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Sparsity Based Feedback Design: A New Paradigm in Opportunistic Sensing

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***Abstract*— We introduce the concept of using compressive sensing techniques to provide feedback in order to control dynamical systems. Compressive sensing algorithms use l_1-regularization for reconstructing data from a few measurement samples. These algorithms provide highly efficient reconstruction for sparse data. For data that is not sparse enough, the reconstruction technique produces a bounded error in the estimate. In a dynamical system, such erroneous state-estimation can lead to undesirable effects in the output of the plant.**

In this work, we present some techniques to overcome the aforementioned restriction. Our efforts fall into two main categories. First, we present some techniques to design feedback systems that sparsify the state in order to perfectly reconstruct it using compressive sensing algorithms. We study the effect of such sparsification schemes on the stability and regulation of the plant. Second, we study the characteristics of dynamical systems that produce sparse states so that compressive sensing techniques can be used for feedback in such scenarios without any additional modification in the feedback loop.

I. INTRODUCTION

Compressive Sensing (CS) is an emerging field based on the fact that a small group of non-adaptive linear projections of a compressible signal contains enough information for reconstruction and processing. In [11], researchers have reported the development of a single pixel camera that uses far fewer pixels than any traditional camera. It works on the principles of compressive sensing. One of the areas in which compressive sensing has shown considerable promise is in the reconstruction of sparse data [7] that is encountered in many important applications, for example, imaging systems [21]. Although, a plethora of work exists that deals with overcoming the limitations of CS techniques in the area of data acquisition and post-processing [2], the limitations of using such sensing techniques to provide feedback in dynamical systems has not yet been addressed. In this work, we address the problem of using compressive sensing techniques for providing feedback in dynamical systems.

The interplay between sparsity and signal recovery have been germinating for many decades in the past. One of the earliest mathematicians to understand this is Constantin Carathéodory [9], [8]. In [3], Arne Beurling proposed a nonlinear extrapolation of the Fourier transform of a signal using l_1 minimization techniques that can be used to construct the entire signal by observing a piece of the Fourier transform. In his 1965 PhD. dissertation [17], Ben Logan of ATT Labs demonstrated that if a bandlimited signal of bandlimit Ω is corrupted on an interval of length less than $2\pi/\Omega$, then, no matter how it was corrupted, we can recover the original bandlimited signal perfectly, simply by finding the L_1-closest bandlimited signal to the corrupted signal. Similar observations were made by researchers in geophysics [25], astronomy and seismology [24] while handling massive amounts of data provided sufficient physical evidence about the relation between sparsity and efficient reconstruction. [10] presents a really nice and detailed anecdote about the aforementioned events and many more instances in scientific and mathematical research that led researchers to explore the deep relationship between sparsity and reconstruction for the last four decades. In the recent past, there has been exciting breakthrough in the study of high dimensional sparse signals. This was initiated with potential applications in the area of computer vision [26], most significant of which is medical imaging [18] and also applications of acoustic and speech signal processing [6] in numerous areas. It has been shown that under broad conditions, a sufficiently sparse linear representation can be correctly and efficiently computed by greedy methods and convex optimization, even though this problem is extremely difficult- NP-hard in the general case. Moreover, studies have shown that such high-dimensional sparse signals can be accurately recovered from drastically smaller number of linear measurements, hence the phrase compressive sensing. These results have already caught the attention of researchers in various fields of mathematics and statistics, signal processing, information theory and theoretical computer science. At present, sparsity promoting and compressive sensing techniques have started to create tremendous impact on a much broader range of engineering fields, including but not limited to pattern recognition [26], machine learning [23], communications [1], sensor networks [27] and imaging sensors [20].

As with most of the reconstruction techniques, compressive sensing techniques too have some shortcomings that need to be addressed before they can be used to design sensors that provide feedback for controlling autonomous dynamical systems. These computationally efficient reconstruction techniques rely on the inherent sparsity present in the input for perfect reconstruction. If the input is not sufficiently sparse these recovery techniques lead to an error in the estimation. Therefore, in order for these sensing systems to have a wide range of applications we need to design techniques so that they can be used even in the face of non-sparsity. The main objective of this work is to study

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the limitations posed by the l_1-reconstruction algorithm, an important component of compressive sensing techniques, in dynamical systems and to some extent, overcome them, without any modification in the fundamental reconstruction algorithms. In some sense, we feel our techniques are *opportunistic* since they create sparsity in the states in order to exploit the benefits provided by the compressive sensing algorithms.

The rest of the paper is organized as follows. In Section II, we formulate our problem and introduce the idea of an ideal compressive sensing device. In Section III, we present some techniques for complete state reconstruction using a compressive sensing device. In Section IV, we study the issues related to stability and regulation of a dynamical system in the face of partial state reconstruction using a compressive sensing device. In Section V, we present the idea of modeling dynamical systems in order to generate sparsity in the states. Finally, we conclude with Section VI after providing future directions in our research.

II. Problem Formulation

In this section, we present the problem formulation. First, we formalize the idea of a *compressed sensing device* and then present the model of the dynamical system that is controlled using feedback from the device. Although, physical devices realizing compressive sensing techniques are currently limited to vision based applications, we introduce the notion of an ideal device that can reconstruct sparse signals of different physical modalities. Our interest lies in understanding the effects of the errors introduced in the measurement process associated with sparse reconstruction techniques in dynamical systems.

Before we formalize the idea of a compressed sensing device, we introduce some definitions in order to quantify sparsity in finite-dimensional vector spaces. Given a vector $x \in \mathbb{R}^n$, the *sparsity* or support of a vector is a function $\mathcal{S} : \mathbb{R}^n \to \mathbb{R}$ defined as $\mathcal{S}(x) = \#\{i | x(i) \neq 0\}$, where $x(i)$ denotes the i-th component of $x(i)$. A vector x is *S-sparse*, if its support contains S non-zero entries. An important point to be noted is that the function $\mathcal{S}$ is not invariant to coordinate transformation and hence it depends on the choice of the basis.

A *Compressive Sensing Device* (CSD) receives as input, a vector $x \in \mathbb{R}^n$ and provides as output a vector $x^{\#} \in \mathbb{R}^n$ with the following property:

$$\|x^{\#} - x\|_{l_2} \leq C \cdot \frac{\|x - x_S\|_{l_1}}{\sqrt{S}} \quad (1)$$

Here x_S represents the S most significant coordinates of x. The constant C is a function of the S and some additional parameters that are functions of the sensing matrix Φ.

The input to a CSD is any vector $x \in \mathbb{R}^n$ in which each entry is a measurement of an observation. This is an important assumption since the output of a dynamical system can correspond to different physical quantities and the aforementioned assumption allows us to measure all the output variables using the CSD. Another important assumption is that the reconstruction error at the output of the device is zero if the input is S-sparse. As far as we know, the sensing matrices that are used in CS techniques can guarantee the error bounds shown in Equation (5) with a very high probability. There are sensing matrices for which this probability is of the order of $1 - O(n^{-M})$, where M is the number of samples, which is a parameter of the CS algorithm. In this analysis, we assume M to be large enough to assume a probability very near to 1. Both assumptions render the CSD more powerful than the currently existing devices that work on CS techniques. In real scenarios, any device that works on l_1 reconstruction techniques will exhibit an inferior performance compared to the CSD and therefore the limitations of using a CSD in order to provide feedback will also be present in all real devices that use CS.

In this work, we consider linear time-invariant discrete-time dynamical systems given by the following equation:

$$x_{k+1} = Ax_k + Bu_k$$

where, $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $B = I_n$ and $u_k \in \mathbb{R}^n$. We assume all the states of the system are accessible to the CSD and therefore, each element of the input vector to the CSD is a state of the system. Since the CSD performs a perfect reconstruction of S-sparse inputs one can sparsify an input vector in order to get accurate values for some of the states. We assume that some of the elements of the input vector can be fixed to be zero in order to achieve adequate sparsity. Mathematically, we can express this by defining a *sparsifying function* $g : \mathbb{R}^n \to \mathbb{R}^n$ that sparsifies an arbitrary vector in its domain. $g(\cdot)$ can be time-varying, time-invariant or state-dependent. In this paper, we restrict our attention to time-invariant linear sparsifing functions. The simplest example of $g(\cdot)$ is a function that retains certain entries of the input vector and fixes the rest of the entries to zero. Since we restrict $g(\cdot)$ to be a linear transformation it can be represented by a matrix. If $\mathcal{I} = \{i_1, \ldots, i_S\}$ denotes the rows of the input vector that are to be retained at the output then $g(\cdot)$ is an $n \times n$ matrix with diagonal entry (i, i) as 1, where $i \in \mathcal{I}$, and zero as rest of the entries. We denote such a matrix by $\mathcal{T}_{\mathcal{I}}$. For example, if $g : \mathbb{R}^4 \to \mathbb{R}^4$ and $\mathcal{I} = \{1, 2, 4\}$ then $\mathcal{T}_{\mathcal{I}}$ is given by the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 1 shows the closed-loop system with the CSD providing a feedback to the controller. It should be noted that sparsifying the input is one way by which a set of S states can be reconstructed perfectly using a CSD. There might be many other techniques for signal recovery and error correction in order to use a CSD for feedback in a dynamical system, which is a topic of our ongoing research.

In the next section, we address the problem of full state reconstruction using compressive sensing devices.

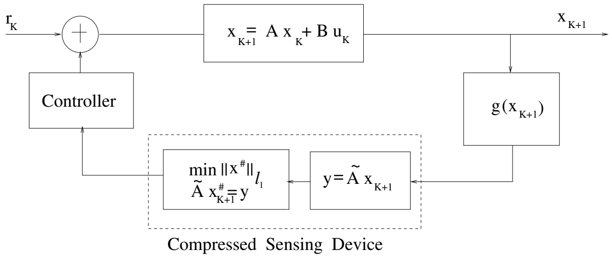

Fig. 1. Closed-loop System with Feedback from the Compressive Sensing Device

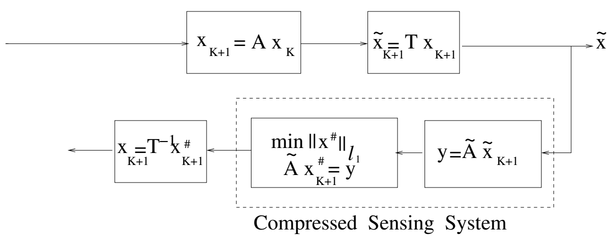

Fig. 2. Invertible Sparsifying Function

III. Full state reconstruction

In this section, we present some techniques for perfect state reconstruction using CSD. If perfect state reconstruction can be achieved then classical techniques in control theory based on full state feedback [19] can be used to control the dynamical system.

Lemma 1: For a discrete-time linear system having n states, a perfect state reconstruction can be achieved by using $\lceil \frac{n}{S} \rceil$ CSDs.

Proof: This involves using S inputs of each CSD for state measurements and fixing the remaining $(n-S)$ inputs at zero using appropriate sparsifying functions for each CSD. In this manner, we can reconstruct the entire state perfectly by concatenating the outputs from all the CSDs. ■

A severe shortcoming of the above technique is that the number of CSDs required is $O(n)$, where n is the dimension of the state space. This naturally leads to the question: What are the scenarios under which we require only one CSD for perfect reconstruction of the full state?

In the next sections, we present techniques to use a single CSD for full state reconstruction using appropriate change of basis.

A. Invertible Sparsifying Function

A technique for full state reconstruction is to use an invertible sparsifying function. If $g(\cdot)$ is invertible, then we can recover the original state after reconstruction from the CSD. Since this work deals with linear sparsifying functions, we are interested in linear transformations that are invertible. Figure 2 shows a scenario in which an invertible sparsifying function $g(\cdot)$ is applied to the state x_{k+1} to obtain an S-sparse vector, $\tilde{x}_{k+1}$. Since the CSD can reconstruct S-sparse inputs perfectly we recover x_{k+1} by applying $g^{-1}(\cdot)$ to the output of the CSD.

In this section, we analyze such sparsifying functions that are linear and time-invariant. Therefore, the invertible sparsifying function $g(\cdot)$ can be represented by a matrix T. The time-invariance of the transformation is dictated by the fact that we do not have any knowledge about x_{k+1} before its recovery from the CSD and therefore, it would be quite difficult to design a $g(\cdot)$ that depends on x_{k+1}.

In the following lemma, $\rho(A)$ denotes the rank of matrix A.

Lemma 2: $\rho(A) \leq n - S \Rightarrow \exists$ invertible matrix T such that TAx is S-sparse $\forall x$.

Proof: $\Rightarrow$ If $\rho(A) \leq n - S$ we can find a collection of orthogonal vectors $V = \{v_1, \cdots, v_S\} \in \mathbb{R}^n$ such that $v_i \perp \text{Range}(A) \quad \forall \quad i \in \{1, \cdots, S\}$. We can construct S linearly independent rows of T from elements of V. The remaining $n - S$ rows of T can be chosen as the basis of Range(A). Therefore, TAx is S-sparse for any x. Moreover, since the rows of T are mutually orthogonal, T is a full rank matrix and therefore invertible. ■

Before concluding, we would like to make a few comments on the design of a full-state observer for our system. As stated before, we assume $g(\cdot)$ to be linear. This leads to the following equivalent representation of the input/output description of the dynamical system:

$$\begin{aligned} x_{k+1} &= Ax_k + u_k \\ y_{k+1} &= Cx_{k+1} \end{aligned} \tag{2}$$

where the matrix C depends on $g(\cdot)$. If a sparsifying function $g(\cdot)$ can be designed such that (A, C) is observable $\Rightarrow$ (A, I, C) is controllable and observable. Therefore, we can design the following full order observer for the system:

$$\hat{x}_{k+1} = A\hat{x}_k + u_k + F[y_k - C\hat{x}_k]$$

Moreover, a suitable gain matrix F can be chosen for arbitrary pole assignment *iff* the pair (A, C) is observable [16]. Therefore, we can design a *deadbeat observer* by placing all the poles of the observer at zero and thus reducing the state error to zero in at most n steps.

In the next section, we present techniques to design linear state feedback using partial observations from the CSD in order to attain a desired objective.

IV. Partial State Reconstruction

In this section, we address the problem of designing feedback strategies for controlling a linear discrete-time plant by partial state reconstruction using a CSD. Since a CSD can reconstruct perfectly an input having sparsity $S \leq n$ we use a sparsifying function of the form $\mathcal{T}_\mathcal{I}$ in order to restrict the support of the input to the CSD to at most S non-zero entries. We assume that the index set $\mathcal{I}$ is time invariant. Therefore, the feedback law is of the following form:

$$u_k = -K\mathcal{T}_\mathcal{I} x_k$$

where, K is a $n \times n$ gain matrix.

Now we address the problem of placing the poles arbitrarily using sparsified input feedback design. The problem of placing eigenvalues of the matrix $(A - K\mathcal{T}_\mathcal{I})$ is equivalent to the problem of constructing a matrix K for designing a static output feedback for arbitrary pole placement for the system description given in (2). In [5], it has been shown that the problem of finding a static output feedback stabilizer from a given bounded set (a hypercube) is NP-complete. It is also known that generic pole placement using static output feedback is not feasible [12], [22], [15], [4]. In [13], the author shows that given a linear time-invariant system and a set of desired poles, the problem of determining if there exists a static output feedback controller such that the closed-loop system contains poles at these desired locations is NP-hard. In the next section, we show that for $A \in \mathbb{R}^{2\times 2}$, the poles can be arbitrarily placed with arbitrary choice of $\mathcal{T}_\mathcal{I}$.

Now we consider the problem of regulation using feedback from the CSD. For the system shown in Equation (2), consider a quadratic cost given by the following expression:

$$\mathcal{J} = \|x_L\|_{R_1} + \sum_{k=0}^{L-1}(\|x_k\|_{R_1} + \|u_k\|_{R_2}) \tag{3}$$

where $\|v\|_{R_1} = v^T R_1 v$, R_1 and R_2 are positive-definite symmetric matrices. We want to design a linear state feedback law of the form $u_k = -K\mathcal{T}_\mathcal{I} x_k$ that minimizes the above cost function assuming x_0 is known.

For a time-invariant feedback K, the closed-loop system can be represented by the following equation:

$$x_{k+1} = (A - K\mathcal{T}_\mathcal{I})x_k \tag{4}$$

whose solution is

$$x_{k+1} = (A - K\mathcal{T}_\mathcal{I})^k x_0 \tag{5}$$

where x_0 is the initial state. First, let us consider the case of a static gain matrix K. Substituting Equation (5) into (3) gives us the following expression for the cost function:

$$\mathcal{J} = \|x_0\|_{R_1} + x_0^T \underbrace{\sum_{k=0}^{L}\left[\|(A - K\mathcal{T}_\mathcal{I})^{k+1}\|_{R_1} + \|K\mathcal{T}_\mathcal{I}(A - K\mathcal{T}_\mathcal{I})^k\|_{R_2}\right]}_{G} x_0$$

For k_{ij} to minimize J, the first order necessary condition is given by the following relation:

$$\frac{\partial \mathcal{J}}{\partial k_{ij}} = 0 \quad \forall i \in [1,\ldots,n] \quad j \in \mathcal{I} \tag{6}$$

Moreover, the second order sufficient condition is given by the positive definiteness of the $Sn \times Sn$ Hessian matrix $\mathcal{H}$:

$$z^T \mathcal{H} z > 0, \quad \forall z \in \mathbb{R}^{n\mathcal{I}} \tag{7}$$

$$\mathcal{H}(ij, st) = \left[\frac{\partial \mathcal{J}(x)}{\partial k_{ij}\partial k_{st}}\right] \quad \forall i, s \in [1,\ldots,n], \quad j, t \in \mathcal{I}$$

where, $\mathcal{H}(ij, st)$ is the element of the Hessian matrix belonging to the row corresponding to k_{ij} and column corresponding to k_{st}. If the condition in Equation (7) is satisfied irrespective of the initial state x_0 then $G \equiv 0$. This leads to n^2 equations in Sn unknowns and therefore, might not have any solutions. On the other hand, if K is assumed to be a function of x_0 this leads to a set of Sn equations in Sn unknowns k_{ij} satisfying $i \in \{1, \cdots, n\}, j \in \mathcal{I}$ and in general, has a unique solution.

$$\begin{aligned} &x_k^T[\|A^T\|_{R_1} + \|K\mathcal{T}_\mathcal{I}\|_{(R_1+R_2)}]\frac{\partial x_k}{\partial k_{ij}} - \\ &x_k^T[(A^T R_1 K\mathcal{T}_\mathcal{I} + \mathcal{T}_\mathcal{I}^T K^T R_1 A)]\frac{\partial x_k}{\partial k_{ij}} + \\ &x_k^T[\|\mathcal{T}_\mathcal{I}^T K^T (R_2 - R_1)\frac{\partial K}{\partial k_{ij}}\mathcal{T}_\mathcal{I} + A^T R_1 \frac{\partial K}{\partial k_{ij}}\mathcal{T}_\mathcal{I}]x_k = 0 \end{aligned} \tag{8}$$

where

$$x_{k+1} = (A - K\mathcal{T}_\mathcal{I})^{(k+1)}x_0$$

$$\frac{\partial x_k}{\partial k_{ij}} = \left\{\sum_{t=0}^{k-1}(A - K\mathcal{T}_\mathcal{I})^t \frac{\partial K}{\partial k_{ij}}\mathcal{T}_\mathcal{I}(A - K\mathcal{T}_\mathcal{I})^{k-1-t}\right\}$$

Next, let us consider a time varying feedback gain matrix K_k. In this case the closed-loop is given by the equation:

$$x_{k+1} = (A - K_k\mathcal{T}_\mathcal{I})x_k$$

The state at time instant k is given by the following expression:

$$x_{k+1} = [\prod_{j=0}^{k}(A - K_{k-j}\mathcal{T}_\mathcal{I})]x_0$$

In this case, the cost $\mathcal{J}$ is given by the following expression:

$$\begin{aligned} \mathcal{J} = x_0^T &\left[\sum_{k=0}^{L}\|\prod_{j=0}^{k+1}(A - K_{k+1-j}\mathcal{T}_\mathcal{I})\|_{R_1}\right] x_0 + \\ x_0^T &\left[\sum_{k=0}^{L}\|K_k\mathcal{T}_\mathcal{I}\prod_{j=0}^{k}(A - K_{k-j}\mathcal{T}_\mathcal{I})\|_{R_2}\right] x_0 + \|x_0\|_{R_1} \end{aligned}$$

Let the entry in the ith row and jth column of K_l be denoted as k_{ij}^l. The first order necessary conditions lead to the following set of SLn equations for all k_{ij}^l satisfying $j \in \mathcal{I}, i \in \{1, \cdots, n\}, l \in \{1, \cdots, L\}$:

$$\begin{aligned} \sum_{k=l}^{L}\Bigg\{ &x_{k+1}(R_1 + \mathcal{T}_\mathcal{I}^T K_k^T R_2 K_k \mathcal{T}_\mathcal{I})\left[\prod_{j=l+1}^{k+1}(A - K_j\mathcal{T}_\mathcal{I})\right] \\ &\frac{\partial K_l}{\partial k_{ij}^l}\mathcal{T}_\mathcal{I} x_{l-1}\Bigg\} + x_l^T \mathcal{T}_\mathcal{I}^T K_l^T R_2 \frac{\partial K_l}{\partial k_{ij}^l}\mathcal{T}_\mathcal{I} x_l = 0 \end{aligned} \tag{9}$$

where

$$x_{k+1} = [\prod_{j=0}^{k}(A - K_{k-j}\mathcal{T}_\mathcal{I})]x_0$$

From Equations (8) and (9), we can deduce that the optimal feedback is a function of the initial state x_0. As pointed out in [14], this undesirable dependence on the initial state is an implication of using static output feedback for optimal regulation of the plant. The paper [14] presents some techniques to overcome this dependence on the initial state in case of static as well dynamical output feedback. For example, in the case of static feedback one might assume a probability distribution over all possible initial states in order to formulate a cost function that minimizes the expected cost over all initial states. One of our future directions in research is to address the aforementioned problem.

In the next section, we consider another perspective of controlling a dynamical systems using a CSD.

V. Sparsity in Dynamics

Compressive sampling algorithms can perform perfect reconstruction of sufficiently sparse data. In the previous sections, we have demonstrated scenarios in which compressive sensing can be used to provide feedback in a dynamical system. The main idea was to sparsify the states of a system by applying appropriate transformations. In this section, we introduce a different perspective of generating sparsity in the output of a dynamical system which involves investigating the properties of the system matrix, A, that retains or generates sparsity in the states. Our motivation to do so arises from the following observation in sampled data control systems.

Consider the following continuous-time system:

$$\dot{x} = A_c x_t \tag{10}$$

where, $x \in \mathbb{R}^n$ and $A_c \in \mathbb{R}^{n\times n}$. A_c can represent the system matrix of an autonomous system or a matrix that represents closed-loop system in the presence of full-state feedback. If we compute only the responses at $t = kT$, then (10) becomes:

$$x_{k+1} = Ax_k \tag{11}$$

where, $A = e^{A_c T}$. Therefore, the properties of A depend on the sampling time T as well as the original system matrix, A_c. By proper choices of T and K, in the case of a full-state feedback, A can have desired properties that preserve or generate appropriate sparsity in x_k so that a single CSD can perform perfect state reconstruction.

Now we present a simple example to illustrate the concept. Consider the autonomous system $x_{k+1} = Ax_k$ where, $x_k \in \mathbb{R}^2$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If x_k is sparse we assume that at least one of its components is zero i.e.,

$$x_k \in \{\text{span}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{span}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$$

In this case, it can be trivially seen that A preserves or generates sparsity *iff* it is in one of the following forms:

$$\underbrace{\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 \\ a_{21} & a_{22} \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix}}_{A_3}, \underbrace{\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}}_{A_4}$$

We can see that each of the above matrices has a unique way of preserving sparsity of the input signal at the output. If the input is not sparse, A_1 and A_4 might not generate sparse output but A_2 and A_3 produce sparse output irrespective of the nature of the inputs. If the input is sparse the output of A_1 has a zero in the same row as that of the input. A_2 and A_3 generate outputs that have zeros in the first and the second rows, respectively, irrespective of the nature of the input. A_4 generates an output that has zero in the row that has a non-zero entry in the input. From the above example we see that A can have a special structure to dictate sparsity in the output irrespective of the nature of the input. Now, we consider a special class of linear transformations that preserve zero in fixed entries of their input.

Definition: A linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is a *Strict Sparsity Preserving Matrix* (SSPM($\mathcal{I}$)) if there is a fixed index set $\mathcal{I} = \{i_1, \dots, i_S\}$ such that the following holds:

$$(x[j] = 0) \wedge (Ax[j] = 0) \implies A^n x[j] = 0, \forall j \in \bar{\mathcal{I}}, \forall n \geq 1$$

Now we present a sufficient condition for a matrix to be SSPM. In order to do so, we need to introduce some more notation. Let $\mathcal{I}$ and $\mathcal{J}$ be an ordered set of indices. Let $\bar{\mathcal{I}}$ denote the ordered set $\{1 \cdots n\}/\mathcal{I}$. Let $A_{\mathcal{I}\times\mathcal{J}}$ denotes the $|\mathcal{I}| \times |\mathcal{J}|$ submatrix of A such that $A_{\mathcal{I}\times\mathcal{J}}[r.k] = A[i_r, i_s]$ where $i_r \in \mathcal{I}, i_s \in \mathcal{J}$. Similarly, $A_{\bar{\mathcal{I}}\times\mathcal{I}}[r,k] = A[i_r, i_s]$ where $i_r \in \bar{\mathcal{I}}, i_s \in \mathcal{I}$. Finally, for a vector x, $x_\mathcal{I}$ denotes a column vector consisting of entries in x located in the index set $\mathcal{I}$.

Lemma 3: A linear transformation $A : \mathbb{R}^n \to \mathbb{R}^n$ is a SSPM($\mathcal{I}$) only if the following condition holds:

$$\text{Range}(A_{\mathcal{I}\times\mathcal{I}}) \subset \mathcal{N}(A^T_{\bar{\mathcal{I}}\times\mathcal{I}})$$

where, $\mathcal{N}(A)$ denotes the nullspace of A.

Proof: Let $y = Ax$ and $y[j] = 0, \quad \forall j \in \bar{\mathcal{I}}$

$$\begin{aligned} x[j] = 0 \quad \forall j \in \bar{\mathcal{I}} \\ \Rightarrow (Ax)_\mathcal{I} &= A_{\mathcal{I}\times[1,\cdots,n]}x \\ &= A_{\mathcal{I}\times\mathcal{I}}x_\mathcal{I} \in \mathcal{N}(A^T_{\bar{\mathcal{I}}\times\mathcal{I}}) \end{aligned} \tag{12}$$

Let $j \in \bar{\mathcal{I}}$.

$$A^2x[j] = Ay[j] = \sum_{l=1}^{n} a_{jl}y_l = \underbrace{\sum_{l\in\mathcal{I}} a_{jl}y_l}_{0} + \underbrace{\sum_{l\in\bar{\mathcal{I}}} a_{jl}y_l}_{0}$$

The first expression is zero due to (12) and the right expression is zero since $y[j] = 0, \quad \forall j \in \bar{\mathcal{I}}$. Moreover, $(Ay)_\mathcal{I} \in \text{Range}(A_{\mathcal{I}\times\mathcal{I}})$. Hence, the rest follows by induction. ■

The above lemma not only provides a sufficient condition for the existence of such matrices but also provides an algorithm to construct them.

Now let us address the problem of choosing a sampling time T for the continuous time system in (10) so that the equivalent sampled-time system matrix A satisfies the sufficient condition presented in Lemma 3. For simplicity, let us consider the case when A_c is diagonalizable i.e, $A_c = Q\Lambda Q^{-1}$ where, Q is the orthogonal matrix comprised of the eigenvectors corresponding to the eigenvalues of A_c, and Λ is a diagonal matrix comprised of all the eigenvalues of A_c namely $\{\lambda_1, \dots, \lambda_n\}$. The expression for A is given below:

$$A = e^{A_c T} = Qe^{\Lambda T}Q^{-1} \tag{13}$$

where $e^{\Lambda T} = \text{diag}\{e^{\lambda_1 T}, \ldots, e^{\lambda_n T}\}$ where diag$\{\cdots\}$ represents a diagonal matrix with arguments as the diagonal elements.

Theorem 1: If T is the sampling time such that A is SSPM($\mathcal{I}$), then T satisfies the following set of equations:

$$\sum_{i_1 \in \mathcal{I}, i_4 \in \mathcal{I}} [\sum_k q_{i_1 k} e^{\lambda_k T} q_{i_2 k}][\sum_l q_{i_3 l} e^{\lambda_l T} q_{i_4 l}] = 0 \quad \begin{matrix} \forall i_2 \in \mathcal{I} \\ \forall i_3 \in \bar{\mathcal{I}} \end{matrix}$$

Proof: Let $\{c_1, \ldots, c_{|\mathcal{I}|}\}$ denote the columns of the matrix $A_{\mathcal{I}\times\mathcal{I}}$. Let $\{r_1, \ldots, r_{|\bar{\mathcal{I}}|}\}$ denote the rows of $A_{\bar{\mathcal{I}}\times\mathcal{I}}$. $c_i \in \text{Range}(A_{\mathcal{I}\times\mathcal{I}}) \quad \forall i \in \{1, \ldots, |\mathcal{I}|\}$. Therefore, if A satisfies the condition of Lemma 3 then $c_i \perp r_j \implies c_i{\cdot}r_j = 0 \quad \forall i, j$. From (13), we get the following expression for a_{ij}:

$$a_{ij} = \sum_{k=1}^{n} q_{ik} e^{\lambda_k T} q_{jk}$$

■

Now consider the case when $A = A_f - B_f K$ in (10) which corresponds to a closed-loop system with full-state feedback of the form $u = -Kx(t)$. If the pair (A_f, B_f) is controllable, then the eigenvalues of A can be placed arbitrarily by choosing appropriate feedback gain matrix K. Therefore, by appropriate choices of $\lambda_1, \ldots, \lambda_n$ and T we can satisfy the sufficient condition in Theorem 1.

VI. Conclusion

In this work, we have presented some techniques to overcome the limitations caused by compressive sensing techniques in order to provide feedback in dynamical systems. First, we presented some techniques to design feedback systems that sparsify the state in order to perfectly reconstruct it using compressive sensing algorithms. Then we studied the effect of such sparsification schemes on the stability of the plant and designed optimal control laws to minimize a finite-horizon quadratic criterion. Next, we studied the characteristics of dynamical systems that produce sparse states so that compressive sensing techniques can be used for feedback in such scenarios without any additional modification in the feedback loop. In this technique, we presented a special kind of matrices called SSPM($\mathcal{I}$) and found sufficient conditions for their existence. Finally, we presented a technique to choose the sampling time and feedback gain matrix in sampled-time control systems in order to generate sparsity in the resulting states.

In the future, we plan to continue our research in dynamical systems in order to understand the relation between sparsity and system theoretic properties like stability, controllability and observability. Another interesting direction of research is to study the effect of different classes of sparsifying functions and also possible switchings among them. This would give rise to interesting questions about the stability of the resulting hybrid system. We also plan to provide techniques for efficient computation of the control laws for high-dimensional systems that use compressive sensing techniques for feedback.

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