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Optimal Sensor Placement for Parametric Identification of Electrical Networks Using Mixed Phasor Measurements

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***Abstract*— In this paper we present an algorithm for placing sensors optimally along the edges of a large network of electrical oscillators to identify a parametric model for the network using a linear combination of three fundamental electrical signals - namely, the magnitude, the phase angle and the frequency of the voltage phasor along each edge, corrupted with Gaussian noise. We pose the identification problem as estimation of four essential parameters for each edge, namely the real and imaginary components of the edge-weight (or, equivalently the resistance and reactance along the transmission line), and the inertias of the two machines connected by this edge. We then formulate the Cramer-Rao bounds for the estimates of these four unknown parameters, and show that the bounds are functions of the sensor locations and of the contribution of each variable in the combined output. We finally state the condition for finding the optimal sensor location and the optimal signal combination to achieve the tightest Cramer-Rao bound.**

***Index Terms*— Power networks, Cramer-Rao bound, swing equation, parameter estimation.**

I. Introduction

Research problems on sensor placement in electric power networks have emerged with renewed interest over the past few years owing to the tremendous advancement of sensing technologies such as Wide-area Measurement Systems (WAMS) [1]. GPS-synchronized, high sampling-rate ($2.5-3$ KHz internal sampling, $6-60$ samples/sec exporting rate) digital devices known as Phasor Measurement Units (PMU) are currently being deployed at different points in the US power transmission network to measure and monitor dynamic electrical signals over distributed geographic spans so that catastrophes such as blackouts and voltage collapses can be avoided. One main concern of current interest among WAMS-researchers is to determine the optimal distribution of PMUs in a large network to ensure sufficient dynamic visibility of the system yet minimizing the number of PMUs and the cost associated with them. The choices for such locations are largely dictated by their driving purposes such as state estimation [2], geometric observability [3], voltage stability [4], etc.

In this paper we address the problem of PMU placement from the standpoint of a completely new application - namely, model identification of electrical networks, especially when the PMU measurements are corrupted with stochastic noise. A small case study of this identification problem, from a completely deterministic point of view without considering any measurement noise effect, was presented in our recent paper [5], but the system under consideration was restricted to a simple two-machine system connected by a single radial line (or equivalently, a two-node graph with only one edge). In this paper we consolidate those results to a much more generalized system of n machines, connected by any arbitrary graph $\mathcal{G}$, and develop an algorithm to identify the electro-mechanical model parameters of this n-node power system using noise-corrupted dynamic measurements available from specific points on the edges of the network. Since the structure of the electro-mechanical dynamics evolving across such power networks is known from Newton's laws of angular motion, we first show that the identification problems can be posed as equivalent parameter estimation problem. For any given transmission line (i.e. an edge in the network graph) four fundamental parameters are of interest to us for solving the identification problem: namely, the resistance and reactance of the line, and the mechanical inertias of the two generators connected by the line. Since these parameters have to be identified using noisy measurements, their deterministic estimates are not available, and the problem has to be posed in terms of estimation error bounds. Such bounds, more commonly referred to as Cramer-Rao bounds (CRB) are widely used in the statistical signal processing literature [6]. We show that the CRB for estimating the four model parameters for any edge in the network is a function of the location of the PMU on that edge as well as a function of the contribution of the type of output variable considered for the estimation, whereby the problem reduces to finding the optimal values of this location and signal-combinations such that the estimation error bound is minimized.

The rest of the paper is organized as follows. In Section II we formulate the placement problem for a multi-nodal electrical network; in Section III we review some statistical preliminaries to facilitate the derivations related to the main results on CRBs, which are presented in Section IV; Section V applies these results to a 2-node network with simulations; Section VI discusses centralized estimation methods. Section VII concludes the paper.

II. Problem Formulation

Consider a network of electrical oscillators with n generators (nodes) connected to each other through m tie-lines (edges) with $m \leq n(n-1)/2$, forming a connected graph with cardinality (n, m), such that atmost one edge exists between any two nodes. This may also be thought of as a power system although we use the word 'power' with reservation as in a real power system generators are not necessarily connected directly but via intermediate buses due to which the network Laplacian becomes extremely complicated, especially for large networks. To avoid this difficulty and in the interest of the specific application discussed in this paper, we restrict our discussion to networks where each dynamic element i.e., a generator is directly connected to its neighbors. An example of such a network consisting of $n = 6$ generators and $m = 9$ tie-lines is shown in Figure 1. The arrows along each edge denote the direction of effective

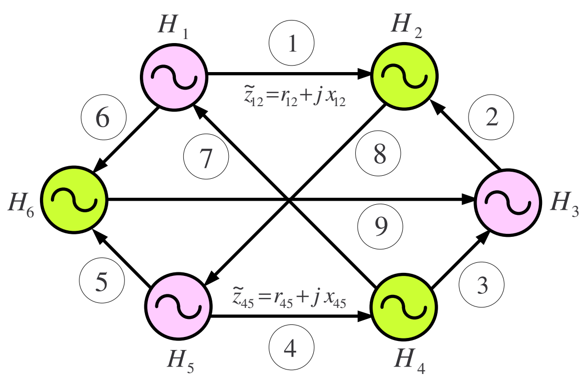


Fig. 1. Network of 6 generator nodes and 9 tie-line edges

power flow. Let the internal voltage phasor of the i^{th} machine be denoted as $\tilde{E}_i = E_i\angle\delta_i, \quad i = 1, 2, \ldots, n$ where, following synchronous machine theory [7], E_i is constant, δ_i is the angular position of the generator rotor, and $E_i\angle\delta_i$ denotes the polar representation $E_i\varepsilon^{j\delta_i}$ ($j = \sqrt{-1}$). The transmission line connecting the p^{th} and the q^{th} machines is assumed to have an impedance $\tilde{z}_{pq} = r_{pq} + jx_{pq}$ where 'r' denotes the resistive part and 'x' denotes the reactive part. Here $p \in \{1, 2, ..., n\}$ and $q \in \mathcal{N}_p$ where $\mathcal{N}_p$ is the set of nodes to which the p^{th} node is connected. It follows that the total number of tuples formed by pairing p and q is m. For the rest of the paper we will denote the edge connecting the p^{th} and the q^{th} node by e_{pq}. $\bar{z}_{pq}$ can also be regarded as a complex *weight* of an edge in the network, and implies that $\tilde{z}_{pq} = \tilde{z}_{qp}$. If two nodes do not share a connection then the impedance corresponding to that non-existing edge is infinite (i.e., open circuit), or equivalently,

$$\tilde{y}_{pq} = \frac{1}{\tilde{z}_{pq}} = \frac{1}{r_{pq} + jx_{pq}} = 0 \quad \forall q \notin \mathcal{N}_p \tag{1}$$

where $\tilde{y}_{pq}$ is the admittance of e_{pq}. The mechanical inertia of the i^{th} machine is denoted as H_i.

The dynamic electro-mechanical model of the i^{th} generator, neglecting damping, can be written as [7]

$$\dot{\delta}_i = \omega_i - \omega_s \tag{2}$$

$$2H_i\dot{\omega}_i = P_{mi} - \sum_{k\in\mathcal{N}_i}\left(\frac{E_i^2 r_{ik} - E_iE_kp_{ik}\cos(\delta_{ik} + \alpha_{ik})}{p_{ik}^2}\right) \tag{3}$$

where $\delta_{ik} = \delta_i - \delta_k$, $\omega_s = 120\pi$ is the synchronous speed for a 60 Hz system, ω_i is the rotor angular velocity, P_{mi} is the mechanical power input, $p_{ik} = \sqrt{r_{ik}^2 + x_{ik}^2}$ and $\alpha_{ik} = \tan^{-1}(x_{ik}/r_{ik})$. All quantities are in per unit except for the phase angles which are in radians. We assume that the network structure is known, i.e., the set $\mathcal{N}_i$ for all $i = 1, 2, \ldots, n$ in (2)-(3) is known, but all other parameters are unknown, namely H_i, E_i, r_{ik} and x_{ik}.

Without any loss of generality, for each edge we fix a hypothetical reference point at the internal node of the generator which is receiving power, i.e., the node where the arrowed end of the edge is incident on. We assume that a PMU is placed at each edge at an impedance $\bar{z}_{pq}$ away from this reference, where $\bar{z}_{pq} = \bar{r}_{pq} + j\bar{x}_{pq}$. Also, assuming that both resistance and reactance along each tie-line is uniformly distributed, we define the *normalized* impedance of e_{pq} as $a_{pq} = \bar{z}_{pq}/\tilde{z}_{pq}$. Due to the uniformity of distribution we can also write $a_{pq} = \bar{r}_{pq}/r_{pq} = \bar{x}_{pq}/x_{pq}$, implying $a_{pq} \in [0, 1]$. If the reference for e_{pq} is at the q^{th} node, then $a_{pq} = 0$ corresponds to this node, $a_{pq} = 1$ corresponds to the p^{th} node, while $0 < a_{pq} < 1$ corresponds to any point in between. The variable a_{pq} may equivalently be regarded as a dimensionless spatial variable measuring the distance of any point from the reference node of e_{pq}. Following a small-signal disturbance entering at any node in the system, time-series data of voltage, corrupted with Gaussian noise, are available from the PMU installed at this point. Our objective is to generate the best possible estimates of all the unknown parameters of the dynamic model (2)-(3) using these noisy voltage measurements. For any e_{pq} these parameters are

$$\mathcal{S}_{pq} = \{E_p,\, E_q,\, H_p,\, H_q,\, r_{pq},\, x_{pq}\} \tag{4}$$

for $p = 1, 2, \ldots, n$ and $q \in \mathcal{N}_p$. However, noting that a

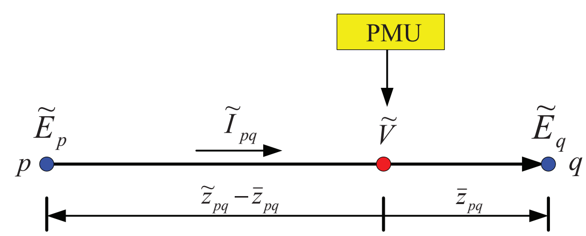


Fig. 2. Voltages and currents on edge e_{pq}

PMU can measure both voltage and current phasors across a line, the two constants E_p and E_q can be readily computed using Ohm's law once r_{pq} and x_{pq} are known:

$$E_p = \mathrm{Re}\,[\tilde{V} - \bar{z}\tilde{I}],\; E_q = \mathrm{Re}\,[\tilde{V} + (\tilde{z} - \bar{z})\tilde{I}] \tag{5}$$

where $\tilde{V}$ and $\tilde{I}_{pq}$ are the AC voltages and currents measured by the PMU located at a known impedance of $\bar{z}_{pq}$ away from the q^{th} node of e_{pq}. The circuit diagram for (5) is shown in Figure 2. Therefore, it suffices to estimate

$$\mathcal{S}_{pq} = \{H_p,\, H_q,\, r_{pq},\, x_{pq}\}. \tag{6}$$

Clearly, $\mathcal{S}_{pq}$ has two *node*-parameters and two *edge*-parameters, that can be estimated by a least squares or auto-regressive moving average with exogenous input (ARMAX) type algorithm [6] using voltage, phase and frquency measurements. However, since the measurements are noisy, unique estimates for $\mathcal{S}_{pq}$ are no longer available, and the problem has to be posed in terms of Cramer-Rao bounds (CRB). We next show that the CRB for (6) is a function of the spatial variable a_{pq} for each edge e_{pq}, and, therefore, there exists an optimal a_{pq} for each edge corresponding to which the estimation error is minimal.

Remark 1 : As the estimation is done for each edge e_{pq}, the inertia H_p of the p^{th} node ($p = 1, 2, \ldots, n$) will be estimated d_p number of times, where d_p is the degree of the p^{th} node. A reasonable estimate for H_p can then be generated by taking the average of these d_p estimates.

III. Statistical Preliminaries

Before addressing the estimation problem for (6), we intend to provide a brief background on CRB and their applications to linear dynamic models in this section. Our discussion is based on the definitions provided in [6].

A. Cramer-Rao Bound

The Cramer-Rao Bound is used to lower bound the second order moments of any stochastic parameter-estimator. A general form of the CRB can be derived by considering the following estimation problem. Given the data record y, let us consider the problem of constructing an estimator $\hat{g}(y)$ to estimate the vector function $\zeta = g(\Theta)$. Let the Fisher Information Matrix (FIM) for the parameter set Θ be $J(\Theta)$ and assume that it is invertible. Assume that the estimator $\hat{g}(y)$ is unbiased and let the estimator covariance matrix be

$$C_{\hat{g}} = E[(\hat{g}(y) - g(\Theta))(\hat{g}(y) - g(\Theta))^T]. \tag{7}$$

The CRB for the estimator $\hat{g}(y)$ is then defined as

$$C_{\hat{g}} \geq J^{-1}(\zeta = g(\Theta)). \tag{8}$$

B. Estimation Error Ellipse

Consider a Λ-dimensional unbiased estimator Θ that is normally distributed as $N[\Theta, \mathcal{C}]$:

$$f_c(\hat{\Theta}) = \frac{1}{(2\pi)^{\Lambda/2}(\det \mathcal{C})^{0.5}} e^{-\frac{\tilde{\Theta}^T \mathcal{C}^{-1} \tilde{\Theta}}{2}} \tag{9}$$

where $\tilde{\Theta} = \Theta - \hat{\Theta}$, and $\det(\mathcal{C})$ denotes the absolute value of the determinant of $\mathcal{C}$. The volume of the Euclidean Λ-space for which $R_c^2 = (\hat{\Theta} - \Theta)^T \mathcal{C}^{-1} (\hat{\Theta} - \Theta) < R^2$ is defined as the volume of the ellipsoid described by $R_c^2 < R^2$. For Λ even, the volume of the ellipsoid is given by

$$\mathcal{V}_{\mathcal{C}} = \mathcal{V}_\Lambda (\det \mathcal{C})^{1/2} R^\Lambda, \quad \mathcal{V}_\Lambda = \frac{\pi^{\Lambda/2}}{\Lambda/2} \tag{10}$$

which, by Hadamard's inequality, and for $R = 1$ implies that

$$\mathcal{V}_{\mathcal{C}}^2 \leq \mathcal{V}_\Lambda^2 \prod_{i=1}^{q} var(\hat{\Theta}_i). \tag{11}$$

Equation (11) indicates that it is always desirable that the volume $V_{\mathcal{C}}^2$ be small since this volume measures the estimator quality in some sense. Since (8) states that $\mathcal{C}_{\hat{g}} > J^{-1}$, it follows that

$$\mathcal{V}_{J^{-1}} = V_q (\det J^{-1})^{1/2} r^\Lambda \leq V_{\mathcal{C}}. \tag{12}$$

In other words, the FIM for an estimator generates the smallest achievable concentration ellipsoid [6]. Equation (12) is of crucial importance for this paper as in the following section we show that for our identification problem (6), the FIM for each e_{pq} is a function of a_{pq}; therefore, it is possible to find an optimal $a_{pq} \in [0, 1]$ for which $V_{J^{-1}}$ is minimum, or equivalently the lower bound for the parameter estimation error with respect to the PMU location is tightest.

C. Construction of FIM for Linear Models

When the measurement model is linear, i.e., $\mathbf{y} = \mathbf{x} + \mathbf{n}$, where $\mathbf{x}$ is a deterministic signal observed in additive Gaussian noise $\mathbf{n}$: $N[0, R]$, then the FIM is of the form:

$$J(\theta) = \left(\frac{\partial \mathbf{x}}{\partial \theta}\right) R^{-1} \left(\frac{\partial \mathbf{x}}{\partial \theta}\right)^T \tag{13}$$

Thus, if we partition the parameter vector as $\theta = col(\mathbf{a}, \mathbf{b})$, assume $R = \sigma^2 \mathbf{I}$, and define

$$\partial \mathbf{x}/\partial \mathbf{a} := \mathbf{H}, \ \partial \mathbf{x}/\partial \mathbf{b} := \mathbf{K} \tag{14}$$

then the FIM can be constructed as

$$J(\mathbf{a}, \mathbf{b}) = \frac{\mathbf{1}}{\sigma^\mathbf{2}} \begin{bmatrix} \mathbf{HH^T} & \mathbf{HK^T} \\ \mathbf{KH^T} & \mathbf{KK^T} \end{bmatrix}. \tag{15}$$

For estimating (6) we will consider $\mathbf{a} = \{r_{pq}, x_{pq}\}$, $\mathbf{b} = \{H_p, H_q\}$, for each e_{pq}, and construct $J(\mathbf{a}, \mathbf{b})$, which we next show is a function of a_{pq}.

IV. Optimal Placement & Signal Combination

Returning to the electrical network of Section II, to derive the FIM $J(\mathbf{a}, \mathbf{b})$ we linearize (2)-(3) about an initial equilibrium $(\delta_{i0}, 0)$ where $0 < \delta_{i0} < 90°$ for all $i = 1, 2, \ldots, n$, and denote the perturbed state variables as

$$\Delta\delta = \text{col}(\Delta\delta_1, \Delta\delta_2, \ldots, \Delta\delta_n) \tag{16}$$

$$\Delta\omega = \text{col}(\Delta\omega_1, \Delta\omega_2, \ldots, \Delta\omega_n). \tag{17}$$

We assume that a disturbance u enters the system (note: the network graph is connected, by assumption) through the j^{th} node, $j \in \{1, 2, \ldots, n\}$, and that u can be modeled as an impulse. The linearized network dynamics then take the form

$$\begin{bmatrix} \Delta\dot{\delta} \\ \hline \Delta\dot{\omega} \end{bmatrix} = \underbrace{\left[\begin{array}{c|c} 0 & I \\ \hline \mathcal{M}^{-1}\mathcal{L} & 0 \end{array}\right]}_{A} \begin{bmatrix} \Delta\delta \\ \hline \Delta\omega \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \hline \mathcal{E}_j \end{bmatrix}}_{B} u \tag{18}$$

where I is the n-dimensional identity matrix, $\mathcal{E}_j$ is the j^{th} unit vector with all elements zero except the j^{th} element which is 1, $\mathcal{M} = \text{diag}(M_1, M_2, \ldots, M_n)$, M_i is the inertia of the i^{th} generator, and $\mathcal{L}$ is the $n \times n$ Laplacian matrix with elements:

$$\mathcal{L}_{ii} = -\sum_{k \in \mathcal{N}_i} \frac{E_i E_k}{p_{ik}} \sin(\delta_{i0} - \delta_{k0} + \alpha_{ik}), \tag{19}$$

$$\mathcal{L}_{ik} = \frac{E_i E_k}{p_{ik}} \sin(\delta_{i0} - \delta_{k0} + \alpha_{ik}), \ \ k \in \mathcal{N}_i, \tag{20}$$

$$\mathcal{L}_{ik} = 0, \ \text{otherwise} \tag{21}$$

for $i = 1, 2, \ldots, n$. It follows that if $M_i = M_j, \forall (i, j)$, then $\mathcal{L} = \mathcal{L}^T$. To construct the FIM, we next consider three different outputs, namely the magnitude, the phase and the frequency of the voltage phasor measured along any edge e_{pq} at a normalized distance a_{pq} from the reference node, and derive their respective expressions in terms of the state variables $(\Delta\delta(t), \Delta\omega(t))$.

A. Voltage Magnitude

Using Ohm's law, the voltage phasor $\tilde{V}_{pq}$ at any point on e_{pq} at an impedance $\bar{z}_{pq}$ away from the reference node, as shown in Figure 2, can be written as

$$\tilde{V}_{pq} \triangleq V_{pq}\angle\theta_{pq} = a_{pq}\tilde{E}_p + (1-a_{pq})\tilde{E}_q \tag{22}$$

from which the voltage magnitude is given as

$$V_{pq}=\sqrt{E_q^2(1-a_{pq})^2+E_p^2a_{pq}^2+2E_pE_qa_{pq}(1-a_{pq})\cos(\delta_{pq})} \tag{23}$$

where, $\delta_{pq} \triangleq \delta_p - \delta_q$. A small change in the voltage, linearized over an existing equilibrium voltage $V_{pq0} \triangleq V_{pq}(\delta_{pq0})$, therefore, can easily be derived as $\Delta V_{pq} = \psi_{pq}l_{pq}(\Delta\delta_q - \Delta\delta_p)$ where, $\psi_{pq}(a_{pq}) = a_{pq}(1-a_{pq})/V_{pq0}$, and $l_{pq} = E_pE_q\sin(\delta_{pq0})$. The discrete-time transfer function for the LTI model (18) with output V_{pq} will then be of the form

$$G_{pq}^V(z) = C_{pq}^V(zI - A_d)^{-1}B_d \tag{24}$$

where $A_d = e^{AT}$, $B_d = \int_0^T e^{A\tau}Bd\tau$, and C_{pq}^V is a $(2n\times 1)$ matrix (i.e., a row vector) whose p^{th} element is $-\psi_{pq}l_{pq}$, q^{th} element is $\psi_{pq}l_{pq}$, and all other elements are zero. Due to this special structure of C_{pq}^V, equation (24) can also be written in the form

$$G_{pq}^V(z) = \psi_{pq}(a_{pq})\tilde{G}_{pq}(z) \tag{25}$$

where $\tilde{G}_{pq}$ is a $2n$-order transfer function with n second-order pole pairs, one due to each node in the network. Equivalently, the pole-residue form of (25) is

$$G_{pq}^V(z) = \psi_{pq}(a_{pq})\sum_{i=1}^{n}\left(\frac{A_{pq,i}}{z-m_i} + \frac{A_{pq,i}^*}{z-m_i^*}\right), \tag{26}$$

where $*$ denotes complex conjugate, from which the output voltage following an impulse input is

$$\Delta V_{pq}(k) = \psi_{pq}(a_{pq})\underbrace{\sum_{i=1}^{n}\left(A_{pq,i}m_i^{k-1} + A_{pq,i}^*m^{*k-1}\right)u(k-1)}_{\xi_{pq}(k)}, \tag{27}$$

for $k = 1,2,\ldots,\infty$, assuming unlimited time-series data are available. It should be noted that as the network is connected the residues and the poles, namely $A_{pq,i}$ and m_i, and their conjugates are functions of the unknown parameters for the entire network and not just for a specific e_{pq}.

B. Voltage Phase

It follows from (22) that the phase angle at any point on e_{pq} at a normalized impedance $a_{pq}\in[0,1]$ away from the reference node is given as

$$\theta_{pq} = \tan^{-1}\frac{a_{pq}E_p\sin(\delta_p) + (1-a_{pq})E_q\sin(\delta_q)}{a_{pq}E_p\cos(\delta_p) + (1-a_{pq})E_q\cos(\delta_q)}. \tag{28}$$

A small change in the phase angle, linearized over an existing equilibrium voltage $\theta_{pq0} \triangleq \theta_{pq}(\delta_{p0},\delta_{q0})$, therefore, can easily be derived as $\Delta\theta_{pq} = C_1\Delta\delta_p + C_2\Delta\delta_q$ where,

$$C_1 = \frac{a_{pq}E_p(a_{pq}E_p + (1-a_{pq})E_q\cos(\delta_{pq0}))}{a_{pq}^2E_p^2+2a_{pq}(1-a_{pq})E_pE_q\cos(\delta_{pq0})+(1-a_{pq})^2E_q^2} \tag{29}$$

$$C_2 = \frac{(1-a_{pq})E_q((1-a_{pq})E_q + a_{pq}E_p\cos(\delta_{pq0}))}{a_{pq}^2E_p^2+2a_{pq}(1-a_{pq})E_pE_q\cos(\delta_{pq0})+(1-a_{pq})^2E_q^2} \tag{30}$$

where, $\delta_{pq0} = \delta_{p0} - \delta_{q0}$. Since $C_1 \neq C_2$, it is clear that the transfer function will not be affine in any isolated function of a_{pq}, as was the case in (25). The phase measurement in response to an impulse input will, therefore, be

$$\Delta\theta_{pq}(k) = \mathcal{Z}^{-1}\big(C_{pq}^\theta(zI - A_d)^{-1}B_d\big) \tag{31}$$

for $k = 1,2,\ldots,\infty$, where $\mathcal{Z}^{-1}$ indicates the inverse z-transform, and C_{pq}^θ is a $(2n\times 1)$ row vector whose p^{th} element is C_1, q^{th} element is C_2, and all other elements are zero. Since C_{pq}^θ is a function of a_{pq}, we refer to the RHS of (31) simply as $\rho_{pq}(k, a_{pq})$.

C. Angular Frequency

Since the angular frequency f_{pq} at any point on e_{pq} is the rate of change of the phase angle at that point, i.e., $f_{pq} = \dot{\theta}_{pq}$, it follows from (28) that

$$f_{pq} = \dot{\theta}_{pq} = C_1\omega_p + C_2\omega_q \tag{32}$$

where the expressions for $C_1 = \partial\theta_{pq}/\partial\delta_p|_{\delta_{p0},\delta_{q0}}$, $C_2 = \partial\theta_{pq}/\partial\delta_q|_{\delta_{p0},\delta_{q0}}$ are as in (29) and (30), respectively. A small change in the frequency, in response of an impulse input will, therefore, be

$$\Delta f_{pq}(k) = \mathcal{Z}^{-1}\big(C_{pq}^f(zI - A_d)^{-1}B_d\big) \tag{33}$$

for $k = 1,2,\ldots,\infty$, where $\mathcal{Z}^{-1}$ indicates the inverse z-transform, and C_{pq}^f is a $(2n\times 1)$ row vector whose $(n+p)^{th}$ element is C_1, $(n+q)^{th}$ element is C_2, and all other elements are zero. It should be noted that equations (27), (31) and (33) assumes the state vector to be arranged in the form $\mathrm{col}(\Delta\delta, \Delta\omega)$. Since C_{pq}^f is a function of a_{pq}, we refer to the RHS of (33) simply as $\vartheta_{pq}(k, a_{pq})$.

D. Linear Combination of Outputs

Considering (27), (31) and (33), we next consider the output measured along e_{pq} $\forall p = 1,..,m$, $q\in\mathcal{N}_p$ for the n-machine system (18), as a linear combination of voltage magnitude, phase and frequency, namely

$$y_{pq}(k) = \mu_1\psi_{pq}(a_{pq})\xi_{pq}(k) + \mu_2\rho_{pq}(k, a_{pq}) + \mu_3\vartheta_{pq}(k, a_{pq}) \tag{34}$$

where μ_1, μ_2 and μ_3 are positive constants denoting the contribution of each measured variable in the output, with $\mu_1+\mu_2+\mu_3 = 1$. The actual measured output is, however,

$$\tilde{y}_{pq}(k) = y_{pq}(k) + w_{pq}(k) \tag{35}$$

where w_{pq} is stationary Gaussian noise with zero mean and variance σ_{pq}^2. To compute the error in estimating the network parameters from $\tilde{y}_{pq}$, we next construct the FIM for e_{pq} as in Section IIIC. We first stack the noise-free outputs as

$$\mathcal{Y}_{pq} = \left[\; y_{pq}(1) \quad y_{pq}(2) \quad \dots \quad y_{pq}(k) \;\right]. \tag{36}$$

With a slight change of notations, we next assign numbers to each edge as $1, 2, \dots, m$, and denote the resistance and reactance of the j^{th} edge simply by the subscript j, using which we define

$$\begin{aligned} \mathbf{a} &= [r_1, x_1, r_2, x_2, \dots, r_m, x_m]^T, && (37) \\ \mathbf{b} &= [H_{1s}, H_{1r}, H_{2s}, H_{2r}, \dots, H_{ms}, H_{mr}]^T && (38) \end{aligned}$$

where H_{js} and H_{jr} are, respectively, the inertias of the sending and receiving nodes of the j^{th} edge. Note that $\mathbf{b}$ may have repeated elements as one node can be connected to multiple edges. The next step is to construct the two matrices $\mathbf{H}$ and $\mathbf{K}$ as in Section IIIC for e_{pq}:

$$\mathbf{H}_{pq} = \frac{\partial \mathcal{Y}_{pq}}{\partial \mathbf{a}}, \quad \mathbf{K}_{pq} = \frac{\partial \mathcal{Y}_{pq}}{\partial \mathbf{b}}. \tag{39}$$

From (34), (36) it follows that (dropping the functional arguments for brevity)

$$\begin{aligned} \mathbf{H}_{pq} &= \mu_1 \left(\frac{\partial \psi_{pq}}{\partial \mathbf{a}} \xi_{pq} + \psi_{pq} \frac{\partial \xi_{pq}}{\partial \mathbf{a}} \right) + \mu_2 \frac{\partial \rho_{pq}}{\partial \mathbf{a}} + \mu_3 \frac{\partial \vartheta_{pq}}{\partial \mathbf{a}} \\ \mathbf{K}_{pq} &= \mu_1 \psi_{pq}(a_{pq}) \frac{\partial \xi_{pq}}{\partial \mathbf{b}} + \mu_2 \frac{\partial \rho_{pq}}{\partial \mathbf{b}} + \mu_3 \frac{\partial \vartheta}{\partial \mathbf{b}}. \end{aligned} \tag{40}$$

Denoting $\mu = \mathrm{col}(\mu_1, \mu_2, \mu_3)$, the final step is to construct the FIM for e_{pq} following (15) as

$$J_{pq}(\mathbf{a}, \mathbf{b}, a_{pq}, \mu) = \frac{1}{\sigma_{pq}^2} \begin{bmatrix} \mathbf{H}_{pq}\mathbf{H}_{pq}^T & \mathbf{H}_{pq}\mathbf{K}_{pq}^T \\ \mathbf{K}_{pq}\mathbf{H}_{pq}^T & \mathbf{K}_{pq}\mathbf{K}_{pq}^T \end{bmatrix}. \tag{41}$$

Our objective is to find the value of a_{pq} and μ for which the determinant of J^{-1} is minimized. Since for any square non-singular matrix J

$$\det(J^{-1}) = \frac{1}{\det(J)}, \tag{42}$$

the problem is equivalent to finding the optimal $(a_{pq}, \mu$ for which the determinant of J is maximized. We summarize this result in the following theorem.

Theorem 1 : Given any edge e_{pq} of the electrical oscillator network (2)-(3), $p = 1, \dots, n$, $q \in \mathcal{N}_p$, the optimal PMU location a_{pq}^* that generates the tightest CRB for $\mathcal{S}_{pq}$ in (6) $\forall\,(p, q)$, is given by the solution of the following optimization problem:

$$\begin{aligned} &\max_{a_{pq},\mu} \det(J_{pq}(\mathbf{a}, \mathbf{b}, a_{pq}), \mu), && (43) \\ &\text{s.t} \;\; a_{pq} \in [0,\, 1], \;\; \mu_1 + \mu_2 + \mu_3 = 1 && (44) \end{aligned}$$

where $\det(\cdot)$ denotes the absolute value of determinant, and $J_{pq}(\mathbf{a}, \mathbf{b}, a_{pq})$ is given as in (41). □

V. Illustrative Example: Two-Node Network

A. Impulse Response and FIM

In this section we illustrate the results of Section IV with a simple network of two oscillators connected by an edge as shown in Figure 3). Assuming power is flowing from node 1 to 2, we set the reference at node 2, and define any point P at any arbitrary impedance of $\bar{z} = \bar{r} + j\bar{x}$ way from this reference. The impedance of the entire edge is denoted as $z = r + jx$. We define $a = \bar{z}/z \in [0, 1]$, and $\alpha = x/r$.

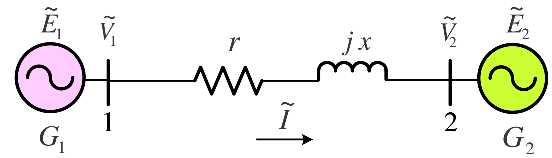
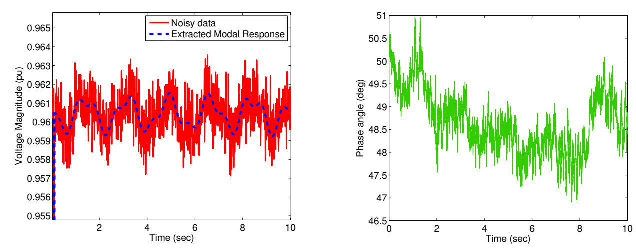


Fig. 3. Two-machine power system



(a) Bus 1 voltage magnitude (b) Bus 1 voltage phase

Fig. 4. Noisy voltage and angle responses at Bus 1

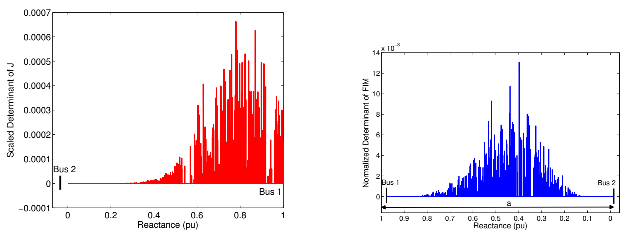
A three-phase short-circuit fault is applied at the midpoint of the line without line switching using Power System Toolbox in Matlab, and voltages and current flows at different points are captured with 5-dbW white noise. Figure 4 shows traces of the voltage magnitude and phase measured at Bus 1. Starting from Bus 2, we apply Algorithm 1 using measurements at different point on the line computed from $\tilde{V}_2$ and $\tilde{I}$. The algorithm converges roughly in 5 iterations, and the optimal measurement position is obtained as approximately $a^* = 0.78$ measured from Bus 2. The estimated and actual values of the unknown parameters are listed in Table 2. The determinant of the FIM at the 5^{th} iteration as a function of a is shown in Figure 5(a). The determinant of the FIM at the 5^{th} iteration as a function of a, when only the voltage magnitude is used as the output, is shown in Figure 5(b). From the ordinates of these two figures it is clear that the value of the determinant reduces significantly when an optimal combination of voltage, phase and frequency is used for estimation compared to the case when only the voltage is used [9]. Observing the final values of μ_1, μ_2 and μ_3 from Table 1, it is also noteworthy that among all three variables V, θ and f, the contribution of the phase angle for generating the best possible parameter estimate is maximum.

VI. Centralized Estimation

We wrap up our discussion with a brief discussion on the situation where the network parameters might be estimated

TABLE I
TWO-MACHINE MODEL PARAMETER ESTIMATION AND MEASUREMENT LOCALIZATION

Parameters	Actual values (per unit)	Estimated values (per unit)				
		Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5
r	0.1	0.05	0.05	0.089	0.081	0.085
x	1	0.50	0.61	0.68	0.73	0.97
H_1	19	11.86	18.91	18.95	18.95	18.98
H_2	13	9.68	11.01	11.16	12.80	12.81
a	-	0.504	0.561	0.711	0.761	0.780
μ_1	-	0.25	0.17	0.15	0.15	0.14
μ_2	-	0.50	0.61	0.65	0.68	0.69
μ_3	-	0.25	0.22	0.20	0.17	0.17



(a) Det(J) with phase

(b) Det(J) without phase

Fig. 5. Determinant of FIM with and without phase measurement in output

using every set of measurement streams available from the m PMUs in a centralized way instead of the edgewise estimation described in Section IV. In that case, assuming that the measurement process of the PMUs are independent of each other, the log-likelihood function of the unknown parameter vector Θ will be [10]

$$\Upsilon(\Theta,\mathcal{A}) = -\sum_{\substack{p=1,2,\ldots,n\\ j\in\mathcal{N}_p}} \int_{T_b} \frac{|\tilde{y}_{pq}(a_{pq},t) - y_{pq}(a_{pq},t,\Theta)|^2}{2\sigma_{pq}^2}dt \tag{45}$$

where $\tilde{y}_{pq}$ is the actual measured value, corrupted with Gaussian noise w_{pq} with mean zero and variance σ_{pq}^2, while y_{pq} is the theoretical value as derived in (34). $\mathcal{A}$ is the set of a_{pq} for all edges stacked together in some specified order. T_b is the final time instant for measurement. The FIM $J(\mathcal{A},\Theta)$ can then be constructed as follows :

$$\begin{aligned} J_{ii} &= E\left\{\left(\frac{\partial\Upsilon(\Theta,\mathcal{A})}{\partial\Theta_i}\right)^2\right\}, \quad i=1,2,\ldots,(2m+n) \\ J_{ij} &= E\left\{\left(\frac{\partial\Upsilon(\Theta,\mathcal{A})}{\partial\Theta_i}\frac{\partial\Upsilon(\Theta,\mathcal{A})}{\partial\Theta_j}\right)\right\}, \quad i\neq j, \end{aligned} \tag{46}$$

where Θ_i denotes the i^{th} element of the vector Θ. The total number of unknown parameters (i.e. the size of Θ) is $2m+n$, namely m resistances, m reactances and n inertias. The approach, thereafter, would be to maximize the determinant of $J(\mathcal{A},\Theta)$ with elements of $\mathcal{A}$ as the optimization variables. The algorithm can be iterated by starting from an initial $\mathcal{A}$, finding Θ from the maximum-likelihood estimate, and then use this estimate to compute a new value for $\mathcal{A}$.

VII. CONCLUSIONS

In this paper we presented several results on the problem of choosing optimal 'measurement points' and 'optimal combination' of measurement variables on the edges of a network of dynamic electrical oscillators such that the noise-corrupted measurements at that point can be used for generating the most accurate estimates for the network parameters. The approach is not to assign edges for PMU placement but rather to find the best location for placing it along any assigned edge. Estimation has been done for the network in *open loop* i.e., there is no feedback from the measured outputs to the network nodes. An interesting observation because of this open-loop structure is that the estimation error bound for each edge parameter in this situation is dependent only on the sensor location on that specific edge, whereby the optimization problem can be addressed in a distributed fashion for each edge separately. Simulation results show significant improvement over our previous results in [9] where only voltage measurements were used for estimation, and, testify that combining the voltage with phase and frequency in an optimal way can reduce the estimation error remarkably.

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