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Distributed Multi-Agent Coordination: A Comparison Lemma Based Approach

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Abstract— **In this paper, we focus on the study of distributed multi-agent coordination, including leaderless consensus and consensus tracking, by using a comparison lemma based approach. First, we investigate and present general comparison lemmas for vector differential equations that are represented in terms of upper right-hand derivatives. By using the general comparison lemmas, the stability of linear/nonlinear closed-loop systems can be guaranteed given that some properly designed linear/nonlinear systems are stable. This provides an important tool in the stability analysis of linear/nonlinear closed-loop systems by making use of known results in linear/nonlinear systems. Second, we apply the general comparison lemmas in the stability analysis of two distributed multi-agent coordination problems, namely, leaderless consensus and consensus tracking. We propose general nonlinear distributed algorithms and derive mild conditions to guarantee convergence by using the general comparison lemmas.**

I. Introduction

The research on distributed multi-agent coordination, including consensus [1]–[3], formation control [4], [5], flocking [6]–[8], has been a very active research topic. The objective of distributed multi-agent coordination is to have a group of agents achieve a global group behavior when each agent in the group receives and acts on information from its local neighbors. Although distributed multi-agent coordination is practical and demonstrates a number of advantages, such as high scalability and robustness, over centralized multi-agent coordination where all agents know the global objective, the nature of distributed multi-agent coordination also brings challenges in the design and analysis of proper distributed control algorithms due to the unavailability of the global objective to all agents.

We next briefly review some existing research in distributed multi-agent coordination. More details on the existing research in distributed multi-agent coordination can be found in [9]–[12], to name a few. One interesting research topic in distributed multi-agent coordination is leaderless consensus whose objective is to design distributed algorithms such that a group of agents can agree on some state of interest. Leaderless consensus has been studied for both single-integrator kinematics [1], [3], [13] and double-integrator dynamics [14]–[16]. The other interesting research topic in distributed multi-agent coordination is consensus tracking whose objective is to design distributed algorithms such that a group of agents can follow a desired (moving) target. Consensus tracking has been studied for single-integrator kinematics [17], [18] and double-integrator dynamics [8], [19]. It is worthwhile to mention that the stability analysis in the previous papers is mainly based on algebraic graph theory, matrix theory, and Lyapunov stability theory.

Although the stability analysis tools used in the aforementioned papers are very useful and interesting, the stability analysis is often given for a specific closed-loop system under a specific distributed algorithm. That is, the same approach might not be applied for another closed-loop system under a different distributed algorithm. In this paper, we try to present an important approach - a comparison lemma based approach - which can be used to analyze the stability of a series of closed-loop systems satisfying certain properties. By using the comparison lemma based approach, the stability of linear/nonlinear closed-loop systems can be guaranteed given that some properly designed linear/nonlinear systems are stable. This provides an important tool in the stability analysis of linear/nonlinear closed-loop systems by making use of known results in linear/nonlinear systems. To better motivate the proposed comparison lemma approach, we study two distributed multi-agent coordination problems, namely, leaderless consensus and consensus tracking. We propose general nonlinear distributed algorithms to solve leaderless consensus and consensus tracking and derive mild conditions to guarantee stability by using the general comparison lemmas.

II. Preliminaries

A. Graph Theory Notions

For a team consisting of n agents, the interaction among all agents can be modeled by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{W})$, where $\mathcal{V} = \{v_1, v_2, \cdots, v_n\}$ and $\mathcal{W} \subseteq \mathcal{V}^2$ represent, respectively, the agent set and the edge set. Each edge denoted as (v_i, v_j) means that agent j can access the state information (*i.e.,* position) of agent i. A directed path is a sequence of edges in a directed graph of the form $(v_1, v_2), (v_2, v_3), \cdots$, where $v_i \in \mathcal{V}$. A directed graph has a directed spanning tree if there exists at least one agent that has directed paths to all other agents. The union of a set of directed graphs $\mathcal{G}_{i_1}, \cdots, \mathcal{G}_{i_m}$ is a directed graph with the edge set given by the union of the edge sets of the directed graphs $\mathcal{G}_{i_j}, j = 1, \cdots, m$.

Mathematically, the interaction graph can be represented by two matrices, namely, adjacency matrix and Laplacian matrix. The adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined such that $a_{ij} > 0$ if agent i can receive the state information from agent j and $a_{ij} = 0$ otherwise. The (nonsymmetric) Laplacian matrix $\mathcal{L} = [\ell_{ij}] \in \mathbb{R}^{n\times n}$ is defined such that

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$\ell_{ii} = \sum_{j=1, j\neq i}^{n} a_{ij}$ and $\ell_{ij} = -a_{ij}, i \neq j$. It is easy to verify that $\mathcal{L}$ has at least one zero eigenvalue with a corresponding eigenvector $\mathbf{1}_n$, where $\mathbf{1}_n$ is an all-one column vector.

B. Notations

We use $\mathbb{R}$ to denote the set of real number. We use $\mathbf{0}_{p\times q} \in \mathbb{R}^{p\times q}$ and $\mathbf{0}_n \in \mathbb{R}^n$ to denote, respectively, $p\times q$ zero matrix and $n \times 1$ zero vector. For two real matrices $A \in \mathbb{R}^{p\times q}$ and $B \in \mathbb{R}^{p\times q}$, $A > B$ (respectively, $A \geq B$) means that each component of $A - B$ is positive (respectively, nonnegative). $\|\cdot\|$ is used to denote the 2-norm. $\varnothing$ is used to denote the empty set. Letting $f : [0, \infty) \mapsto J \subseteq \mathbb{R}^n$ be a continuous function, the upper right-hand derivative of $f(t)$ is given by $D^+ f(t) = \limsup_{h\to 0^+} \frac{1}{h}[f(t+h) - f(t)]$.

III. Comparison Lemmas for Vector Differential Equations

In this section, we will present several comparison lemmas for vector differential equations. Before moving on, we need the comparison lemma for scalar differential equations.

Lemma 3.1: [20, Comparison Lemma] Consider the scalar differential equation $\dot{z} = f(t, z),\ z(t_0) = \mu_0$, where $f(t, z)$ is continuous in t and locally Lipschitz in z for all $t \geq 0$ and all $z \in J \subset \mathbb{R}$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution z, and suppose that $z \in J$ for all $t \in [t_0, T)$. Let ω be a continuous function whose upper right-hand derivative $D^+\omega$ satisfies the differential inequality $D^+\omega \leq f(t, \omega),\ \omega(t_0) \leq \mu_0$, where $\omega \in J$ for all $t \in [t_0, T)$. Then $\omega(t) \leq z(t)$ for all $t \in [t_0, T)$.

Note that Lemma 3.1 can only be applied to scalar differential equations. We next present comparison lemmas which can be used for vector differential equations. Before moving on, we need the following lemma.

Lemma 3.2: [20, Theorem 3.5] Let $f(t, x, \lambda)$ be continuous in (t, x, λ) and locally Lipschitz in x (uniformly in t and λ) on $[t_0, t_1] \times J \times \{\|\lambda - \lambda_0\| \leq c\}$, where $J \subset \mathbb{R}^n$ is an open connected set. Let $y(t, \lambda_0)$ be a solution of $\dot{x} = f(t, x, \lambda_0)$ with $y(t_0, \lambda_0) = y_0 \in J$. Suppose that $y(t, \lambda_0)$ is defined and belongs to J for all $t \in [t_0, t_1]$. Then, given $\epsilon > 0$, there is $\delta > 0$ such that if $\|z_0 - y_0\| < \delta$ and $\|\lambda - \lambda_0\| < \delta$, then there is a unique solution $z(t, \lambda)$ of $\dot{x} = f(t, x, \lambda)$ defined on $[t_0, t_1]$, with $z(t_0, \lambda) = z_0$, and $z(t, \lambda)$ satisfies $\|z(t, \lambda) - y(t, \lambda_0)\| < \epsilon$ for all $t \in [t_0, t_1]$.

With Lemma 3.2, we next present the comparison lemma for vector differential equations.

Lemma 3.3: Consider the following vector differential equation

$$\dot{z} = f(t, z), \quad z(t_0) = \mu_0,$$

where $z = [z_1, \cdots, z_p]^T \in \mathbb{R}^p$, $f(t, z) = [f_1(t, z), \cdots, f_p(t, z)]^T$ is defined such that $f_i(t, z),\ i = 1, \cdots, p$, is continuous in t and locally Lipschitz in $z_i,\ i = 1, \cdots, p$, for all $t > 0$ and all $z \in J \subset \mathbb{R}^p$. Let $[t_0, T)$ (T could be infinity) be the maximal interval of existence of the solution z, and suppose that $z \in J$ for all $t \in [t_0, T)$. Let $\omega \in \mathbb{R}^p$ be a continuous function whose upper right-hand derivative $D^+\omega$ satisfies the inequality

$$D^+\omega \leq f(t, \omega), \quad \omega(t_0) \leq \mu_0,$$

where $\omega \in J$ for all $t \in [t_0, T)$. Then $\omega(t) \leq z(t)$ for all $t \in [t_0, T)$.

Proof: The proof is motivated by that of Lemma 3.1 (see [20]). Consider the following vector differential equation

$$\dot{x} = f(t, x) + \lambda, \quad x(t_0) = z(t_0) \tag{1}$$

for $i = 1, \cdots, p$, where $x \in \mathbb{R}^p$ and $\lambda = [\lambda_1, \cdots, \lambda_p]^T$ is a positive constant vector. For $t \in [t_0, t_1]$, where $t_1 > t_0$, it follows from Lemma 3.2 that for any $\epsilon > 0$, there is $\delta > 0$ such that if $\|\lambda\| < \delta$, (1) has a unique solution $\xi(t, \lambda)$ defined on $[t_0, t_1]$ and $\|\xi(t, \lambda) - z(t)\| < \epsilon, \forall t \in [t_0, t_1]$. Therefore, we have that

$$\|\xi_i(t, \lambda) - z_i(t)\| < \epsilon, \forall t \in [t_0, t_1]. \tag{2}$$

Claim 1: $\omega_i(t) \leq \xi_i(t, \lambda)$ for all $t \in [t_0, t_1]$. We prove this by contradiction. Assume that there exist times $a, b \in (t_0, t_1]$ such that $\omega_i(a) = \xi_i(a, \lambda)$ and $\omega_i(t) > \xi_i(t, \lambda)$ for $a < t \leq b$. Accordingly, we have that $\omega_i(t) - \omega_i(a) > \xi_i(t, \lambda) - \xi_i(a, \lambda), \forall t \in (a, b]$, which implies that $D^+\omega_i(a) \geq D^+\xi_i(a, \lambda) = \dot{\xi}_i(a, \xi) = f_i(a, \xi) + \lambda_i > f_i(a, \xi)$. This contradicts the inequality $D^+\omega \leq f(t, \omega)$.

Claim 2: $\omega_i(t) \leq z_i(t)$ for all $t \in [t_0, t_1]$. Again, we prove this by contradiction. Assume that there exists $a \in (t_0, t_1]$ such that $\omega_i(a) > z_i(a)$. Letting $\epsilon = \frac{\omega_i(a) - z_i(a)}{2}$ and using (2), we obtain that

$$\begin{aligned}\omega_i(a) - \xi_i(a, \lambda) =& \omega_i(a) - z_i(a) + z_i(a) - \xi_i(a, \lambda)\\ =& 2\epsilon + z_i(a) - \xi_i(a, \lambda) \geq \epsilon,\end{aligned}$$

which contradicts the statement of Claim 1. ■

In Lemma 3.3, it is assumed that z is continuously differentiable. We next present a general comparison lemma for vector differential equations where z is continuous with the upper right-hand derivative of z, D^+z, well defined for $t \in [0, T)$ while $\dot{z}$ might not be well defined for all $t \in [0, T)$.

Lemma 3.4: Suppose that $F(t, z) : [t_0, T) \times J \subseteq \mathbb{R}^p \mapsto \mathbb{R}^q$, is a continuous function satisfying that

$$D^+F = f(t, z),$$

where $z \in \mathbb{R}^p$, and $f(t, z)$ is piecewise continuous in t and is locally Lipschitz in z when $f(t, z)$ is continuous at t. Let $G(t, \omega) : [t_0, T) \times J \subseteq \mathbb{R}^p \mapsto \mathbb{R}^q$ be a continuous function whose upper right-hand derivative D^+G satisfies the differential inequality

$$D^+G \leq f(t, \omega), \quad G[t_0, \omega(t_0)] \leq F[t_0, z(t_0)].$$

Then $G(t) \leq F(t)$ for all $t \in [t_0, T)$.

Proof: The proof of the lemma can be divided into three cases:

Case 1: $q = p$. Without loss of generality, assume that $f(t, z)$ is continuous in t for $t \in [t_i, t_{i+1}),\ i = 0, 1, \ldots$. For $t \in [t_0, t_1)$, consider a new vector differential equation given by

$$\dot{x} = f(t, x), \quad x(t_0) = F[t_0, z(t_0)]. \tag{3}$$

Because $D^+F = f(t, z) \leq f(t, z)$ and $F(t_0) = x(t_0) \leq x(t_0)$ are trivially satisfied, it follows from Lemma 3.3 that

$F(t) \le x(t)$ for all $t \in [t_0, t_1)$. Noting also that $D^+(-F) = -f(t, z) \le -f(t, z)$ and $-F(t_0) = -x(t_0) \le -x(t_0)$ are trivially satisfied, it follows from Lemma 3.3 that $-F(t) \le -x(t)$ for all $t \in [t_0, t_1)$. Combining the two arguments shows that $F(t) = x(t)$ for all $t \in [t_0, t_1)$. Note that $D^+G \le f(t, \omega)$ and $G[t_0, z(t_0)] \le F[t_0, \omega(t_0)] = x(t_0)$. It thus follows from Lemma 3.3 that $G(t) \le x(t)$ for all $t \in [t_0, t_1)$. Because $F(t) = x(t)$ for all $t \in [t_0, t_1)$, it follows that $G(t) \le F(t)$ for all $t \in [t_0, t_1)$. Because $F(t)$ is a continuous function, by employing a similar analysis for $t \in [t_i, t_{i+1}),\ i = 1, \ldots,$ it can be shown that $G(t) \le F(t)$ for all $t \in [t_i, t_{i+1}),\ i = 1, \ldots$. Therefore $G(t) \le F(t)$ for all $t \in [t_0, T)$.

Case 2: $1 \le q < p$. Define $\widetilde{F} \triangleq [F^T, \mathbf{1}_{p-q}^T]^T$ and $\widetilde{G} \triangleq [G^T, \mathbf{1}_{p-q}^T]^T$. By letting $\widetilde{F}$ and $\widetilde{G}$ play the role of, respectively, F and G, it follows from a similar analysis to that of the case when $q = p$ that $\widetilde{G}(t) \le \widetilde{F}(t)$ for all $t \in [t_0, T)$, which implies that $G(t) \le F(t)$ for all $t \in [t_0, T)$.

Case 3: $q > p$. Note that there exists a positive integer m such that $q = mp + q_d$, where $0 \le q_d < p$ is a nonnegative integer. When $q_d = 0$, G and F can be written as $G = [G_1^T, \ldots, G_m^T]^T$ and $F = [F_1^T, \ldots, F_m^T]^T$, where $G_i \in \mathbb{R}^p$ and $F_i \in \mathbb{R}^p$ for all $i = 1, \ldots, m$. By applying the result of Case 1 repeatedly, we have $G_i(t) \le F_i(t), \forall i = 1, \ldots, m$, for all $t \in [t_0, T)$. This implies $G(t) \le F(t)$ for all $t \in [t_0, T)$. When $q_d \neq 0$, G and F can be written as $G = [G_1^T, \ldots, G_m^T, G_{m+1}^T]^T$ and $F = [F_1^T, \ldots, F_m^T, F_{m+1}^T]^T$, where $G_i \in \mathbb{R}^p$ and $F_i \in \mathbb{R}^p$ for all $i = 1, \ldots, m$, and $G_{m+1} \in \mathbb{R}^{q_d}$ and $F_{m+1} \in \mathbb{R}^{q_d}$. By applying the result of Case 1 repeatedly, we also have $G_i(t) \le F_i(t), \forall i = 1, \ldots, m$, for all $t \in [t_0, T)$. By applying the result of Case 2, we have $G_{m+1}(t) \le F_{m+1}(t)$ for all $t \in [t_0, T)$. This implies $G(t) \le F(t)$ for all $t \in [t_0, T)$.

Combining the previous cases completes the proof. ■

The proposed comparison lemmas provide a powerful tool in the stability analysis of nonlinear closed-loop systems. Due to the existence of nonlinear mechanics, the stability analysis of nonlinear closed-loop systems is much more challenging than that of linear systems. By using the comparison lemmas, the stability of nonlinear closed-loop systems can be guaranteed given that some properly designed linear systems are stable. This provides an important tool in the stability analysis of nonlinear closed-loop systems by making use of known results in linear systems.

IV. Consensus with Nonlinear Mechanics

In this section, we focus on the study of leaderless consensus and consensus tracking for single-integrator kinematics with nonlinear mechanics. We consider a team of n agents with single-integrator kinematics given by

$$\dot{r}_i = u_i, \quad i = 1, \cdots, n, \tag{4}$$

where $r_i \in \mathbb{R}^m$ and $u_i \in \mathbb{R}^m$ represent, respectively, the state and the control input for the ith agent. Let $r_{i(k)} \in \mathbb{R}$ be the kth component of r_i for $k = 1, \cdots, m$. Define $r_{(k)} \triangleq [r_{1(k)}, \cdots, r_{m(k)}]^T$.

A. Leaderless Consensus

The objective of leaderless consensus is to design u_i such that $\|r_i(t) - r_j(t)\| \to 0$ as $t \to \infty$. Compared with the commonly used leaderless consensus algorithms in [1], [21], [22], we propose the following nonlinear leaderless consensus algorithm for (4) as

$$u_i = -\sum_{j=1}^{n} a_{ij}(t) f_{i,j}(t, r_i, r_j), \quad i = 1, \cdots, n, \tag{5}$$

where $a_{ij}(t)$ is the (i, j)th entry of the adjacency matrix $\mathcal{A}(t)$ associated with the interaction graph $\mathcal{G}(t)$ characterizing the interaction among the n agents at time t, and $f_{i,j}(t, r_i, r_j) \triangleq [f_{i,j}(t, r_{i(1)}, r_{j(1)}), \cdots, f_{i,j}(t, r_{i(m)}, r_{j(m)})]^T$ is defined such that:
If $\left|r_{i(k)} - \max_j r_{j(k)}\right| \ge \sigma$ and $\left|r_{i(k)} - \min_j r_{j(k)}\right| \ge \sigma$,

$$f_{i,j}(t, r_{i(k)}, r_{j(k)}) = \begin{cases} \epsilon_1(t) > \epsilon, & r_{i(k)} > r_{j(k)}, \\ -\epsilon_2(t) < -\epsilon, & r_{i(k)} < r_{j(k)}, \\ 0, & x = y, \end{cases} \tag{6}$$

or

$$f_{i,j}(t, r_{i(k)}, r_{j(k)}) = \begin{cases} \varsigma_1(t)(r_{i(k)} - r_{j(k)}) \ge \varsigma(r_{i(k)} - r_{j(k)}), & r_{i(k)} > r_{j(k)}, \\ -\varsigma_2(t)(r_{i(k)} - r_{j(k)}) \le \varsigma(r_{i(k)} - r_{j(k)}), & r_{i(k)} < r_{j(k)}, \\ 0, & r_{i(k)} = r_{j(k)}, \end{cases} \tag{7}$$

where $\epsilon_1(t)$, $\epsilon_2(t)$, $\varsigma_1(t)$, and $\varsigma_2(t)$ are continuous in t, and σ, ϵ, and ς are any positive constant scalars.
If $\left|r_{i(k)} - \max_j r_{j(k)}\right| < \sigma$ or $\left|r_{i(k)} - \min_j r_{j(k)}\right| < \sigma$, $f_{i,j}(t, r_{i(k)}, r_{j(k)})$ satisfies (7).

Remark 4.1: The leaderless consensus algorithm proposed in [1], [21], [22] is a special case of (5) when $f_{i,j}(t, x, y) = x - y$.

The closed-loop system of (4) using (5) is given by

$$\dot{r}_i = -\sum_{j=1}^{n} a_{ij}(t) f_{i,j}(t, r_i, r_j), \quad i = 1, \cdots, n. \tag{8}$$

Note that the existence of the solution to (8) can be guaranteed by Proposition 3 in [23].

We next present the main result for leaderless consensus with nonlinear mechanics.

Theorem 4.2: Suppose that there exists positive $\overline{t}$ such that the directed graph associated with $\int_t^{t+\overline{t}} \mathcal{A}(t)dt$ has a directed spanning tree for any $t \ge 0$ and $a_{ij}(t)$ is lower-bounded and upper-bounded if $a_{ij}(t) \neq 0$. Using (5) for (4), consensus is reached ultimately. That is, for all $r_i(0)$, $\|r_i(t) - r_j(t)\| \to 0$ as $t \to \infty$.

Proof: In order to show that $\|r_i(t) - r_j(t)\| \to 0$ as $t \to \infty$, it is equivalent to show that $\left|r_{i(k)}(t) - r_{j(k)}(t)\right| \to 0$ for all $k = 1, \cdots, m$, as $t \to \infty$. Consider the nonnegative function $F(t, r_{(k)}) = \max_i r_{i(k)} - \min_i r_{i(k)}$. Note that $F(t, r_{(k)})$ is continuous if $\max_i r_{i(k)}$ and $\min_i r_{i(k)}$ are continuous. The upper right-hand derivative of $F(t, r_{(k)})$ is given by

$$\begin{aligned} &D^+F(t, r_{(k)}) \\ &= \limsup_{h\to 0^+} \frac{1}{h}[F(t+h, r_{(k)}(t+h)) - F(t, r_{(k)}(t))] \\ &= \limsup_{h\to 0^+} \left\{ \frac{1}{h}[\max_i r_{i(k)}(t+h) - \max_i r_{i(k)}(t)] \right. \\ &\quad \left. - \frac{1}{h}[\min r_{i(k)}(t+h) - \min r_{i(k)}(t)] \right\} \\ &= \max_{i\in \arg\max r_{i(k)}} D^+ r_{i(k)} - \max_{i\in \arg\min r_{i(k)}} D^+ r_{i(k)}. \end{aligned}$$

Considering the properties of $f_{i,j}(t,x,y)$ in (6) and (7), it follows that $\max_{i\in\arg\max r_{i(k)}} D^+ r_{i(k)} \le 0$ because $\dot{r}_{i(k)} < 0, \forall i \in \arg\max r_{i(k)}$, when $r_{i(k)}$ increases to be greater than $\max r_{i(k)}$. Similarly, $\max_{i\in\arg\min r_{i(k)}} D^+ r_{i(k)} \ge 0$ because $\dot{r}_{i(k)} > 0, \forall i \in \arg\min r_{i(k)}$, when $r_{i(k)}$ decreases to be smaller than $\min r_{i(k)}$. Therefore, $D^+F(t, r_{(k)}) \le 0$. By letting $F(t, r_{(k)})$ play the role of ω, $z(0) = \max_i r_{i(k)}(0) - \min_i r_{i(k)}(0)$, and $f(t,z) = 0$ in Lemma 3.1, it then follows from Lemma 3.1 that $F(t, r_{(k)}) = \max_i r_{i(k)}(t) - \min_i r_{i(k)}(t) \le \max_i r_{i(k)}(0) - \min_i r_{i(k)}(0), \forall t \ge 0$. Noting that $\left|r_{i(k)}(t) - r_{i(k)}(t)\right| \le F(t, r_{(k)})$, it follows that $\left|r_{i(k)}(t) - r_{i(k)}(t)\right| \le \max_i r_{i(k)}(0) - \min_i r_{i(k)}(0)$. Define $\eta \triangleq \min\{\varsigma, \frac{\epsilon}{\max_i r_{i(k)}(0) - \min_i r_{i(k)}(0)}\}$, where ϵ and ς are defined in, respectively, (6) and (7). It then follows from (8) that

$$\begin{aligned} D^+F(t, r_{(k)}) \le& \max_{i\in\arg\max r_{i(k)}} \left[-\sum_{j=1}^n a_{ij}(t)\eta(r_{i(k)} - r_{i(k)}) \right] \\ &- \max_{i\in\arg\min r_{i(k)}} \left[-\sum_{j=1}^n a_{ij}(t)\eta(r_{i(k)} - r_{i(k)}) \right], \end{aligned}$$

where we have used the properties of $f_{i,j}(t,x,y)$ in (6) and (7) and the fact that $\max_i r_{i(k)}$ is a nonincreasing function, $\max_i r_{i(k)}$ is a nondecreasing function, and $\max_i r_{i(k)} \ge r_{j(k)}$ and $\min_i r_{i(k)} \le r_{j(k)}$ for all $j = 1, \cdots, n$.

Consider the closed-loop dynamics

$$\dot{\xi}_i = -\sum_{j=1}^n a_{ij}(t)\eta(\xi_i - \xi_j), \quad i = 1, \cdots, n, \tag{9}$$

where $\xi_i(0) = r_{i(k)}(0)$ and $a_{ij}(t)$ is defined as in (8). With (9), consider the nonnegative function $G(t,\xi) \triangleq \max_i \xi_i - \min_i \xi_i$. It can be computed that

$$\begin{aligned} D^+G(t,\xi) =& \max_{i\in\arg\max \xi_i} \left[-\sum_{j=1}^n a_{ij}(t)\eta(\xi_i - \xi_j) \right] \\ &- \max_{i\in\arg\min \xi_i} \left[-\sum_{j=1}^n a_{ij}(t)\eta(\xi_i - \xi_j) \right]. \end{aligned}$$

Note that $D^+F(t, r_{(k)}) \le D^+G(t, r_{(k)})$. When $\xi_i(0) = r_{i(k)}(0)$, it follows that $F(0, r_{(k)}(0)) = G(0, r_{(k)}(0))$. It then follows from Lemma 3.4 that $F(t) \le G(t)$. Given (9), if there exists positive $\bar{t}$ such that the directed graph associated with $\int_t^{t+\bar{t}} \mathcal{A}(t)dt$ has a directed spanning tree for any $t \ge 0$, then it follows from Theorem 1 in [22] that $G(t) \to 0$ as $t \to \infty$. Note from the definition of $F(t)$ that $F(t) \ge 0$ for all $t \ge 0$. It then follows from the fact $F(t) \le G(t)$ that $F(t) \to 0$ as $t \to \infty$, which implies that $\max_i r_{i(k)}(t) - \min_i r_{i(k)}(t) \to 0$ as $t \to \infty$. Therefore, $\left|r_{i(k)}(t) - r_{j(k)}(t)\right| \to 0$ as $t \to \infty$. This completes the proof. ∎

Remark 4.3: Note that in [22], the closed-loop system is given by (9) when $\eta = 1$. Compared with the *linear* algorithm proposed in [22], we propose a more general *nonlinear* algorithm (5). Furthermore, the stability of some algorithms cannot be guaranteed by the results in [22] while can be guaranteed by Theorem 4.2. For instance, when $f_{i,j}(t,x,y)$ satisfies (6), we can get that $a_{ij}(t)f_{i,j}(t,x,y) = a_{ij}(t)\frac{\epsilon_1(t)}{|x-y|}(x-y)$ or $a_{ij}(t)f_{i,j}(t,x,y) = -a_{ij}(t)\frac{\epsilon_2(t)}{|x-y|}(x-y)$. By considering $a_{ij}(t)\frac{\epsilon_1(t)}{|x-y|}$ or $a_{ij}(t)\frac{\epsilon_2(t)}{|x-y|}$ as $b_{ij}(t)$ (corresponding to $a_{ij}(t)$ in [22]), it follows that $b_{ij}(t)$ is not upper-bounded if $a_{ij}(t) > 0$ is lower-bounded and $|x-y|$ approaches zero, which contradicts the assumption in [22] that $a_{ij}(t)$ is both lower-bounded and upper-bounded if $a_{ij}(t) \ne 0$. Therefore, the results presented in this section extend the results in [22].

Remark 4.4: Another interesting observation is that the computation of $D^+F(t, r_{(k)})$ and $D^+G(t, r_{(k)})$ only relies on $a_{ij}(t)$, $i \in \arg\min_i r_{i(k)} \bigcup \arg\max_i r_{i(k)}$, $j \in \mathcal{N}_i$, based on the proof of Theorem 4.2. For some special initial states and interaction graphs, $D^+F(t, r_{(k)}) \le D^+G(t, r_{(k)})$ holds even if $f_{i,j}(t,x,y)$ does not satisfy (6) or (7). For example, consider a group of three agents, labeled as 1, 2, and 3 in a one-dimensional space, with $r(0) = [1,2,3]^T$. Let agent 3 be a neighbor of agent 1 with $a_{13} = 1$ and $f_{1,3}(t,x,y) = x - y$ for a period of time $t^\star$ such that $r_2(t^\star) = 2$. Then let agent 2 be a neighbor of agents 1 and 3 with $a_{12} = 1$, $a_{32} = 1$, and $f_{1,2}(t,x,y) = f_{1,3}(t,x,y) = x - y$ for $t > t^\star$. According to [22], consensus can be achieved ultimately. Alternatively, consider the case when the interaction graph and $f_{i,j}(t,x,y)$ are the same except that $f_{1,3}(t,x,y) = 0$ for $t > t^\star$. Note that although $f_{1,3}(t,x,y) = 0$ does not satisfy (6) or (7), $D^+F(t, r_{(k)}) < D^+G(t, r_{(k)})$ holds for all $t \ge 0$. Note also that the interaction graph satisfies the conditions in Theorem 4.2, which implies that consensus can be achieved ultimately.

B. Consensus Tracking

In this subsection, we assume that in addition to the n agents, also called followers 1 to n, there exists a leader, labeled as agent 0. Note that the leader could be physical or virtual. We assume that $\|D^+ r_0\| \le \gamma$. Let $\overline{\mathcal{G}}$ be the directed graph for the n followers and the leader.

For a group of n followers with the dynamics given by (4), we propose the following distributed consensus tracking algorithm as

$$u_i = -\alpha \sum_{j=0}^{n} a_{ij}(t) f_{i,j}(t, r_i, r_j) - \beta \mathrm{sgn}\left[\sum_{j=0}^{n} a_{ij}(t) f_{i,j}(t, r_i, r_j)\right], i = 1, \cdots, n \quad (10)$$

where α and β are nonnegative constant scalars, $\mathrm{sgn}(\cdot)$ is the signum function defined componentwise, $a_{ij}(t)$ is the (i,j)th entry of the adjacency matrix $\overline{\mathcal{A}}(t)$ associated with $\overline{\mathcal{G}}(t)$ characterizing the interaction among the n followers and the leader at time t, and $f_{i,j}(t, r_i, r_j) \triangleq [f_{i,j}(t, r_{i(1)}, r_{j(1)}), \cdots, f_{i,j}(t, r_{i(m)}, r_{j(m)})]^T$ is defined such that for $x, y \in \mathbb{R}$, $f_{i,j}(t, x, y)$ satisfies (6) or (7), $i = 1, \cdots, n$, $j = 1, \cdots, n$, and $f_{i,0}(t, r_i, r_j) \triangleq [f_{i,0}(t, r_{i(1)}, r_{j(1)}), \cdots, f_{i,0}(t, r_{i(m)}, r_{j(m)})]^T$ is defined such that for $x, y \in \mathbb{R}$, $f_{i,0}(t, x, y)$ satisfies (6) or (7) or

$$f_{i,0}(t, x, y) \begin{cases} = \epsilon_1(t) > \epsilon, & x > y, \\ = -\epsilon_2(t) < -\epsilon, & x < y, \\ \in [-\epsilon_2(t), \epsilon_1(t)], & x = y. \end{cases} \quad (11)$$

The objective of (10) is to guarantee that $\|r_i(t) - r_0(t)\| \to 0$ as $t \to \infty$.

Define $\tilde{r}_i \triangleq r_i - r_0$. Using (10), (4) can be written as

$$\dot{\tilde{r}}_i = -\alpha \sum_{j=0}^{n} a_{ij}(t) f_{i,j}(t, r_i, r_j) - \beta \, \mathrm{sgn}\left[\sum_{j=0}^{n} a_{ij}(t) f_{i,j}(t, r_i, r_j)\right] - \dot{r}_0, \quad i = 1, \cdots, n. \quad (12)$$

Note that (12) is a nonlinear time-varying system and it is generally difficult to analyze the stability. We next study the stability of (12) by using the comparison lemmas in Section III.

Theorem 4.5: Suppose that the leader in the directed graph $\overline{\mathcal{G}}(t)$ has directed paths to followers 1 to n. For (15), when $\alpha > 0$ and $\beta \ge \gamma$, $\tilde{r}_i(t) \to \mathbf{0}_m$ as $t \to \infty$.

Proof: To show that $\tilde{r}_i(t) \to \mathbf{0}_m$ as $t \to \infty$, it is equivalent to show that $\max_i \left|\tilde{r}_{i(k)}\right| \to 0$ as $t \to \infty$ for all $k = 1, \cdots, m$. Consider the nonnegative function $F(t, \tilde{r}_{(k)}) = \max_i \left|\tilde{r}_{i(k)}\right|$. For the case when $F(t^\star, \tilde{r}_{(k)}) = 0$, it follows that $\tilde{r}_{i(k)}(t^\star) = 0$. For $t \ge t^\star$, note that $\dot{\tilde{r}}_{i(k)}(t) < 0$ if $\tilde{r}_{i(k)}(t)$ increases to be greater than 0. Therefore, $\tilde{r}_{i(k)}(t)$ is a nonincreasing function. Similarly, $\tilde{r}_{i(k)}(t)$ is a nondecreasing function. Therefore, $\tilde{r}_{i(k)}(t) = 0$ for any $t \ge t^\star$ when $F(t^\star, \tilde{r}_{(k)}) = 0$. We next focus on studying the case when $F(t, \tilde{r}_{(k)}) \neq 0$.

When $F(t, \tilde{r}_{(k)}) \neq 0$, the upper right-hand derivative of $F(t, \tilde{r}_{(k)})$ is given by

$$\begin{aligned} &D^+F(t, \tilde{r}_{(k)}) \\ &= \limsup_{h \to 0^+} \frac{1}{h}[F(t+h, \tilde{r}_{(k)}(t+h)) - F(t, \tilde{r}_{(k)}(t))] \\ &= \limsup_{h \to 0^+} \frac{1}{h}\left[\max_i \left|\tilde{r}_{i(k)}(t+h)\right| - \max_i \left|\tilde{r}_{i(k)}(t)\right|\right]. \end{aligned}$$

We next study $D^+F(t, \tilde{r}_{(k)})$ in the following three cases:

Case 1: $\max_i \left|\tilde{r}_{i(k)}\right| = \max_i \tilde{r}_{i(k)}$. Then there exists at least one agent, labeled as j, such that $\tilde{r}_{j(k)} > 0$ and $\max_i \left|\tilde{r}_{i(k)}\right| = \tilde{r}_{j(k)}$. In this case, it can be computed that $D^+F(t, \tilde{r}_{(k)}) = \max_{i \in \arg\max_i \tilde{r}_{i(k)}} D^+\tilde{r}_{i(k)}$. Note that for any agent j satisfying that $\tilde{r}_{j(k)} = \max_i \tilde{r}_{i(k)}$, $\dot{\tilde{r}}_{j(k)} < 0$ if $\tilde{r}_{j(k)}$ increases to be greater than $\max_i \tilde{r}_{i(k)}$. Therefore, $\max_i \left|\tilde{r}_{i(k)}\right|$ is a nonincreasing function.

Case 2: $\max_i \left|\tilde{r}_{i(k)}\right| = -\min_i \tilde{r}_{i(k)}$. Then there exists at least one agent, labeled as h, such that $\tilde{r}_{h(k)} < 0$ and $\max_i \left|\tilde{r}_{i(k)}\right| = -\tilde{r}_{h(k)}$. In this case, it can be computed that $D^+F(t, \tilde{r}_{(k)}) = \max_{i \in \arg\min_i \tilde{r}_{i(k)}} -D^+\tilde{r}_{i(k)}$. Note that for any agent h satisfying that $\tilde{r}_{h(k)} = \min_i \tilde{r}_{i(k)}$, $\dot{\tilde{r}}_{h(k)} > 0$ if $\tilde{r}_{h(k)}$ decreases to be smaller than $\max_i \tilde{r}_{i(k)}$. Therefore, $\max_i \left|\tilde{r}_{i(k)}\right|$ is a nonincreasing function.

Case 3: $\max_i \left|\tilde{r}_{i(k)}\right| = \max_i \tilde{r}_{i(k)} = -\min_i \tilde{r}_{i(k)}$. Then there exist at least one agent, labeled as j, such that $\tilde{r}_{j(k)} > 0$ and $\max_i \left|\tilde{r}_{i(k)}\right| = \tilde{r}_{j(k)}$ and at least one agent, labeled as h, such that $\tilde{r}_{h(k)} < 0$ and $\max_i \left|\tilde{r}_{i(k)}\right| = -\tilde{r}_{h(k)}$. In this case, it can be computed that $D^+F(t, \tilde{r}_{(k)}) = \max_{i \in \arg\max_i \tilde{r}_{i(k)}, j \in \arg\min_j \tilde{r}_{j(k)}} \{D^+\tilde{r}_{i(k)}, -D^+\tilde{r}_{j(k)}\}$. By following the analysis in Cases 1 and 2, it follows that $\max_i \left|\tilde{r}_{i(k)}\right|$ is a nonincreasing function.

Define $\eta \triangleq \min\{\varsigma, \frac{\epsilon}{\max_i \left|r_{i(k)}(0)\right|}\}$, where ϵ and ς are defined in, respectively, (6) and (7). Let $D^+r_{0(k)}$ be the kth component of D^+r_0. For Case 1, it can be computed that

$$\begin{aligned} D^+F(t, \tilde{r}_{(k)}) &= \max_{i \in \arg\max_i \tilde{r}_i} D^+\tilde{r}_{i(k)} \\ &\le \max_{i \in \arg\max_i \tilde{r}_{i(k)}} \Bigg\{ -\alpha \sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{r}_{i(k)} - \tilde{r}_{j(k)}) \\ &\quad - \beta \mathrm{sgn}\left[\sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{r}_{i(k)} - \tilde{r}_{j(k)})\right] - D^+r_{0(k)} \Bigg\}, \quad (13) \end{aligned}$$

where we have used the properties of $f_{i,j}(t, x, y)$ in (6) and (7) and the fact that $\max_i r_{i(k)}$ is a nonincreasing function, $\max_i \tilde{r}_{i(k)}$ is a nondecreasing function, $\max_i \tilde{r}_{i(k)} \ge \tilde{r}_{j(k)}$ for all $j = 1, \cdots, n$, and $\tilde{r}_{0(k)} = 0$. For Case 2, it can also be computed that

$$\begin{aligned} D^+F(t, \tilde{r}_{(k)}) &= \max_{i \in \arg\min_i \tilde{r}_{i(k)}} -D^+\tilde{r}_{i(k)} \\ &\le \max_{i \in \arg\min_i \tilde{r}_{i(k)}} \Bigg\{ \alpha \sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{r}_{i(k)} - \tilde{r}_{j(k)}) \\ &\quad + \beta \mathrm{sgn}\left[\sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{r}_{i(k)} - \tilde{r}_{j(k)})\right] + D^+r_{0(k)} \Bigg\}. \quad (14) \end{aligned}$$

For Case 3, it can be computed that $D^+F(t, \tilde{r}_{(k)})$ satisfies (13) and (14).

Consider the closed-loop dynamics given by

$$\dot{\tilde{\xi}}_i = -\alpha \sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{\xi}_i - \tilde{\xi}_j) - \beta \mathrm{sgn}\left[\sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{\xi}_i - \tilde{\xi}_j)\right] - D^+r_{0(k)}, \quad i = 1, \cdots, n, \quad (15)$$

where $\tilde{\xi}_i(0) = \tilde{r}_{i(k)}(0)$. Define $G(t, \tilde{\xi}) \triangleq \max_i \left|\tilde{\xi}_i\right|$, where $\tilde{\xi} = [\tilde{\xi}_1, \cdots, \tilde{\xi}_n]^T$. We also study $D^+G(t, \tilde{\xi})$ in three cases: Case 1: $\max_i \left|\tilde{\xi}_i\right| = \max_i \tilde{\xi}_i$. Then there exists at least one agent j such that $\tilde{\xi}_j > 0$ and $\max_i \left|\tilde{\xi}_i\right| = \tilde{\xi}_j$. In this case, it can be computed that

$$D^+G(t, \tilde{\xi}) = \max_{i \in \arg\max_i \tilde{\xi}_i} \left\{ -\alpha \sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{\xi}_i - \tilde{\xi}_j) - \beta \mathrm{sgn}\left[\sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{\xi}_i - \tilde{\xi}_j)\right] - D^+ r_{0(k)} \right\}.$$

Case 2: $\max_i \left|\tilde{\xi}_i\right| = -\min_i \tilde{\xi}_i$. Then there exists at least one agent h such that $\tilde{\xi}_h < 0$ and $\max_i \left|\tilde{\xi}_i\right| = -\tilde{\xi}_h$. In this case, it can be computed that

$$D^+G(t, \tilde{\xi}) = \max_{i \in \arg\min_i \tilde{\xi}_i} \left\{ \alpha \sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{\xi}_i - \tilde{\xi}_j) + \beta \mathrm{sgn}\left[\sum_{j=0}^{n} a_{ij}(t)\eta(\tilde{\xi}_i - \tilde{\xi}_j)\right] + D^+ r_{0(k)} \right\}.$$

Case 3: $\max_i \left|\tilde{\xi}_i\right| = \max_i \tilde{\xi}_i = -\min_i \tilde{\xi}_i$. Then there exist at least one agent, labeled as j, such that $\tilde{\xi}_j > 0$ and $\max_i \left|\tilde{\xi}_i\right| = \tilde{\xi}_j$ and at least one agent, labeled as h, such that $\tilde{\xi}_h < 0$ and $\max_i \left|\tilde{\xi}_i\right| = -\tilde{\xi}_h$. In this case, it can be computed that

$$D^+G(t, \tilde{\xi}) = \max_{i \in \arg\max_i \tilde{\xi}_i, j \in \arg\min_i \tilde{\xi}_i} \{D^+\tilde{\xi}_i, -D^+\tilde{\xi}_j\}.$$

Note that $D^+F(t, \tilde{r}_{(k)}) \leq D^+G(t, \tilde{r}_{(k)})$. When $\tilde{r}_{(k)}(0) = \xi(0)$ [*i.e.*, $F(0, \tilde{r}_{(k)}(0)) = G(0, \tilde{r}_{(k)}(0))$], it then follows from Lemma 3.4 that $F(t) \leq G(t)$ for any $t \geq 0$. Given (9), if the directed graph $\mathcal{G}(t)$ has a directed spanning tree, then it follows from Theorem 3.1 in [24] that $G(t) \to 0$ as $t \to \infty$. Note from the definition of $F(t)$ that $F(t) \geq 0$ for all $t \geq 0$. It then follows from the fact $F(t) \leq G(t)$ that $F(t) \to 0$ as $t \to \infty$. Therefore, we have that $\max_i \tilde{r}_{i(k)}(t) = \min_i \tilde{r}_{i(k)}(t) = 0$ as $t \to \infty$, which implies that $\tilde{r}_{i(k)}(t) \to 0$ as $t \to \infty$. This completes the proof. ■

V. Conclusion

This paper studied distributed multi-agent coordination by using a comparison lemma based approach. First, we presented general comparison lemmas for vector differential equations. Then the application of the general comparison lemmas was illustrated by studying leaderless consensus and consensus tracking. Compared with the traditional approaches used in the stability analysis, the comparison lemma based approach provides an important tool in the stability analysis, especially for nonlinear closed-loop systems, by making use of known results in linear/nonlinear systems.

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