_1 dw_2 \cdots dw_k \\
= \; & 1 - \int_{-\infty}^{b(k)}\int_{-\infty}^{b(k-1)}\cdots\int_{-\infty}^{b(1)} f_{W(k)|W(k-1)}(w_k|w_{k-1}) \\
& \cdot f_{W(k-1)|W(k-2)}(w_{k-1}|w_{k-2}) \cdots f_{W(2)|W(1)}(w_2|w_1) \\
& \cdot f_{W(1)}(w_1)\ dw_1 dw_2 \cdots dw_k \\
= \; & 1 - \int_{-\infty}^{b(k)}\int_{-\infty}^{b(k-1)}\cdots\int_{-\infty}^{b(1)} \\
& (2\pi\sigma^2 T_s)^{-\frac{k}{2}} \exp\left(-\frac{w_1^2 + \sum_{j=2}^{k}(w_j - w_{j-1})^2}{2\sigma^2 T_s}\right) \\
& dw_1 dw_2 \cdots dw_k.
\end{aligned} \tag{1}
$$

Using the theorem of substitution for multiple variables [16], we can obtain an inferior bound of the k-order integral and thus a superior bound of the probability. Firstly we define the new coordinates

$$
\mathbf{v} = \begin{bmatrix} v_k \\ v_{k-1} \\ \vdots \\ v_1 \end{bmatrix} = \varphi(\mathbf{w}) = \begin{bmatrix} w_k - w_{k-1} \\ w_{k-1} - w_{k-2} \\ \vdots \\ w_1 \end{bmatrix}, \tag{2}
$$

where $\mathbf{w} = [w_k, w_{k-1}, \cdots w_1]^T$.

Calculate the Jacobian matrix of the function φ and its determinant:

$$
\begin{aligned}
(D\varphi)(\mathbf{w}) &= \begin{bmatrix} \frac{dv_k}{dw_k} & \frac{dv_k}{dw_{k-1}} & \cdots & \frac{dv_k}{dw_1} \\ \frac{dv_{k-1}}{dw_k} & \frac{dv_{k-1}}{dw_{k-1}} & \cdots & \frac{dv_{k-1}}{dw_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{dv_1}{dw_k} & \frac{dv_1}{dw_{k-1}} & \cdots & \frac{dv_1}{dw_1} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix},
\end{aligned}
$$

obviously $\det((D\varphi)(\mathbf{w})) = 1$.

Define the integral domain $T = \prod_{j=1}^{k}(-\infty, b(j))$ as in the original coordinates, and it can be transformed in the new coordinates as $\varphi(T) = (-\infty, b(1)) \times \prod_{j=2}^{k}(-\infty, b(j) - w_{j-1})$. Since $\varphi(T)$ contains independent variables of the integral as $\{w_{k-1}, w_{k-2}, \dots, w_1\}$, the original multiple integral will be transformed as an iterated integral in the new coordinates. In addition, define $f(\mathbf{v}) = (2\pi\sigma^2 T_s)^{-k/2} \exp(-\sum_{j=1}^{k} v_j^2/(2\sigma^2 T_s))$, which makes it possible to continue the calculation of the multiple integral shown in (1):

$$
\begin{aligned}
& \text{the integral in (1)} = \int_T f(\varphi(\mathbf{w}))\ d\mathbf{w} \\
= \; & \int_T f(\varphi(\mathbf{w}))|\det((D\varphi)(\mathbf{w}))|\ d\mathbf{w} \\
= \; & \int_{\varphi(T)} f(\mathbf{v})\ d\mathbf{v} \\
= \; & \int_{-\infty}^{b(1)}\int_{-\infty}^{b(2)-w_1}\cdots\int_{-\infty}^{b(k)-w_{k-1}} \frac{\exp\left(-\frac{\sum_{j=1}^{k} v_j^2}{2\sigma^2 T_s}\right)}{(2\pi\sigma^2 T_s)^{\frac{k}{2}}} \\
& dv_k dv_{k-1} \cdots dv_1 \\
= \; & \int_{-\infty}^{b(1)}\int_{-\infty}^{b(2)-w_1}\cdots\int_{-\infty}^{b(k-1)-w_{k-2}} \\
& (2\pi\sigma^2 T_s)^{-\frac{k-1}{2}} \exp\left(-\frac{\sum_{j=1}^{k-1} v_j^2}{2\sigma^2 T_s}\right) \Phi\left(\frac{b(k) - w_{k-1}}{\sqrt{\sigma^2 T_s}}\right) \\
& dv_{k-1} dv_{k-2} \cdots dv_1,
\end{aligned}
$$

where $\Phi(x)$ denotes the cumulative distribution function (cdf) of the standard natural distribution $N(0, 1)$ at x.

It is obvious that in the domain $T(\mathbf{w})$ or $\varphi(T(\mathbf{v}))$, we have $w_j \leq b(j)$ for $j = 1, 2, \ldots, k$, which helps generate

$$\begin{aligned}
&\text{the integral in (1)} \geq \Phi\left(\frac{b(k)-b(k-1)}{\sqrt{\sigma^2 T_s}}\right) \\
&\cdot \int_{-\infty}^{b(1)} \int_{-\infty}^{b(2)-w_1} \cdots \int_{-\infty}^{b(k-1)-w_{k-2}} \\
&(2\pi\sigma^2 T_s)^{-(k-1)/2} \exp\left(-\sum_{j=1}^{k-1} v_j^2/(2\sigma^2 T_s)\right) \\
&dv_{k-1} dv_{k-2} \cdots dv_1 \\
\geq \;& \Phi\left(\frac{b(k)-b(k-1)}{\sqrt{\sigma^2 T_s}}\right) \Phi\left(\frac{b(k-1)-b(k-2)}{\sqrt{\sigma^2 T_s}}\right) \\
&\cdot \int_{-\infty}^{b(1)} \int_{-\infty}^{b(2)-w_1} \cdots \int_{-\infty}^{b(k-2)-w_{k-3}} (2\pi\sigma^2 T_s)^{-\frac{k-2}{2}} \\
&\cdot \exp\left(-\sum_{j=1}^{k-2} v_j^2/(2\sigma^2 T_s)\right) dv_{k-2} \cdots dv_1 \\
\geq \;& \cdots \\
\geq \;& \Phi\left(\frac{b(1)}{\sqrt{\sigma^2 T_s}}\right) \prod_{j=2}^{k} \Phi\left(\frac{b(j)-b(j-1)}{\sqrt{\sigma^2 T_s}}\right). \quad (3)
\end{aligned}$$

Equivalently, the probability of fault detection with in the time length kT_s, i.e. the cumulative distribution function of the FHT, has a superior bound

$$\begin{aligned}
& P(0 < T_h \leq kT_s) \\
= \;& P_{\{W(j):1\leq j\leq k\}}(\exists j : W(j) \geq b(j)) \\
\leq \;& 1 - \Phi\left(\frac{b(1)}{\sqrt{\sigma^2 T_s}}\right) \prod_{j=2}^{k} \Phi\left(\frac{b(j)-b(j-1)}{\sqrt{\sigma^2 T_s}}\right). \quad (4)
\end{aligned}$$

Although a superior bound of the detection delay CDF is computed, its distance to the actual detection delay probability $P(0 < T_h \leq kT_s)$ has not been quantized yet, and $P(0 < T_h \leq kT_s)$ is not an analytical result convenient to be calculated. Considering the fact that more analysis techniques exist in continuous boundary hitting, we switch the research focus to continuous FHT distribution. The resulting conclusion is comparable with the simulation results, as continuous FHT distribution is the limit case when $T_s \to 0$.

B. Continuous Likelihood Ratio Based Detection

Here we firstly figure the extension of GLR detection to the continuous time domain (CGLR), referring to the discrete prototype used in [1]. We may assume the recordable signal $Y(t)$ is a zero-mean Gaussian white noise with the variance σ^2 before the fault happens, and the fault affects its mean to a non-zero unknown value ν in a step manner. i.e.,

$$Y(t) \sim \begin{cases} N(0, \sigma^2), & \text{when } t < t_s \\ N(\nu, \sigma^2), & \text{when } t \geq t_s, \end{cases} \quad (5)$$

where t_s denotes the time instant when the fault starts to affect the signal. For simplicity, only $\nu > 0$ is considered in this paper.

Use y(t) to denote the recorded sample of $Y(t)$. As the likelihood ratio at time t concerning this research topic is

$$\Lambda(t) = \ln \frac{f_{Y(t)}^{(\nu)}(y(t))}{f_{Y(t)}^{(0)}(y(t))} = \frac{\nu}{\sigma^2}\left(y(t) - \frac{\nu}{2}\right), \quad (6)$$

where the probability density function (PDF) centered with m is assumed following the normal distribution:

$$f_X^{(m)}(x) \triangleq \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

.

Define the cumulative integral of likelihood ratio from t_j to t_k:

$$S_{t_j}^{t_k} = \int_{t_j}^{t_k} \Lambda(\tau) d\tau = \frac{\nu}{\sigma^2} \int_{t_j}^{t_k} \left(y(\tau) - \frac{\nu}{2}\right) d\tau. \quad (7)$$

Note that ν is unknown, and one solution is to find a reasonable estimate $\hat{\nu}$ which generates maximum likelihood for each t_k. Following the thought of double maximization of discrete GLR regarding both ν and t_j [1], we may likewise define the decision function $g(t)$ at $\forall t > t_0$:

$$\begin{aligned}
g(t) &= \sup_{t_0 < t_j < t} \sup_{\nu > 0} S_{t_j}^{t} \\
&= \sup_{t_0 < t_j < t} \sup_{\nu > 0} \int_{t_j}^{t} \left(\frac{\nu y(\tau)}{\sigma^2} - \frac{\nu^2}{2\sigma^2}\right) d\tau. \quad (8)
\end{aligned}$$

Select a $h > 0$ as the threshold of FD. When $g(t) \geq h$, the detection equipment will judge the system as faulted; otherwise it will not.

Although (8) provides a constant boundary h for $g(t)$ to cross, the complex form of $g(t)$ limits further analysis of the performance of the standard. Now we start to simplify $g(t)$. Note that the detection standard is equivalent to the proposition: for a fixed t, $\exists t_j \in (t_0, t)$, s.t. $\sup_\nu S_{t_j}^t \geq h$. It can be further transformed like in [1]:

$$\begin{aligned}
& \sup_\nu S_{t_j}^t \geq h \\
\Leftrightarrow \;& \sup_\nu \left\{\int_{t_j}^{t} \left(\frac{y(\tau)}{\sigma} - \frac{\nu}{2\sigma}\right) d\tau - \frac{h\sigma}{\nu}\right\} \geq 0 \\
\Leftrightarrow \;& \frac{1}{\sigma} \int_{t_j}^{t} y(\tau) d\tau \geq \inf_\nu \left\{\frac{\nu}{2\sigma}(t - t_j) + \frac{h\sigma}{\nu}\right\} \\
\Leftrightarrow \;& \frac{1}{\sigma} \int_{t_j}^{t} y(\tau) d\tau \geq \sqrt{2h(t - t_j)}. \quad (9)
\end{aligned}$$

Referring to the properties of Wiener processes [17], we may define the left part of (9) as

$$w(t) = \frac{1}{\sigma} \int_{t_j}^{t} y(\tau) d\tau \quad (10)$$

as a sample of a Wiener process $W(t)$ formed by taking the integration of $y(t)$. Here $W(t)$ starts at t_j and satisfies

$$W(t) \sim \begin{cases} N(0, t - t_j), & \text{when } t < t_s \\ N(\nu(t - t_s)/\sigma, (t - t_j)), & \text{when } t \geq t_s \end{cases} \quad (11)$$

The right part of (9) is a square root curve. Obviously, the CGLR in (9) is a moving window detection method, with the convex of the square root boundary moving along the wiener process sample w.

Note that (9) has already removed the superior bound caused by the drift ν via optimization, and we may remove the superior bound caused by t_j. As our goal is the FD delay, so the fault start time t_s and the time of detection t are important, whereas t_j is not. Then it is important to notice that for fixed t_1, $\exists t > t_1$

$$\frac{1}{\sigma}\int_{t_1}^{t} y(\tau)d\tau = \sqrt{2h(t-t_1)},$$

then from the perspective of GLR, t_1 is one possible candidate of t_j when referred to t, but the possibility for t to have alternative $t_j \in (t_1, t)$ still cannot be excluded. Hence, we may define a GLR-like, but simpler continuous likelihood ratio standard (CLR) with fixed square root bound convex, and it is not as sensitive as the original CGLR regarding wiener processes with the identical distribution. In the mathematical form, it is expressed as

$$P_{\mathrm{CLR}}(t_0 < T_h < t) \leq P_{\mathrm{CGLR}}(t_0 < T_h < t), \tag{12}$$

where P denotes probability, and T_h denotes the FHT.

Reference [18],[19] has concluded the expression of the probability of FHT within a range concerning a zero mean wiener process and a monotonically non-increasing linear boundary. Especially, [19],[1] have extended the conclusion to any monotonically non-increasing concave boundary $b(t)$ differentiable on (t_0, ∞) satisfying $b(t_0^+) \geq 0$, where (13) is summarized as an upper bound of the probability due to the concavity:

$$P(t_0 < T_h < t) \leq \int_{t_0}^{t} \frac{b(\tau) - \tau b'(\tau)}{\sqrt{2\pi\tau^3}} e^{-\frac{b^2(\tau)}{2\tau}} d\tau, \text{ if } b'(\tau) \leq 0 \tag{13}$$

Reference [7] concluded the integral term (i.e., the upper bound of probability density function (PDF)) will keep the same form in the case of linear increasing boundaries, and thus (13) is applicable to all the concave boundaries $b(t)$ satisfying $b(t_0^+) \geq 0$. As a result, it can be applied to analyze the distribution of the FHT of CLR standard.

At first we concentrate on the distribution of the detection delay T_h. Fix $t_s = 0$ for simplicity, treat it as the start time of CLR detection, and rewrite the boundary $b(t)$ in (9):

$$b(t) = \sqrt{2ht}. \tag{14}$$

As the research covers statistical characterization rather than specific calculation, we may add $-\nu t/\sigma$ to both $w(t)$ and the boundary so that the original problem can be transformed into an equivalent problem [18], i.e. the detection of the zero mean Wiener process $w_0(t) \sim N(0, t)$ hitting the bound

$$b_0(t) = \sqrt{2ht} - \frac{\nu}{\sigma}t. \tag{15}$$

Note that both $b(t)$ and $b_0(t)$ are concave, implying that (13) can be used for computing the FHT distribution for $w(t)$ to cross $b(t)$ and $b_0(t)$.

Following the discussion above, we may provide an upper bound of the probability of detection delay (which is FHT) T_h from $t_0 > 0$ to t as in (16), regarding a zero mean wiener process sample w_0 and the boundary b_0. As $w_0(0) = 0$ and $b_0(0) = 0$ imply the meaningless "initial hitting" rather than FHT mentioned above, (16) should be used with the cases of $t_0 > 0$.

$$\begin{aligned} P_{\mathrm{CLR}} \quad (t_0 < \quad T_h < t) &\leq \int_{t_0}^{t} \frac{b_0(\tau) - \tau b_0'(\tau)}{\sqrt{2\pi\tau^3}} e^{-\frac{b_0^2(\tau)}{2\tau}} d\tau \\ &= \quad \int_{0}^{t} \frac{\sqrt{h}}{2\sqrt{\pi\tau}} e^{-h + \frac{\sqrt{2h}\nu}{\sigma}\sqrt{\tau} - \frac{\nu^2\tau}{2\sigma^2}} d\tau. \end{aligned} \tag{16}$$

False alarm covers the case that $\nu = 0$ but $w(t)$ hits the bound $b(t)$ at some $t > t_s = 0$. As a result, the probability distribution of the first false alarm time F_{CLR} between (t_0, t) with $t_0 > 0$ satisfies

$$\begin{aligned} F_{\mathrm{CLR}}(t_0 < T_h < t) &\leq \int_{t_0}^{t} \frac{\sqrt{h}}{2\sqrt{\pi\tau}} e^{-h} d\tau \\ &= \frac{\sqrt{h}}{2\sqrt{\pi}} e^{-h} (\ln t - \ln t_0). \end{aligned} \tag{17}$$

Nevertheless, the PDF in (16) and (17) tend to be infinite when $t_0 \to 0$, which is trivial and makes (16) and (17) lose its value of analysis. It is also noticeable in the simulation part that most FHT happens in the first several time instants, whereas a considerable proportion of FHT can be caused by temporary spikes in $w(t)$. In order to avoid it, we introduce a constant bias $\beta > 0$ to the original CLR, forming the biased CLR (BCLR), which will make the detection less sensitive but also not easily affected by temporary spikes in $w(t)$. Under BCLR the bound becomes

$$b(t) = \sqrt{2ht} + \beta, \tag{18}$$

$$b_0(t) = \sqrt{2ht} - \frac{\nu}{\sigma}t + \beta. \tag{19}$$

Then the distribution of detection delay $P_{BCLR}(t_0 < T_h < t)$ and false alarm $F_{BCLR}(t_0 < T_h < t)$ become

$$P_{BCLR}(t_0 < T_h < t) \leq \int_{t_0}^{t} \frac{\sqrt{2}\beta + \sqrt{h\tau}}{2\sqrt{\pi\tau^3}} e^{-\frac{(\sqrt{2h\tau} - \frac{\nu}{\sigma}\tau + \beta)^2}{2\tau}} d\tau, \tag{20}$$

and

$$F_{BCLR}(t_0 < T_h < t) \leq \int_{t_0}^{t} \frac{\sqrt{2}\beta + \sqrt{h\tau}}{2\sqrt{\pi\tau^3}} e^{-\frac{(\sqrt{2h\tau} + \beta)^2}{2\tau}} d\tau. \tag{21}$$

Obviously the problem of infinity PDF in (16) and (17) is solved in (20) and (21).

As algorithms able to be examined in simulation and practice, CLR and BCLR are comparable with the CUSUM algorithm, which is a mature detection standard so far. Considering the concavity and the start point of CLR and BCLR, we may determine that the detection probability (and also the false alarming probability) with CLR/BCLR is lower than that of CUSUM before the time at the intersection of the two bounds, but higher than that of CUSUM after the time at the intersection. As a result, the PDF of the detection delay (and also the time between false alarms) with CLR/BCLR is more concentrated around some $t > 0$ than that with CUSUM, which is more "separated". The

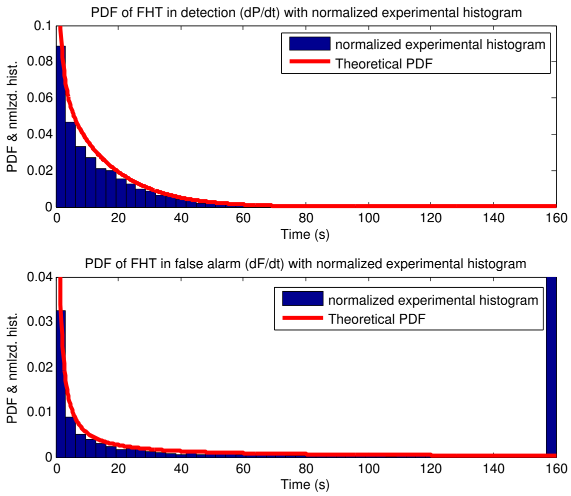

Fig. 1. PDF of FHT in detection (upper) and false alarming (bottom) with normalized experimental histograms.

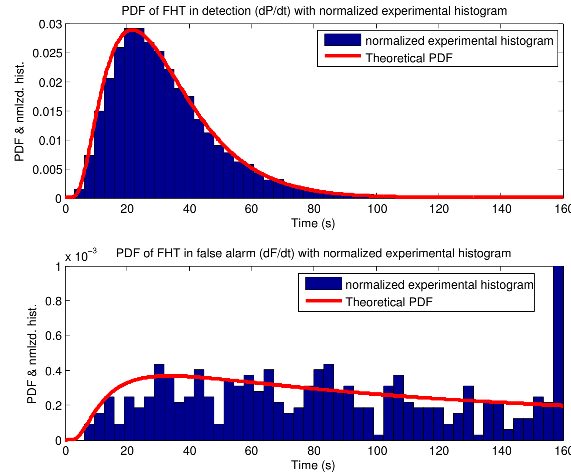

Fig. 2. PDF of FHT in detection (upper) and false alarming (bottom) with normalized experimental histograms.

explication of this phenomenon is positive: compared with CUSUM, the CLR/BCLR standard is not easily disturbed by outlier spikes but will have higher probability to response in a timely manner if a real fault occurs.

III. Simulation

The simulation has been carried out using MATLAB in two parts, respectively covering CLR and BCLR bounds. In each part, both the normal fault detection (fault coefficient ν occurs at $t = 0$) and the false alarm case are discussed, where the PDF of FHT and the normalized experimental histogram (50 divisions) are compared.

Random walk samples are used to approximate Wiener processes, where the sampling interval $T_s = 0.2$s. The Gaussian white noise signal source without fault is selected as $\sim N(0, 1)$, generating the undrifted random walk $W_0(k) \sim N(0, kT_s)$. The fault drifts $W_0(k)$ to $W(k) \sim N(\nu kT_s, kT_s)$. FHT is tested with 10000 randomly generated Gaussian random walk samples with the same distribution, so that the histogram can tend to the real distribution.

The time length of observation is set to 160s; in the histogram all the FHT beyond that time are classified as those "larger than 160" and annexed to the rightmost bar.

A. CLR

Select the parameters as $h = 2$ and $\nu = 0.5$ and simulate both the detection and false alarm test respectively with given 10000 random walk samples, resulting in Fig. 1:

Fig. 1 has shown two monotonic decreasing PDF, and the distributions of experimental FHT match the corresponding PDF well, implying the correctness of (16) and (17). The first hitting tends to occur more frequently in the first several seconds since t_s, reflecting the infinity problem in (16) and (17), as well as the necessity to push the bound away at least in the first several seconds. Note that the cases that the first hitting does not happen in the first 156.8s (the rightmost bar) occupy 7423 times over 10000 random walk samples, regarding the false alarm problem. The false alarm rate is reasonable concerning the bound itself but still high in practice; it must be improved, for instance, with a positive bias β, before used in the industrial signal monitoring.

B. BCLR

Select the parameters as $h = 2$, $\nu = 0.5$, and the bias $\beta = 5$. Simulate both the detection and false alarm test respectively with given 10000 random walk samples, resulting in Fig. 2:

Fig. 2 has shown two single peak PDF, and the distributions of experimental FHT match the corresponding PDF well, implying the correctness of (20) and (21). The visually obvious deviation of histogram from the PDF in the bottom figure is caused by the fact that the number of FHT valued from 0 to 156.8 is much smaller than that larger than 156.8, resulting in the different scaling. The peaks are around 21s (detection) and 25s (false alarming test), and it can be concluded that the BCLR is not as sensitive as CLR by comparing the normalized histogram or the PDF with the counterpart in Fig. 1. Note that the cases that the first hitting does not happen in the first 156.8s occupy for 9647 times over 10000 random walk samples, reflecting a reasonable and acceptable false alarm rate in practice.

IV. Conclusion

This paper has characterized a new continuous likelihood ratio test standard, which detects the abrupt change of the mean of the monitored data sequence, based on the GLR algorithm. The CLR and the improved BCLR standards have provided the probability distributions of detection delay and false alarm interval in analytical forms. The boundary curve shape provides the CLR/BCLR standard with advantages over CUSUM. The simulation has validated the analytical-formed theoretical results and shown that BCLR has much less false alarm rate despite the longer detection delay on average, which is still acceptable.

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