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Hybrid Output Regulation for Minimum Phase Linear Systems

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***Abstract*— We consider the hybrid output regulation problem for minimum phase linear systems with relative degree greater than one. In the hybrid output regulation problem, the disturbances, which are assumed to be matched with the control input, evolve through a hybrid dynamical system that experiences jumps periodically. In constructing an output regulator for this problem, we combine recent work on necessary conditions for hybrid output regulation with classical ideas related to high-gain output feedback and high-gain observers. The output regulator we design achieves global exponential stability of a compact set in which the output of the system is zero.**

I. INTRODUCTION

The problem of output regulation, which consists of controlling the output of a plant to reject disturbances generated by an exosystem, has a long history including the seminal work of Davison, Francis and Wonham, [1], [2], and [3]. Recently, an effort has been made to extend output regulation theory to the case where the disturbances are generated by a linear exosystem that experiences linear jumps periodically. In this setting, the control objective has been labeled "hybrid output regulation", and several aspects of the theory have been developed in [7]. One issue that is not completely resolved in [7] is how to construct the stabilizing part of the hybrid output regulator. This part is challenging because it involves the design of both continuous-time and discrete-time components simultaneously. In [7], the stabilization problem characterizing the hybrid output regulation problem is solved completely for relative degree one, minimum phase systems; here we extend that construction to minimum phase systems with higher relative degree. To accomplish this task, we use a high-gain observer.

Like in [7], we rely on the hybrid systems framework summarized in [6]. The control ideas we use have connections to the results in [9]. Other work in the literature where switching or hybrid systems is combined with output regulation include [4], [8], and [5].

In Section II we present the class of systems under investigation. In Section III we review the general regulator construction proposed in [7]. In Section IV there is a closed-loop analysis of the system with the hybrid regulator. An example appears in Section V. The appendix contains a proof of an intermediate proposition.

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II. FRAMEWORK

The system we work with conforms to the following framework:

$$\left.\begin{array}{rcl} \dot{z} &=& Az + Bx_1 \\ \dot{x}_1 &=& x_2 \\ &\vdots& \\ \dot{x}_{m-1} &=& x_m \\ \dot{x}_m &=& \alpha_1 x_1 + \ldots + \alpha_m x_m \\ && +B_z z + b(u - R(\tau)\omega) \\ \dot{\tau} &=& 1 \\ \dot{\omega} &=& S\omega \end{array}\right\} \quad (1)$$
$$(z,x,\tau,\omega) \in \mathbb{R}^n \times \mathbb{R}^m \times [0,T] \times \mathcal{W},$$

and:

$$\left.\begin{array}{rcl} z^+ &=& z \\ x^+ &=& x \\ \tau^+ &=& 0 \\ \omega^+ &=& J\omega \end{array}\right\} \quad (2)$$
$$(z,x,\tau,\omega) \in \mathbb{R}^n \times \mathbb{R}^m \times \{T\} \times \mathcal{W},$$

where $\mathcal{W} \subset \mathbb{R}^s$. The state $y = x_1$ corresponds to the output of the plant, while input $u \in \mathbb{R}$. The dimension of x, m, is the relative degree of the plant. The state z represents additional internal states of the system. The exosystem has the state ω and enters the plant additively with the control input. The clock variable, τ, guarantees that the system jumps periodically. For the purposes of this paper it is assumed that τ is a known state. This may not always be reasonable, and estimation of the clock variable is an interesting problem that is addressed in [10].

Assumption 1. *The matrix A is Hurwitz, the set $\mathcal{W}$ is compact, $|b| \geq b_0 > 0$, and $T > 0$.*

III. REGULATOR

Building on [7], we construct a regulator in two parts, including an internal model of the form:

$$\begin{array}{rcl} \dot{\xi} &=& (F + G\Gamma(\tau))\xi + \Phi_\Delta \eta \\ \xi^+ &=& \Sigma_{im}\xi \end{array} \qquad \xi \in \mathbb{R}^\nu, \quad (3)$$

and a stabilizer of the form:

$$\begin{array}{rcl} \dot{\eta} &=& \Phi_{st}\eta + \Lambda_{st} y \\ \eta^+ &=& \eta \end{array} \qquad \eta \in \mathbb{R}^m, \quad (4)$$

with the feedback:

$$u = \Gamma(\tau)\xi + K\eta. \tag{5}$$

The following assumption must be made on F and Σ_{im}, the necessity of which becomes clear in Section IV.

Assumption 2. *F, G, and Σ_{im} are such that the eigenvalues of $\Sigma_{im}\exp(FT)$ are within the unit disk and the pair (F, G) is controllable. Also, $\nu \geq s$.*

Note that the feedback term $\Gamma(\tau)$ need not always be τ-dependent; for such cases the regulator design can be significantly simplified.

A. τ-Independent Γ

The feedback term Γ can be taken to be τ-independent if the following assumption holds.

Assumption 3. *The parameter R is not dependent on τ, F is Hurwitz, and the folllowing set of equations:*

$$\begin{aligned} F\Pi - \Pi S + GR &= 0, \\ \Sigma_{im}\Pi &= \Pi J, \end{aligned}$$

has a solution (Π, Σ_{im}).

In this scenario take $\Gamma = R\Pi^{\dagger}$, where $\Pi^{\dagger}$ denotes the Moore-Penrose pseudo-inverse. Occasionally, Assumption 3 fails; the following is an example of such a case.

Example 1. *Consider the exosystem with the following matrix parameters:*

$$S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{6}$$

Suppose that (F, G) is a controllable pair with F Hurwitz, such that the equation $F\Pi - \Pi S + GR = 0$ has a unique solution Π. The resulting solution, $\Pi = \begin{bmatrix} -F^{-1}G & 0 \end{bmatrix}$, does not admit a solution Σ_{im} to the equation $\Sigma_{im}\Pi = \Pi J$. Attempting to find one gives:

$$\begin{aligned} \Sigma_{im}\begin{bmatrix} -F^{-1}G & 0 \end{bmatrix} &= \begin{bmatrix} -F^{-1}G & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \\ \begin{bmatrix} -\Sigma_{im}F^{-1}G & 0 \end{bmatrix} &= \begin{bmatrix} 0 & -F^{-1}G \end{bmatrix}. \end{aligned}$$

This relationship implies that $F^{-1}G = 0$, which leads to $G = 0$. Therefore, (F, G) is not controllable. This is a contradiction because (F, G) was chosen to be controllable via Assumption 2. It can be concluded that there is no solution, Σ_{im}.

On the other hand, it is trivial to show that the exosystem described by (6) fits into the framework for the τ-dependent case, which is outlined in the following section. So, the result can still be pursued by choosing Γ to be τ-dependent.

B. τ-Dependent Γ

When Assumption 3 fails we continue with a τ-dependent parameter $\Gamma(\tau)$. In this scenario we choose $\Gamma(\tau) = R(\tau)\Pi(\tau)^{\dagger}$, where $\Pi(\tau)$ is the solution to the following set of equations:

$$\begin{aligned} \frac{d\Pi(\tau)}{d\tau} &= F\Pi(\tau) - \Pi(\tau)S + GR(\tau), \\ 0 &= \Sigma_{im}\Pi(T) - \Pi(0)J, \\ R(\tau) &= \Gamma(\tau)\Pi(\tau). \end{aligned} \tag{7}$$

The first two equations of (7) can be solved by taking $L : [0, T] \to \mathbb{R}^{\nu \times s}$ and $\Pi : [0, T] \to \mathbb{R}^{\nu \times s}$ to be continuously differentiable functions satisfying $L(0) = 0$, where $\nu \geq s$,

$$\frac{dL(\tau)}{d\tau} = FL(\tau) + GR(\tau)\exp(S\tau)$$

and

$$\Pi(\tau) = (\exp(F\tau)\Pi(0) + L(\tau))\exp(-S\tau),$$

where $\Pi(0)$ satisfies

$$\Sigma_{im}\exp(FT)\Pi(0) - \Pi(0)J\exp(ST) + \Sigma_{im}L(T) = 0.$$

As shown in [7], the last equation of (7) can be solved if there exists a positive $r \leq \nu$ such that the rank of $\Pi(\tau) = r$ for all $\tau \in [0, T]$. This scenario is pursuable under the following assumption, which is less stringent than Assumption 3.

Assumption 4. *F and Σ_{im} are such that the eigenvalues of $\Sigma_{im}\exp(F\tau)$ and $J\exp(S\tau)$ are disjoint. Furthermore, there exists a positive $r \leq \nu$ such that the rank of $\Pi(\tau) = r$ for all $\tau \in [0, T]$.*

C. Parameter Choices for Global Exponential Stability

We now propose some parameter choices for the regulator and feedback, (3)-(4) and (5), respectively. These choices allow the system (1)-(2) with state $(z, x, \tau, \omega, \xi, \eta)$ to have the set $\{0\} \times \{0\} \times \Upsilon \times \{0\}$ globally exponentially stable (GES), where $\Upsilon = \{(\tau, \omega, \xi) \in [0, T] \times \mathcal{W} \times \mathbb{R}^{\nu} : \xi = \Pi(\tau)\omega\}$. Ideas on the use of high-gain observers and feedback are drawn from [11], which discusses these concepts in a non-linear, non-hybrid setting. Choose:

$$\begin{aligned} K &= -sgn(b)\kappa \begin{bmatrix} k_1 & \dots & k_{m-1} & 1 \end{bmatrix}, \ \Phi_{\Delta} = GK, \\ \Lambda_{st} &= \begin{bmatrix} c_1 l \\ \vdots \\ c_m l^m \end{bmatrix}, \ \Phi_{st} = \begin{bmatrix} -\Lambda_{st} & \begin{matrix} I_{m-1} \\ 0_{1\times(m-1)} \end{matrix} \end{bmatrix}, \end{aligned} \tag{10}$$

where the coefficients c_i and k_i are designed such that $s^m + c_1 s^{m-1} + \ldots + c_{m-1}s + c_m$ and $s^{m-1} + k_1 s^{m-2} + \ldots + k_{m-2}s + k_{m-1}$ are Hurwitz polynomials.

Recall that F, G, and Σ_{im} are chosen to satisfy Assumption 2, and either Assumption 3 or 4. Furthermore, $\Gamma(\tau) = R(\tau)\Pi(\tau)^{\dagger}$, with its τ dependence determined by the validity of Assumption 3.

Theorem 1. *There exists $\kappa^* > 0$ and for each $\kappa \geq \kappa^*$ there exists $l^* > 0$ such that for each $l \geq l^*$ and with the choices in (10) and Assumptions 2-4, the closed-loop system with regulator (3)-(4) and feedback (5) applied to the system (1)-(2) with the state $(z, x, \tau, \omega, \xi, \eta)$ has the set $\{0\} \times \{0\} \times \Upsilon \times \{0\}$ GES.*

$$
\begin{aligned}
p &:= \begin{bmatrix} \alpha_1 - k_1(\alpha_m + k_{m-1}) & \ldots & \alpha_{m-1} - k_{m-1}(\alpha_m + k_{m-1}) \end{bmatrix} + \begin{bmatrix} 0 & k_1 & k_2 & \ldots & k_{m-2} \end{bmatrix} && (8)\\
q(x, z, \chi, \tilde{x}_m) &:= -\alpha_1 x_1 - \ldots - \alpha_{m-1} x_{m-1} + \alpha_m(k_1 x_1 + \ldots + k_{m-1} x_{m-1}) \\
&\quad -B_z z - b\Gamma(\tau)\chi + (\kappa - \alpha_m - \Gamma(\tau)G)\tilde{x}_m && (9)
\end{aligned}
$$

In the following section, we prove Theorem 1 through an analysis of the closed-loop system.

IV. Closed-Loop System Analysis

In this section we provide a proof of Theorem 1. In proceeding with this proof we drop the ω dynamics from the closed-loop analysis, since it is not affected by and does not affect the other states after an appropriate coordinate transformation.

A. Linear Transformation

To begin with, perform the following change of variables:

$$
\begin{aligned}
\tilde{x}_m &:= x_m + k_1 x_1 + \ldots + k_{m-1} x_{m-1}, \\
\chi &:= \xi - b^{-1} G \tilde{x}_m - \Pi(\tau)\omega.
\end{aligned}
$$

Note that $(z, [x_1 \ldots x_{m-1}], \tilde{x}_m, \chi) = 0$, $\tau \in [0, T]$, and $\omega \in \mathcal{W}$ if and only if $(z, x, \tau, \omega, \xi) \in \{0\} \times \{0\} \times \Upsilon$.

Bearing in mind (7), pick $\Phi_\Delta = GK$, as in (10), to eliminate the presence of η in the χ dynamics. During flow this gives:

$$
\begin{aligned}
\dot{z} &= Az + Bx_1 \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{m-2} &= x_{m-1} \\
\dot{x}_{m-1} &= \tilde{x}_m - k_1 x_1 - \ldots - k_{m-1} x_{m-1} \\
\dot{\tilde{x}}_m &= (\Gamma(\tau)G + k_{m-1} + \alpha_m)\tilde{x}_m \\
&\quad + \begin{bmatrix} p & 0 \end{bmatrix} x + B_z z + b\Gamma(\tau)\chi + bK\eta \\
\dot{\chi} &= F\chi - b^{-1} G(\begin{bmatrix} p & 0 \end{bmatrix} x + B_z z) \\
&\quad + b^{-1}(FG - G(\alpha_m + k_{m-1}))\tilde{x}_m \\
\dot{\eta} &= \Phi_{st}\eta + \Lambda_{st} x_1,
\end{aligned}
$$

where p is defined as in (8). Since x is not measured we use the dynamics of η, which corresponds to a high-gain observer, to estimate x. In particular, define $\tilde{\eta}$ as follows:

$$
\tilde{\eta} := D_l(\eta - x),
$$

where $D_l = \mathrm{diag}(l^{m-1}, \ldots, l^0)$.

With K, Φ_{st} and Λ_{st} chosen as in (10), this change of variables results in the following closed-loop system, where $C := \mathbb{R}^n \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^\nu \times \mathbb{R}^m \times [0, T]$ and $D := \mathbb{R}^n \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^\nu \times \mathbb{R}^m \times \{T\}$:

$$
\left.
\begin{aligned}
\dot{z} &= Az + Bx_1 \\
\dot{x}_1 &= x_2 \\
&\vdots \\
\dot{x}_{m-2} &= x_{m-1} \\
\dot{x}_{m-1} &= \tilde{x}_m - k_1 x_1 - \ldots - k_{m-1} x_{m-1} \\
\dot{\tilde{x}}_m &= (\alpha_m + k_{m-1} + \Gamma(\tau)G - |b|\kappa)\tilde{x}_m \\
&\quad + \begin{bmatrix} p & 0 \end{bmatrix} x + bKD_l^{-1}\tilde{\eta} \\
\dot{\chi} &= F\chi - b^{-1} G(\begin{bmatrix} p & 0 \end{bmatrix} x + B_z z) \\
&\quad + b^{-1}(FG - G(\alpha_m + k_{m-1}))\tilde{x}_m \\
\dot{\tilde{\eta}} &= lH\tilde{\eta} + \begin{bmatrix} 0_{(m-1)\times 1} \\ 1 \end{bmatrix} \cdot \\
&\quad \cdot (q(x, z, \chi, \tilde{x}_m) - bKD_l^{-1}\tilde{\eta}) \\
\dot{\tau} &= 1
\end{aligned}
\right\} \quad (11)
$$

$$
(z, \begin{bmatrix} x_1 & \ldots & x_{m-1} \end{bmatrix}, \tilde{x}_m, \chi, \tilde{\eta}, \tau) \in C,
$$

and:

$$
\left.
\begin{aligned}
z^+ &= z \\
x_1^+ &= x_1 \\
&\vdots \\
x_{m-1}^+ &= x_{m-1} \\
\tilde{x}_m^+ &= \tilde{x}_m \\
\chi^+ &= \Sigma_{im}\chi + (\Sigma_{im} - I)b^{-1}G\tilde{x}_m \\
\tilde{\eta}^+ &= \tilde{\eta} \\
\tau^+ &= 0
\end{aligned}
\right\} \quad (12)
$$

$$
(z, \begin{bmatrix} x_1 & \ldots & x_{m-1} \end{bmatrix}, \tilde{x}_m, \chi, \tilde{\eta}, \tau) \in D,
$$

where, $q(x, z, \chi, \tilde{x}_m)$ is defined in (9) and H is Hurwitz by the choice of $c_1, \ldots, c_m$. Specifically:

$$
H = \begin{bmatrix} -c_1 & I_{m-1} \\ \vdots & \\ -c_m & 0_{1\times(m-1)} \end{bmatrix}.
$$

In the next section a Lyapunov analysis shows that the set $(z, \begin{bmatrix} x_1 & \ldots & x_{m-1} \end{bmatrix}, \tilde{x}_m, \chi, \tilde{\eta}, \tau) \in \{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times [0, T]$ is GES.

B. Lyapunov Analysis

The following proposition is used in both steps of this proof.

Proposition 1. *Consider the system*

$$
\left.
\begin{aligned}
\dot{v}_1 &= A_1 v_1 + B_1(\ell) v_2 \\
\dot{v}_2 &= \ell A_2 v_2 + M(\ell) v_2 + B_2(\ell) v_1 \\
\dot{\tau} &= 1
\end{aligned}
\right\} \quad (13)
$$

$$
(v_1, v_2, \tau) \in \mathbb{R}^\rho \times \mathbb{R}^\sigma \times [0, T]
$$

$$\left.\begin{array}{rcl} v_1^+ &=& J_1 v_1 + L(\ell) v_2 \\ v_2^+ &=& J_2 v_2 \\ \tau^+ &=& 0 \end{array}\right\} \tag{14}$$
$$(v_1, v_2, \tau) \in \mathbb{R}^\rho \times \mathbb{R}^\sigma \times \{T\},$$

where $(v_1, v_2, \tau) \in \mathbb{R}^\rho \times \mathbb{R}^\sigma \times [0, T]$ and $T > 0$. If the matrices $M(\ell)$, $B_1(\ell)$, $B_2(\ell)$ and $L(\ell)$ are bounded uniformly in ℓ, the eigenvalues of $J_1 \exp(A_1 T)$ are inside the unit disk and A_2 is Hurwitz, then there exists $\ell^* > 0$ such that, for each $\ell \geq \ell^*$ the set $\{0\} \times [0, T] \subset \mathbb{R}^{\rho+\sigma+1}$ is GES.

The closed-loop system, described by (11)-(12), has a desirable structure, which allows for easy application of Proposition 1. The Lyapunov analysis of the closed-loop system is performed in two steps. First, ignore the $\tilde{\eta}$ dynamics, and choose κ to be large enough to stabilize the $(x_1, \ldots, x_{m-1}, z, \chi, \tilde{x}_m, \tau)$ dynamics. Then, re-account for $\tilde{\eta}$ and choose l to be large enough such that the overall closed-loop system, (11)-(12), with the state $(z, x_1, \ldots, x_{m-1}, \tilde{x}_m, \chi, \tilde{\eta}, \tau)$ has the set $\{0\} \times \{0\} \times \ldots \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times [0, T]$ GES.

1) Ignoring $\tilde{\eta}$: First, pick $v_1 = (x_1, \ldots, x_{m-1}, z, \chi)$ and $v_2 = \tilde{x}_m$ and ignore $\tilde{\eta}$. The v_1 and v_2 dynamics can be written as:

$$\left.\begin{array}{rcl} \dot{v}_1 &=& \begin{bmatrix} A_C & 0 & 0 \\ B & A & 0 \\ -b^{-1}Gp & -b^{-1}GB_z & F \end{bmatrix} v_1 \\ && + \begin{bmatrix} B_C \\ 0_{n\times 1} \\ b^{-1}(FG - G(\alpha_m + k_{m-1})) \end{bmatrix} v_2 \\ \dot{v}_2 &=& (\alpha_m + k_{m-1} + \Gamma(\tau)G - |b|\kappa) v_2 \\ && + \begin{bmatrix} p & 0 & 0 \end{bmatrix} v_1 \\ \dot{\tau} &=& 1 \end{array}\right\} \tag{15}$$
$$(v_1, v_2, \tau) \in \mathbb{R}^{(m-1)+n+\nu} \times \mathbb{R} \times [0, T],$$

and:

$$\left.\begin{array}{rcl} v_1^+ &=& \begin{bmatrix} I_{m-1} & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & \Sigma_{im} \end{bmatrix} v_1 \\ && + \begin{bmatrix} 0 \\ 0 \\ (\Sigma_{im} - I_s) b^{-1} G \end{bmatrix} v_2 \\ v_2^+ &=& v_2 \\ \tau^+ &=& 0 \end{array}\right\} \tag{16}$$
$$(v_1, v_2, \tau) \in \mathbb{R}^{(m-1)+n+\nu} \times \mathbb{R} \times \{T\},$$

where $A_C = \begin{bmatrix} 0_{(m-2)\times 1} & & I_{m-2} & \\ -k_1 & -k_2 & \ldots & -k_{m-1} \end{bmatrix}$ is Hurwitz and $B_C = \begin{bmatrix} 0_{(m-1)\times 1} \\ 1 \end{bmatrix}$. The system described by (15)-(16) fits in the framework of Proposition 1, where κ fills the role of ℓ and $-|b|$ fills the role of A_2. Therefore, κ can be chosen large enough such that the system with state (v_1, v_2) has the set $\{0\} \times \{0\}$ GES. This leads to the conclusion that the eigenvalues of $J_{cl} \exp(A_{cl} T)$ lie within the unit disk, where $A_{cl} =$

$$\begin{bmatrix} A_C & 0 & 0 & B_C \\ B & A & 0 & 0 \\ -\frac{1}{b}Gp & -\frac{1}{b}GB_z & F & \frac{1}{b}(FG - G(\alpha_m + k_{m-1})) \\ p & 0 & 0 & \alpha_m + k_{m-1} + \Gamma(\tau)G - |b|\kappa \end{bmatrix}$$

and $J_{cl} =$

$$\begin{bmatrix} I_{m-1} & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \\ 0 & 0 & \Sigma_{im} & (\Sigma - I_s) b^{-1} G \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

With this established there is one last step to show global exponential stability for the entire closed-loop system described by (11)-(12), namely the $\tilde{\eta}$ dynamics must be re-accounted for.

2) Re-accounting for $\tilde{\eta}$: Take $v_2 = \tilde{\eta}$ and $v_1 = (x_1, \ldots, x_{m-1}, z, \chi, \tilde{x}_m)$. Then:

$$\left.\begin{array}{rcl} \dot{v}_1 &=& A_{cl} v_1 + \begin{bmatrix} 0_{(m-1+n+s)\times 1} \\ bKD_l^{-1} \end{bmatrix} v_2 \\ \dot{v}_2 &=& lHv_2 \\ && + \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix} (q(x, z, \chi, \tilde{x}_m) - bKD_l^{-1} v_2) \\ \dot{\tau} &=& 1 \end{array}\right\}$$
$$(v_1, v_2, \tau) \in \mathbb{R}^{m+n+\nu} \times \mathbb{R}^m \times [0, T]$$

$$\left.\begin{array}{rcl} v_1^+ &=& J_{cl} v_1 \\ v_2^+ &=& v_2 \\ \tau^+ &=& 0 \end{array}\right\}$$
$$(v_1, v_2, \tau) \in \mathbb{R}^{m+n+\nu} \times \mathbb{R}^m \times \{T\}.$$

Once again, this system fits into the framework of Proposition 1. Therefore, it can be concluded that the closed-loop system described by (11)-(12) with state $([x_1, \ldots, x_{m-1}], z, \chi, \tilde{x}_m, \tilde{\eta}, \tau)$ has the set $\{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times [0, T]$ GES.

V. EXAMPLE

As an example of how to apply the regulator designed here, we provide the following. Consider a plant with the relative degree two transfer function:

$$\frac{Y(s)}{\tilde{U}(s)} = \frac{a_2 s^2 + ds + k}{s^2(a_1 a_2 s^2 + k(a_1 + a_2))}.$$

With $a_2, d, k > 0$, the plant is minimum phase. For our simulations we take $a_1 = 10$, $a_2 = 1$, $k = 1$ and $d = 1$. Assume that there is a disturbance additive with the control signal, such that:

$$\tilde{u} = u - \omega,$$

where u is the control signal and ω is a disturbance generated by the exosystem:

$$\left.\begin{array}{rcl} \dot{\omega} &=& 0 \\ \dot{\tau} &=& 1 \end{array}\right\} \quad (\omega, \tau) \in \mathbb{R} \times [0, T],$$

$$\left.\begin{array}{rcl} \omega^+ &=& -\omega \\ \tau^+ &=& 0 \end{array}\right\} \quad (\omega, \tau) \in \mathbb{R} \times \{T\},$$

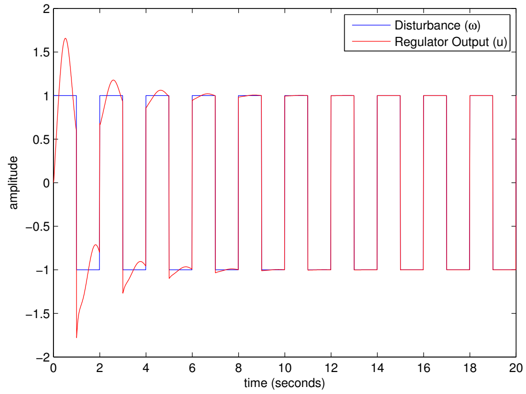

Fig. 1. Disturbance and Regulator Output

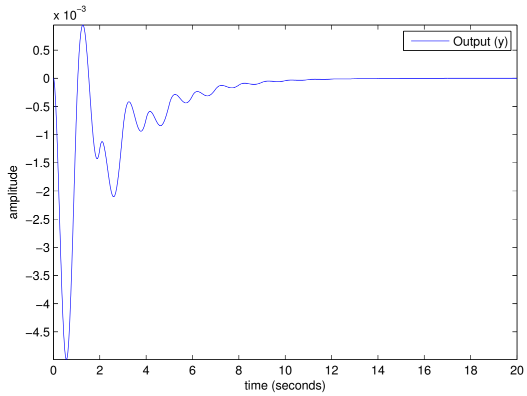

Fig. 2. Plant Output and Internal State

where τ is a clock variable governing the exosystem's jumps.

Following the steps laid out in this paper, we can design a regulator to achieve global exponential stability of the origin of the plant in the presence of this disturbance.

Begin by choosing the pair (F, G) as:

$$F = \begin{bmatrix} -10 & -50 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The considered exosystem satisfies Assumption 3, so we can take Γ to be τ-invariant, where:

$$\Pi = \begin{bmatrix} 0 \\ \frac{1}{50} \end{bmatrix}, \quad \Sigma_{im} = -1, \quad \Gamma = \begin{bmatrix} 0 & 50 \end{bmatrix}.$$

Then, pick the Hurwitz polynomial coefficients $k_1 = 1$ and $(c_1, c_2) = (4, 4)$. Finally, guided by Theorem 1, we pick κ sufficiently large and, subsequently, l sufficiently large. By simulation, we find that $\kappa = 50$ and $l = 70$ is sufficient for stability. The remainder of the regulator is constructed based on these choices. The results are shown in Figure 1 and Figure 2.

VI. Conclusions

The problem of extending hybrid output regulation to relative degree greater than one, minimum phase systems has been addressed. This has been accomplished by using a high-gain observer, in conjunction with high-gain control. The regulator builds on the framework already put in place in [7] for output regulation of linear systems with hybrid exosystems, and also on the high-gain results of [11]. Future goals include handling cases where the jump clock, τ, is not measured, but the duration of flows is known, based on the relative degree one results for this case reported in [10].

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Appendix I
Proof of Proposition 1

This proof breaks the system (13)-(14) into two subsystems. Simply take the v_1 dynamics as the first system and treat v_2 as its input, then do the reverse for the second system. Finally, the two subsystems are interconnected and it is shown that the interconnection is GES.

For a function β depending on a state ϕ that satisfies $\phi^+ = g(\phi)$, we define the shorthand notation $\beta^+ := \beta(g(\phi))$. Similarly, if β is continuously differentiable and ϕ satisfies $\dot{\phi} = f(\phi)$, we define the shorthand notation $\dot{\beta} := \langle \nabla\beta(\phi), f(\phi) \rangle$.

To show that the interconnection is GES, we find a Lyapunov function, Ψ, and positive constants, $\alpha_1, \ldots, \alpha_4$, such that:

$$\begin{aligned} \alpha_1 ||v||^2 \leq \quad \Psi(v, \tau) & \leq \alpha_2 ||v||^2, \\ \Psi^+ - \Psi \quad & \leq \quad -\alpha_3 ||v||^2, \\ \dot{\Psi} \quad & \leq \quad -\alpha_4 ||v||^2, \end{aligned} \tag{17}$$

where $|| \cdot ||$ denotes the Euclidean norm and $v = (v_1, v_2)$. This implies that the system with state v has the origin GES, since this means that:

$$\begin{aligned} \Psi^+ & \leq \exp(-\lambda)\Psi, \\ \dot{\Psi} & \leq -\lambda\Psi, \end{aligned}$$

for some $\lambda > 0$. It then follows that $\Psi(v(t,j),\tau(t,j)) \leq \exp(-\lambda(t+j))\Psi(v(0,0),\tau(0,0))$. In turn, $||v(t,j)||$ can be bounded using the first inequality of (17).

A. System 1

Consider the system:

$$\left.\begin{array}{rcl}\dot{v}_1 &=& A_1 v_1 + B_1(\ell) u_1 \\ \dot{\tau} &=& 1\end{array}\right\} \quad (v_1,\tau) \in \mathbb{R}^\rho \times [0,T],$$

and:

$$\left.\begin{array}{rcl}v_1^+ &=& J_1 v_1 + L(\ell) u_1 \\ \tau^+ &=& 0\end{array}\right\} \quad (v_1,\tau) \in \mathbb{R}^\rho \times \{T\}.$$

Choose the Lyapunov function $W(v_1,\tau) =$

$$\exp(-\epsilon\tau) v_1^T \exp\left(A_1(T-\tau)\right)^T X \exp\left(A_1(T-\tau)\right) v_1,$$

where $X = X^T > 0$ is specified later. First, look at the behavior of W along jumps:

$$\begin{aligned}W^+ - W \;\; = \;\; & v_1^T J_1^T \exp(A_1T)^T X \exp(A_1T) J_1 v_1 \\ & + 2 v_1^T J_1^T \exp(A_1T)^T X \exp(A_1T) L(\ell) u_1 \\ & + u_1^T L(\ell)^T \exp(A_1T)^T X \exp(A_1T) L(\ell) u_1 \\ & - \exp(-\epsilon T) v_1^T X v_1.\end{aligned}$$

Then:

$$\begin{aligned}W^+ - W \;\; \leq \;\; & \exp(-\epsilon T) v_1^T (M^T X M - X) v_1 \\ & + (1+\gamma_2^2) u_1^T L(\ell)^T \exp(A_1T)^T \cdot \\ & \cdot X \exp(A_1T) L(\ell) u_1,\end{aligned}$$

where, $M = \sqrt{\frac{1+\frac{1}{\gamma_2^2}}{\exp(-\epsilon T)}}\exp(A_1T)J_1$. Now, specify that X satisfies the discrete Lyapunov equation $M^TXM - X = -Q$, where $Q = Q^T > 0$. Here, it is important that the eigenvalues of $\exp(A_1T)J_1$ are inside the unit circle. This allows the eigenvalues of M to remain inside the unit circle, when $\gamma_2, \epsilon > 0$ are chosen such that $\sqrt{\frac{1+\frac{1}{\gamma_2^2}}{\exp(-\epsilon T)}} > 1$ is arbitrarily close to one. Thus:

$$\begin{aligned}W^+ - W \;\; \leq \;\; & -\exp(-\epsilon T) v_1^T Q v_1 + \\ & (1+\gamma_2^2) u_1^T L(\ell)^T \exp(A_1T)^T \cdot \\ & \cdot X \exp(A_1T) L(\ell) u_1,\end{aligned}$$

or, simply:

$$W^+ - W \;\; \leq \;\; -d_1||v_1||^2 + d_2||u_1||^2, \qquad (18)$$

where, d_1 and d_2 are positive scalar constants. Next, observe $W(v_1,\tau)$ along flows:

$$\begin{aligned}\dot{W} \;\; = \;\; & -\epsilon W + 2\exp(-\epsilon\tau) v_1^T \exp(A_1(T-\tau))^T X \cdot \\ & \cdot \exp(A_1(T-\tau)) B_1(\ell) u_1.\end{aligned}$$

Furthermore:

$$\begin{aligned}\dot{W} \;\; \leq \;\; & \left(-\epsilon + \frac{1}{\gamma_1^2}\right) W + \gamma_1^2 u_1^T B_1(\ell)^T \exp(A_1(T-\tau))^T \cdot \\ & \cdot X \exp(A_1(T-\tau)) B_1(\ell) u_1.\end{aligned}$$

Pick γ_1 such that $-\epsilon + \frac{1}{\gamma_1^2} < 0$. Note that there exist constants $c_1, c_2 > 0$, such that:

$$\dot{W} \;\; \leq \;\; -c_1||v_1||^2 + c_2||u_1||^2. \qquad (19)$$

B. System 2

Consider the system:

$$\left.\begin{array}{rcl}\dot{v}_2 &=& \ell A_2 v_2 \\ && + M(\ell) v_2 + B_2(\ell) u_2 \\ \dot{\tau} &=& 1\end{array}\right\} \quad (v_2,\tau) \in \mathbb{R}^\sigma \times [0,T],$$

and:

$$\left.\begin{array}{rcl}v_2^+ &=& J_2 v_2 \\ \tau^+ &=& 0\end{array}\right\} \quad (v_2,\tau) \in \mathbb{R}^\sigma \times T.$$

Choose the Lyapunov function $V(v_2,\tau) = \exp(\mu\tau) v_2^T P v_2$, where $P = P^T > 0$ and μ are specified later. As done for the first Lyapunov function, $W(v_1,\tau)$, begin by studying $V(v_2,\tau)$ along jumps:

$$V^+ - V \;\; = \;\; v_2^T J_2^T P J_2 v_2 - \exp(\mu T) v_2^T P v_2,$$

where $\mu > 0$ is chosen large enough such that:

$$V^+ - V \;\; \leq \;\; -d_3||v_2||^2. \qquad (20)$$

Now, look at the behavior of V along flows:

$$\begin{aligned}\dot{V} \;\; = \;\; & \exp(\mu\tau) v_2^T (\mu P + \ell(A_2^T P + P A_2) + 2PM(\ell)) v_2 \\ & + 2\exp(\mu\tau) v_2^T \ell P B_2(\ell) \ell^{-1} u_2.\end{aligned}$$

Then:

$$\begin{aligned}\dot{V} \;\; \leq \;\; & \exp(\mu\tau) v_2^T (\mu P + \ell(A_2^T P + P A_2) + 2PM(\ell)) v_2 \\ & + \exp(\mu\tau)(\ell v_2^T v_2 + \frac{4}{\ell} u_2^T B_2(\ell)^T P^T P B_2(\ell) u_2), \\ \dot{V} \;\; \leq \;\; & \exp(\mu\tau) v_2^T (\mu P + \ell(A_2^T P + P A_2 - I) + \\ & 2PM(\ell)) v_2 + \exp(\mu\tau) \frac{4}{\ell} u_2^T B_2(\ell)^T P^T P B_2(\ell) u_2.\end{aligned}$$

Here, specify that P satisfies $A_2^T P + P A_2 = -2I$. Then:

$$\dot{V} \;\; \leq \;\; (c_4 - \ell c_3)||v_2||^2 + \frac{c_5}{\ell}||u_2||^2, \qquad (21)$$

for some $c_3, c_4, c_5 > 0$.

C. Interconnection

Substitute $u_1 = v_2$ and $u_2 = v_1$, to obtain the overall system, (13)-(14), presented in Proposition 1. To construct a Lyapunov function, $\Psi(v_1,v_2,\tau)$, for (13)-(14) use a weighted sum of $W(v_1,\tau)$ and $V(v_2,\tau)$:

$$\Psi(v_1,v_2,\tau) \;\; := \;\; W(v_1,\tau) + kV(v_2,\tau).$$

Explore the behavior of $\Psi(v_1,v_2,\tau)$ along jumps:

$$\Psi^+ - \Psi \;\; = \;\; W^+ - W + k(V^+ - V),$$

or:

$$\Psi^+ - \Psi \;\; \leq \;\; -d_1||v_1||^2 + (d_2 - kd_3)||v_2||^2.$$

Therefore, the second inequality in (17) is satisfied for $k > \frac{d_2}{d_3}$. Along flows:

$$\dot{\Psi} \;\; = \;\; \dot{W} + k\dot{V},$$

so:

$$\dot{\Psi} \;\; \leq \;\; \left(\frac{c_5}{\ell} - c_1\right)||v_1||^2 + (c_2 + c_4 - \ell c_3)||v_2||^2.$$

Therefore, the last inequality in (17) is satisfied for $\ell > \max\left(\frac{c_5}{c_1}, \frac{c_2+c_4}{c_3}\right)$. QED.