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On Positive Invariance for Delay Difference Equations

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Abstract— **In this paper a new concept of set invariance, called $\mathcal{D}$-invariance, is introduced for dynamical systems described by delay difference equations. We will be interested in the definition and computation of such invariant sets in a specified bounded region of the state-space. The Minkowski algebra will be used to define mappings over the set of compact sets in direct relationship with the invariance of time-delay systems. Set-iterates based on these mappings can be used for the construction of a non-decreasing sequence of $\mathcal{D}$-invariant sets.**

I. INTRODUCTION

The invariant set theory is an important topic in mathematics and receives an increased attention in control literature in relationship with constrained systems or robust control design (see the monograph [1], the survey paper [2] and the references therein). The link between invariant sets and classical stability theory is well understood and comes back as an active research topic particularly in optimization-based control design and related feasibility or reachability problems.

An important family of invariant sets is represented by the class of polyhedral sets. Even if the complexity of their representation is higher than in the ellipsoidal case [3], polyhedral sets have the advantage to follow more accurate the shape of the limit (maximal/minimal) invariant set in different frameworks. Due to safety guaranties for the system evolution in the presence of constraints the invariant sets were studied in the literature [4], [5], [6].

From the system dynamics point of view, the reaction of real systems and physical processes to exogenous signals can rarely be described as "instantaneous". One of the classical ways of modelling such phenomena is by using *time-delays*. Roughly speaking, the delays (constant or time-varying) describe coupling between the dynamics, propagation and transport phenomena, heredity and competition in population dynamics. Various motivating examples and related discussions ca be found in [7], [8], [9]. Networking (congestion mechanisms, consensus algorithms, tele-operation and networked control systems) is one of the classical examples among numerous application domains where the *delay* is a *critical parameter* in understanding dynamics behavior and/or improving (overall) system's behavior. Independently of the mathematical problems related to the appropriate representation of such dynamics, the delay systems are known to rise challenging control problems due to the instabilities introduced in the closed loop by the presence of delays.

In the framework of time-delay systems the invariance conditions are often difficult to characterize. In [10], [11] the existence conditions for set invariance of continuous-time time-delay systems are derived using similar arguments to the nominal linear time invariant case (see [12] and [13]).

In the discrete-time case, the set invariance wih respect to time-delay systems has been addressed recently [14], [15], [16], [17]. It was shown that for an uncertain polytopic system affected by delays, a stabilizing feedback gain and an invariant set can be obtained in an extended state-space, where all the retarded control (or state) entries must be stored. Although finite (due to the discrete time framework), the dimension of the augmented state-space depends on the delay and sampling period and can lead to intractable analysis and design problems. In order to avoid this inconvenient several stabilization methods concentrate on the original state-space, based on Lyapunov-Krasovskii candidates (as in [18]), but the invariant set treatment is still performed in an augmented state space. In the reference [17], the relationship between Lyapunov-Krasovskii and Lyapunov-Razumikhin constructions was discussed, the associated concept of invariance associated with the Lyapunov-Razumikhin stability being close to the proprieties we are interested in here.

The present paper concentrates on set-invariance properties for discrete-time systems with state delays in the non-augmented state space. The concept of $\mathcal{D}$-invariance understood as set-invariance in both the current and retarded state space is introduced and the main properties are discussed in a set-theoretic framework. As a main contribution we propose an algorithm for the construction of $\mathcal{D}$-invariant sets. The theoretic foundation for this construction is provided by a non-decreasing mapping on the space of compact convex sets, related to the Minkowski algebra based condition of $\mathcal{D}$-invariance. The sequence of $\mathcal{D}$-invariant sets is limited by the original bounded region of the state space thus placing a limitation on the iterative procedure.

This paper is structured as follows. Section II presents some preliminary mathematical notations and the description of the dynamics. The concept of $\mathcal{D}$-invariance is defined in Sec. III while further properties are presented in Sec. IV with a focus on the related set-iterates, along with the description of an effective procedure for $\mathcal{D}$-invariant sets construction. Sec. V presents a numerical example and Sec. VI draws some concluding remarks.

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II. PRELIMINARIES

A. Basic notions and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every subset Π of $\mathbb{R}$ we define $\mathbb{R}_\Pi := \mathbb{R} \cap \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$.

A polyhedron (or a bounded polyhedral set) in $\mathbb{R}^n$ is a set obtained as the intersection of a finite number of open and/or closed half-spaces. The set of vertices of a polyhedron $\mathcal{A} \subseteq \mathbb{R}^n$ is denoted $\mathcal{V}(\mathcal{A})$.

For two arbitrary sets $\mathcal{A} \subseteq \mathbb{R}^n$ and $\mathcal{B} \subseteq \mathbb{R}^n$

$$\mathcal{A} \oplus \mathcal{B} := \{x + y \mid x \in \mathcal{A}, y \in \mathcal{B}\}$$

denotes their Minkowski sum,

$$\mathcal{A} \sim \mathcal{B} := \{x \in \mathbb{R}^n \mid x + \mathcal{B} \subseteq \mathcal{A}\}$$

denotes their Pontryagin difference, $Co(\mathcal{A}, \mathcal{B}) = Co(\mathcal{A} \cup \mathcal{B})$ denotes the convex hull of $\mathcal{A}$ and $\mathcal{B}$ and

$$\mathcal{B} \setminus \mathcal{A} := \{x \in \mathcal{B} \mid x \notin \mathcal{A}\}$$

denotes the set difference between $\mathcal{B}$ and $\mathcal{A}$.

For an arbitrary set $\mathcal{A} \subseteq \mathbb{R}^n$ and $\alpha \in \mathbb{R}_+$, we define

$$\alpha\mathcal{A} := \{\alpha x \mid x \in \mathcal{A}\}.$$

For an arbitrary set $\mathcal{A} \subseteq \mathbb{R}^n$, $int(\mathcal{A})$ denotes the interior of $\mathcal{A}$. In set theory, a set $\mathcal{A}$ is a subset of a set $\mathcal{B}$ if $\mathcal{A} \subset \mathcal{B}$. Correspondingly, set $\mathcal{B}$ is a superset of $\mathcal{A}$.

For the Euclidian space $\mathbb{R}^n$, we denote by $Com(\mathbb{R}^n)$ the space of compact subsets of $\mathbb{R}^n$. A convex and compact set in $\mathbb{R}^n$ that contains the origin in its interior is called a $\mathcal{C}$-set. $ComC(\mathbb{R}^n)$ denotes the space of $\mathcal{C}$-subsets of $\mathbb{R}^n$ containing the origin. $\mathbb{B}^n_{0,r}$ denotes the ball of radius r in Euclidean norm, centered in the origin of $\mathbb{R}^n$.

The spectrum of a matrix $A \in \mathbb{R}^{n\times n}$ is the set of eigenvalues of A, denoted $\lambda(A)$, while the spectral radius is defined as

$$\rho(A) := \max_{\lambda \in \lambda(A)} (|\lambda|).$$

The spectral norm will be denoted $\sigma(A)$ and is defined as:

$$\sigma(A) := \sqrt{\rho(A^T A)}$$

B. System Dynamics

We consider delay difference equations of the form:

$$x(k+1) = A_0 x(k) + A_d x(k-d) \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector at the time $k \in \mathbb{Z}_+$. $d \in \mathbb{Z}_+$ is the *fixed* time-delay, the matrices $A_j \in \mathbb{R}^{n\times n}$, for $j \in \mathbb{Z}_{[0,d]}$ and the initial conditions $x(-i) = x_{-i} \in \mathbb{R}^n$, for $i \in \mathbb{Z}_{[0,d]}$.

III. $\mathcal{D}$-INVARIANCE RELATED DEFINITIONS AND BASIC PROPERTIES

A. Definitions

Definition III.1 A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called $\mathcal{D}$-invariant for the system (1) with initial conditions $x_{-i} \in \mathcal{P}$ for all $i \in \mathbb{Z}_{[0,d]}$ if the state trajectory satisfies $x_k \in \mathcal{P}, \forall k \in \mathbb{Z}_+$. □

The concept of $\mathcal{D}$-invariance introduced in the previous definition will be used extensively in this paper. It is related to the *dynamics of systems affected by time-delay* and to the state constraints which describe a region with invariant properties with respect to the given dynamics. It is noteworthy the difference between the $\mathcal{D}$-invariance introduced here and the invariance of *systems with disturbance inputs* as defined in [5], [6] and related works. In the present framework, the dynamics are autonomous and the presence of the retarded argument is not interpreted as a disturbance signal but is treated as a state dependence in a nominal manner.

Theorem III.2 *The following affirmations are equivalent:*

i) A set $\mathcal{P} \subseteq \mathbb{R}^n$ is $\mathcal{D}$-invariant for system (1).
ii) $A_0\mathcal{P} \oplus A_d\mathcal{P} \subseteq \mathcal{P}$.

Proof: $i) \to ii)$ $\mathcal{P}$ is $\mathcal{D}$-invariant $\Rightarrow x_1 \in \mathcal{P}$ for all $x_0 \in \mathcal{P}$ and $x_{-d} \in \mathcal{P}$, which is equivalent to $A_0x_0 + A_dx_{-d} \in \mathcal{P}, \forall x_0 \in \mathcal{P}$ and $x_{-d} \in \mathcal{P}$.

$ii) \to i)$ $\forall x_0, x_{-d} \in \mathcal{P}, x_1 = A_0x_0 + A_dx_{-d} \in A_0\mathcal{P} \oplus A_d\mathcal{P} \subseteq P$. Then, $x_k \in \mathcal{P}, \forall k$ follows by induction. ■

In the following, several properties are reviewed in order to fix a set of basic relations to be used in the algorithmic construction of $\mathcal{D}$-invariant sets.

Proposition III.3 *The following properties hold:*

P1 If $\mathcal{P} \in \mathbb{R}^n$ is $\mathcal{D}$-invariant then $\alpha\mathcal{P}$ is $\mathcal{D}$-invariant for any $\alpha \in \mathbb{R}_{>0}$.

P2 Let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \mathbb{R}^n$ be two $\mathcal{D}$-invariant sets for the dynamics (1). Then $\mathcal{P}_1 \cap \mathcal{P}_2$ is a $\mathcal{D}$-invariant set for the same dynamical system.

P3 Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a convex set containing the origin. If $\mathcal{P}$ is $\mathcal{D}$-invariant with respect to (1) then $\mathcal{P}$ is positive invariant with respect to the time invariant linear dynamics:

$$x(k+1) = A_0 x(k), \tag{2}$$

$$x(k+1) = A_d x(k). \tag{3}$$

Equivalently, $A_0\mathcal{P} \subseteq \mathcal{P}$, $A_d\mathcal{P} \subseteq \mathcal{P}$.

P4 Given a $\mathcal{D}$-invariant set $\mathcal{P} \in \mathbb{R}^n$ for the system

$$x(k+1) = A_0 x(k) + A_d x(k-d) \tag{4}$$

then $\mathcal{P}$ is $\mathcal{D}$-invariant for

$$x(k+1) = A_d x(k) + A_0 x(k-d) \tag{5}$$

P5 For some $d \in \mathbb{Z}_{>0}$, given a $\mathcal{D}$-invariant set $\mathcal{P} \in \mathbb{R}^n$ for the system

$$x(k+1) = A_0 x(k) + A_d x(k-d) \tag{6}$$

then $\mathcal{P}$ is $\mathcal{D}$-invariant for

$$x(k+1) = A_0 x(k) + A_d x(k - \bar{d}) \tag{7}$$

for any $\bar{d} > 0$.

Proof:

P1 Using basic properties of the Minkowski addition (see [19]) for $\forall \mathcal{P} \subset ComC(\mathbb{R}^n)$ and Theorem III.2 one obtains:

$$\alpha A_0 \mathcal{P} \oplus \alpha A_d \mathcal{P} = \alpha(A_0 \mathcal{P} \oplus A_d \mathcal{P}) \subseteq \alpha \mathcal{P}. \tag{8}$$

P2 Consider two points $x \in \mathcal{P}_1 \cap \mathcal{P}_2$ and $y \in \mathcal{P}_1 \cap \mathcal{P}_2$. The set $\mathcal{P}_1$ is $\mathcal{D}$-invariant and $x, y \in \mathcal{P}_1$, thus:

$$A_0 x + A_d y \in \mathcal{P}_1.$$

Similarly $\mathcal{P}_2$ is $\mathcal{D}$-invariant and $x, y \in \mathcal{P}_2$ imply:

$$A_0 x + A_d y \in \mathcal{P}_2$$

and thus observing that x, y were chosen arbitrarily it completes the proof:

$$A_0 x + A_d y \in \mathcal{P}_1 \cap \mathcal{P}_2.$$

P3 By using the fact that $\{0\} \in \mathcal{P}$ and the $\mathcal{D}$-invariance property with respect to (1) then for any $i \in \mathbb{Z}_{[0,d]}$ the following set inclusions hold:

$$A_d \mathcal{P} = A_0\{0\} \oplus A_d \mathcal{P} \subset A_0 \mathcal{P} \oplus A_d \mathcal{P} \subset \mathcal{P},$$
$$A_0 \mathcal{P} = A_0 \mathcal{P} \oplus A_d\{0\} \subset A_0 \mathcal{P} \oplus A_d \mathcal{P} \subset \mathcal{P},$$

which corresponds to the definition of a positive invariant set $\mathcal{P}$ with respect to the dynamics in (2).

P4 This represents a direct implication of Theorem III.2 since the set theoretic description of the $\mathcal{D}$-invariance with respect to (4) and (5) is identical due to commutativity of the Minkowski addition.

P5 The set $\mathcal{P}$ is $\mathcal{D}$-invariant for the system (6) with $d = 1$ if $A_0 \mathcal{P} \oplus A_d \mathcal{P} \subset \mathcal{P}$. When $d = 2$ the set $\mathcal{P}$ is $\mathcal{D}$-invariant if $A_0 \mathcal{P} \oplus A_d \mathcal{P} \subset \mathcal{P}$ and by induction, the set $\mathcal{P}$ is $\mathcal{D}$-invariant if $A_0 \mathcal{P} \oplus A_d \mathcal{P} \subset \mathcal{P}$ for all $d > 0$.

■

The properties P4 and P5 underline the fact that $\mathcal{D}$-invariance is related to the algebraic properties of pair (A_0, A_d) and thus delay-independent from a structural point of view. The effective link with the specificity of dynamical systems affected by delays is resumed by the conditions $x_{-i} \in \mathcal{P}, i \in \mathbb{Z}_{[1,d]}$ (with their dimensional and complexity implications).

B. Necessary Conditions and Sufficient Conditions

The goal of the present paper is to present a *constructive* procedure for $\mathcal{D}$-invariant set descriptions. It is obvious that the existence of a nondegenerate and bounded $\mathcal{D}$-invariant set[1] is related to the stability of the discrete-time dynamical system affected by delay (1). As shown by the previous example, these prove to be only necessary conditions for the existence of a $\mathcal{D}$-invariant set. In the following we enumerate a set of necessary conditions and, alternatively, a set of sufficient conditions which are easily checkable using classical numerical routines for the eigenvalues problems.

[1]Note that sets like $\{0\}$ or $\mathbb{R}^n$ are $\mathcal{D}$-invariant but they do not satisfy the non-degenerate or boundedness conditions.

Let us introduce the following notation for the extended state-space matrix:

$$A_\xi = \begin{bmatrix} A_0 & 0 & \dots & 0 & A_d \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix} \tag{9}$$

1) Necessary conditions: Considering the system (1), the existence of a $\mathcal{D}$-invariant $\mathcal{C}$-set P implies:

N1 The spectral radii of the matrices A_0 and A_d are subunitary: $\rho(A_i) \leq 1,\ \forall i \in \{0, d\}$;

N2 The spectral radius of the matrix $(A_0 + A_d)$ is subunitary: $\rho(A_0 + A_d) \leq 1$;

N3 The spectral radius of the extended state-space matrix is subunitary:

$$\rho(A_\xi) \leq 1.$$

2) Sufficient conditions: Considering the system (1), the existence of a $\mathcal{D}$-invariant $\mathcal{C}$-set P is guaranteed if one of the following conditions hold:

S1 The sum of the spectral radius of A_0 and the spectral radius of A_d is subunitary:

$$\sigma(A_0) + \sigma(A_d) < 1.$$

S2 In the case of nonsingular matrix A_0 (or A_d)

$$(1 + \sigma(A_0^{-1} A_d))\sigma(A_0) \leq 1,$$

or alternatively

$$(1 + \sigma(A_d^{-1} A_0))\sigma(A_d) \leq 1.$$

IV. SET DYNAMICS AND $\mathcal{D}$-INVARIANCE

The equivalence between the $\mathcal{D}$-invariance in its nominal description and the set-theoretic counterpart evidenced in Theorem III.2 lead us to the treatment of the constructive algorithms for $\mathcal{D}$-invariance in a set-theoretic framework [20]. Considering the matrices $A_0, A_d \in \mathbb{R}^{n \times n}$ as in (1) we define the mappings:

$$\begin{aligned} \Phi: \quad & ComC(\mathbb{R}^n) \to ComC(\mathbb{R}^n) \\ & \Phi(\mathcal{P}) = A_0 \mathcal{P} \oplus A_d \mathcal{P}; \end{aligned} \tag{10}$$

$$\begin{aligned} \Psi: \quad & ComC(\mathbb{R}^n) \to ComC(\mathbb{R}^n) \\ & \Psi(\mathcal{P}) = Co(\mathcal{P}, A_0 \mathcal{P} \oplus A_d \mathcal{P}) = Co(\mathcal{P}, \Phi(\mathcal{P})). \end{aligned} \tag{11}$$

We remark that using (10) and as a direct consequence of the Theorem III.2, a given compact set $\mathcal{P}$ is $\mathcal{D}$-invariant if $\Phi(\mathcal{P}) \subseteq \mathcal{P}$.

The mappings (10)-(11) can be further used to define k-iterates over the family of $\mathcal{C}$-sets:

$$\begin{aligned} \Phi^k(\mathcal{P}) &= \Phi(\Phi^{k-1}(\mathcal{P})), k \geq 0 \text{ with } \Phi^0(\mathcal{P}) = \mathcal{P}, \\ \Psi^k(\mathcal{P}) &= \Psi(\Psi^{k-1}(\mathcal{P})), k \geq 0 \text{ with } \Psi^0(\mathcal{P}) = \mathcal{P}. \end{aligned} \tag{12}$$

The next remark points out several properties of the iterative set mappings.

Remark IV.1 For the mappings defined in (10)-(11), the following properties hold[2]:

i $\Psi^k(\mathcal{P})$ is set-wise non-decreasing, i.e.

$$\Psi^k(\mathcal{P}) \supseteq \Psi^{k-1}(\mathcal{P}), \forall k \geq 1,$$

for any $\mathcal{P} \in ComC(\mathbb{R}^n)$.

ii If $\Phi(\mathcal{P}) \subseteq \mathcal{P}$ then $\Phi^k(\mathcal{P})$ with $k > 0$ is set-wise non-increasing, i.e. $\Phi^k(\mathcal{P}) \subseteq \Phi^{k-1}(\mathcal{P}), \forall k \geq 1$.

iii If $\mathcal{P}$ is $\mathcal{D}$-invariant and convex then $\mathcal{P}$ is a fixed point for $\Psi(.)$, i.e.:

$$\Psi(\mathcal{P}) = \mathcal{P}. \tag{13}$$

The space $ComC(\mathbb{R}^n)$ is endowed with the Hausdorff distance. Consider two non-empty arbitrary sets $\mathcal{P}_1 \subseteq \mathbb{R}^n$ and $\mathcal{P}_2 \subseteq \mathbb{R}^n$. The Hausdorff distance is defined as:

$$d_H(\mathcal{P}_1, \mathcal{P}_2) = \max\left(\max_{x\in\mathcal{P}_1}\min_{y\in\mathcal{P}_2} d(x,y),\ \ \max_{x\in\mathcal{P}_2}\min_{y\in\mathcal{P}_1} d(x,y)\right),$$

where $d(x, y)$ is the Euclidian distance between the points x and y in $\mathbb{R}^n$.

The property iii in Remark IV.1 points out that a $\mathcal{D}$-invariant set is related to one of the *fixed points* of the set-dynamics described by (11). Starting with a set $\mathcal{P} \in ComC(\mathbb{R}^n)$, the convergence of the iterates $\Psi^k(\mathcal{P})$ to a *fixed point* is directly related to the convergence of the Hausdorff distance between the iterates and the *fixed point* to zero.

The mapping $\Phi(\cdot)$ in (10) is a *contraction* [20] if there exists an $\alpha \in \mathbb{R}_{[0,1)}$ such that for two arbitrary sets $\mathcal{P}_1 \subseteq \mathbb{R}^n$ and $\mathcal{P}_2 \subseteq \mathbb{R}^n$:

$$d_H(\Phi(\mathcal{P}_1), \Phi(\mathcal{P}_2)) \leq \alpha d_H(\mathcal{P}_1, \mathcal{P}_2),$$

The contractive behavior of the mapping $\Psi(\cdot)$ in (11) is defined similarly.

Next, we propose an algorithmic procedure for the computation of a non-decreasing sequence of $\mathcal{D}$-invariant sets for the system (1). This procedure considers as input argument a predefined region in the state space which confines the $\mathcal{D}$-invariant candidates and exploits the contractive properties of the mappings (10)-(11).

A. A generic procedure for $\mathcal{D}$-invariant set construction

In this subsection, we describe the main steps of an iterative construction of $\mathcal{D}$-invariant sets. One of the main concepts to be exploited is the upper boundedness of the set-wise non-decreasing sequence of sets $\Psi(\mathcal{P})$ in case of the existence of a bounded $\mathcal{D}$-invariant set. The following results are instrumental in this sense.

Lemma IV.2 *If $\mathcal{P} \in ComC(\mathbb{R}^n)$ is a $\mathcal{D}$-invariant set for* (1) *then for any subset $\mathcal{S} \subseteq \mathcal{P}$ we have:*

$$\Phi^k(\mathcal{S}) \subseteq \mathcal{P}, \forall k \geq 0.$$

[2]The definition of non-increasing/non-decreasing sequence of iterates is similar to the one in [20].

Proof: By assumption, the set $\mathcal{P}$ is $\mathcal{D}$-invariant which assures $\Phi(\mathcal{P}) \subseteq \mathcal{P}$. Since $\mathcal{S} \subseteq \mathcal{P}$ we have $\Phi(\mathcal{S}) \subseteq \Phi(\mathcal{P}) \subseteq \mathcal{P}$. The result hold for all $k \geq 0$ by induction. ■

Theorem IV.3 *Given a convex set $\mathcal{P} \in ComC(\mathbb{R}^n)$, the sequence $\Psi^k(\mathcal{P}), k \geq 0$ converges toward a $\mathcal{D}$-invariant convex superset.*

Proof: Let $\mathcal{D} \subseteq \mathbb{R}^n$ be a convex $\mathcal{D}$-invariant superset of $\mathcal{P}$ (a set which contains $\mathcal{P}$). Using Lemma IV.2, $\mathcal{P} \subseteq \mathcal{D}$ implies $\Phi(\mathcal{P}) \subseteq \mathcal{D}$. Exploiting the fact that we are dealing with convex sets, we have $Co(\mathcal{P}, \Phi(\mathcal{P})) \subseteq \mathcal{D}$ which further implies $\Psi(\mathcal{P}) \subseteq \mathcal{D}$. By induction $\Psi^k(\mathcal{P}) \subseteq \mathcal{D}$ and in the same time $\Psi^k(\mathcal{P})$ is a non-decreasing sequence of sets. Thus the following inclusion holds:

$$\mathcal{P} \subseteq \Psi(\mathcal{P}) \subseteq \Psi^2(\mathcal{P}) \subseteq \cdots \subseteq \Psi^k(\mathcal{P}) \subseteq \cdots \subseteq \mathcal{D}. \tag{14}$$

The relationship (14) holds for any $\mathcal{D}$-invariant $\mathcal{C}$-superset of $\mathcal{P}$.

At this point, we proved the existence of a non-decreasing sequence of sets upperbounded by a $\mathcal{D}$-invariant set. In order to complete the proof we have to show that the limit set $\bar{\mathcal{D}} = \lim_{k\to\infty} \Phi^k(\mathcal{P})$ is a $\mathcal{D}$-invariant set. This is immediate by the fact that $\Phi(\Psi^k(\mathcal{P})) \subseteq \Psi^{k+1}(\mathcal{P})$ and thus

$$\lim_{k\to\infty} \Phi(\Psi^k(\mathcal{P})) \subseteq \lim_{k\to\infty} \Psi^{k+1}(\mathcal{P}) \Rightarrow \Phi(\bar{\mathcal{D}}) \subseteq \bar{\mathcal{D}} \tag{15}$$

■

This main result offers a basis for a generic construction of convex $\mathcal{D}$-invariant sets based on set-iterations. Our main objectives will be to enlarge this set all by preserving the $\mathcal{D}$-invariance properties and remaining in the pre-defined region $\mathcal{X} \subseteq \mathbb{R}^n$.

Algorithm IV.4

Input*: A convex set $\mathcal{X} \in ComC(\mathbb{R}^n)$, $A_0, A_d \in \mathbb{R}^{n\times n}$*

Output*: A $\mathcal{D}$-invariant set in $\mathcal{X}$*

1) Obtain $\mathcal{R}_0$, a initial $\mathcal{D}$-invariant $\mathcal{C}$-set in $\mathcal{X}_0$. Set $i = 0$;

2) Compute a set $\mathcal{T} = Co(\mathcal{R}_i, x)$ with $x \in int(\mathcal{X} \setminus \mathcal{R}_i)$;

3) Compute the limit set $\mathcal{D} = \lim_{k\to\infty} \Phi^k(\mathcal{T})$;

if $\mathcal{D} \subseteq \mathcal{X}$ **then**

  $\mathcal{R}_{i+1} = \mathcal{D}$;

  if $\mathcal{R}_i$ *has to be improved (enlarged)* **then**

    Start over from 2) with $i = i + 1$;

  else

    return $\mathcal{R}_i$

  end

else

  Start over from 2) with a new point x and note that the previous selected point is not contained in the final $\mathcal{D}$-invariant set;

end

An advantage of this algorithm is that at each intermediary step we dispose of a $\mathcal{D}$-invariant set and generate a non-decreasing sequence of $\mathcal{D}$-invariant sets in $\mathcal{X}$.

In order to have an effective implementation of this algorithm, there are two issues that have to be clarified :

- How to estimate the Hausdorff distance between the current $\mathcal{D}$-invariant set and the limit $\mathcal{D}$-invariant set (the former one being à priori unknown). In fact, this estimation will be used in the evaluation of a stopping criteria for the set-iteration.
- How to chose points in $\mathcal{X} \setminus \mathcal{R}_i$ and how to use them in the approximation of a new $\mathcal{D}$-invariant set.

B. Auxiliary routines

We define in an algorithmic form several operations which are instrumental for the proposed practical $\mathcal{D}$-invariant set construction.

In order to evaluate the converge rate, a pair of inner-outer approximations has to be used. If the inner approximation is mainly based on the numerical exploitation of the Theorem IV.3, for the outer approximation we propose the use of the complementary set for a $\mathcal{D}$-invariant set $\mathcal{R}$ with respect to a given subspace of the state space $\mathcal{X}_0$. This will be defined as the collection of points in $\mathcal{X}_0$, which mapped by either A_0 or A_d can be summed up (in the Minkowski sense) with any point in $A_d\mathcal{R}$ or $A_0\mathcal{R}$ and remain in the collection.

- $\mathcal{P} = outer(\mathcal{R}, \mathcal{X}_0, A_0, A_d)$: An iterative algorithm which use a $\mathcal{D}$-invariant set $\mathcal{R}$ to construct the largest set $\mathcal{P} \subseteq \mathcal{P}_0 = \mathcal{X}_0 \subseteq \mathbb{R}^n$ that verifies:

$$\begin{array}{l} A_0\mathcal{P} \oplus A_d\mathcal{R} \subseteq \mathcal{P} \\ A_0\mathcal{R} \oplus A_d\mathcal{P} \subseteq \mathcal{P} \end{array}.$$

Algorithm IV.5

Input: *Convex sets* $\mathcal{R} \subseteq \mathcal{P}_0 \in Com\mathbb{R}^n$, $A_0, A_d \in \mathbb{R}^{n\times n}$
Output: $\mathcal{P}$
$\mathcal{P}_1 = \{x \in \mathcal{P}_0 \mid \begin{array}{l} A_0 x \oplus A_d\mathcal{R} \subseteq \mathcal{P}_0; \\ A_0\mathcal{R} \oplus A_d x \subseteq \mathcal{P}_0\} \end{array};$
$i = 1$;
while $\mathcal{P}_i \neq \mathcal{P}_{i-1}$ **do**
  $\mathcal{P}_{i+1} = \{x \in \mathcal{P}_i \mid \begin{array}{l} A_0 x \oplus A_d\mathcal{R} \subseteq \mathcal{P}_i; \\ A_0\mathcal{R} \oplus A_d x \subseteq \mathcal{P}_i\} \end{array};$
  $i = i + 1$;
end
return $\mathcal{P} = \mathcal{P}_i$

The routine providing a $\mathcal{D}$-invariant set by set-iterations from an initial arbitrary set $\mathcal{S}$:

- $\mathcal{R} = get\_Dinv(\mathcal{S})$: an algorithmic construction of the limit set $\lim\limits_{k\to\infty} \Psi^k(\mathcal{S})$.

Algorithm IV.6

Input: *A convex set* $\mathcal{S} \in ComC(\mathbb{R}^n)$, $A_0, A_d \in \mathbb{R}^{n\times n}$
Output: $\mathcal{R}$
$\mathcal{R}_0 = \mathcal{S}$;
$\mathcal{R}_1 = \Psi(\mathcal{R}_0)$;
$i = 1$;
while $\mathcal{R}_i \neq \mathcal{R}_{i-1}$ **do**
  $\mathcal{R}_{i+1} = \Psi(\mathcal{R}_i)$;
  $i = i + 1$;
end
return $\mathcal{R} = \mathcal{R}_i$

In order to adapt the outer approximation after each refinement of the inner approximation, one has to introduce new points in the step 2) of Algorithm IV.4. In our implementation, at each iteration we will make use of the vertex of the outer approximation corresponding to the point which generated the Hausdorff distance between the inner and the outer approximation.

- $[dH, v_H] = d_H(\mathcal{P}_1, \mathcal{P}_2)$: is a function which returns the Hausdorff distance between $\mathcal{P}_1 \supseteq \mathcal{P}_2$. The second output argument is the point $v_H \in \mathcal{P}_1$ for which $d_H(\mathcal{P}_1, \mathcal{P}_2) = d_H(v_H, \mathcal{P}_2)$ [3].

C. Non-decreasing sequence of $\mathcal{D}$-invariant sets

Once all the generic steps are defined and the specific routines are available, we can bring together all the elements and describe an effective algorithm for the construction of a sequence of non-decreasing $\mathcal{D}$-invariant set up to a $\mathcal{E}$-improvement.

Algorithm IV.7

Input: *Convex set* $\mathcal{X}_0 \in Com\mathbb{R}^n$, $A_0, A_d \in \mathbb{R}^{n\times n}$ *and a* $\mathcal{E} \in \mathbb{R}$
Output: $\mathcal{R}$, *a convex* $\mathcal{D}$*-invariant set in* $\mathcal{X}_0$
$\mathcal{S} = get\_Dinv(\mathcal{X}_0)$;
Find γ *such that* $\gamma\mathcal{S} \subseteq \mathcal{X}_0$;
$R_0 = \gamma\mathcal{S}$;
$\mathcal{P}_0 = outer(\mathcal{X}_0, \mathcal{R}, A_0, A_d)$;
$i = 1$;
$[dH, v_H] = d_H(\mathcal{P}_0, \mathcal{R}_0)$;
while $dH \leq \mathcal{E}$ **do**
  Choose $\lambda \in \mathbb{R}_{(0,1)]}$ *such that* $\lambda v_H \in int(\mathcal{P}_i \setminus \mathcal{R}_i)$;
  $\mathcal{T} = Co(\lambda v_H, \mathcal{R}_i)$;
  $\mathcal{R}_{i+1} = get\_Dinv(\mathcal{T})$;
  $\mathcal{P}_{i+1} = outer(\mathcal{P}_i, \mathcal{R}_{i+1}, A_0, A_d)$;
  if $\mathcal{R}_{i+1} \not\subseteq \mathcal{P}_{i+1}$ **then**
    $\mathcal{R}_{i+1} = \mathcal{R}_i$;
    $\mathcal{P}_{i+1} = Co(\mathcal{R}_i, \lambda v_H, \mathcal{V}(\mathcal{P}_i) \setminus \{v_H\})$;
  end
  $i = i + 1$;
  $[dH, v_H] = d_H(\mathcal{P}_i, \mathcal{R}_i)$;
end
return $\mathcal{R}_i$

The Algorithm IV.7 constructs a sequence of sets which satisfies the following property:

$$\mathcal{X}_0 = \mathcal{P}_0 \supseteq \mathcal{P}_1 \supseteq \cdots \supseteq \mathcal{P}_i \supseteq \cdots \supseteq \mathcal{R}_i \supseteq \mathcal{R}_1 \supseteq \mathcal{R}_0. \quad (16)$$

In terms of Hausdorff distance:

$$d_H(\mathcal{P}_0, \mathcal{R}_0) \geq d_H(\mathcal{P}_1, \mathcal{R}_1) \geq \cdots \geq d_H(\mathcal{P}_i, \mathcal{R}_i) \geq \ldots \quad (17)$$

and due to the finite number of vertices and the the fact the points λv_H are in the strict interior of $\mathcal{P}_i \setminus \mathcal{R}_i$ we have

$$d_H(\mathcal{P}_i, \mathcal{R}_i) \longrightarrow 0. \quad (18)$$

[3]The Hausdorff distance can be found by solving a quadratic program for each vertex of the set $\mathcal{A}$.

Thus Algorithm IV.7 has an effective stopping criteria in the condition $d_H < \mathcal{E}$. A transparent tuning parameter for the proposed routine is the scalar λ which place the points to be treated closer to the inner or to the outer approximation of the limit $\mathcal{D}$-invariant set. It is interesting to remark the fact that λ can be adapted at each iteration.

V. ILUSTRATIVE EXAMPLE

Consider a dynamical system as in (1), with:

$$A_0 = \begin{bmatrix} 0.0809 & -0.0588 \\ 0.0588 & 0.0809 \end{bmatrix}, \quad A_d = \begin{bmatrix} 0.8257 & -0.1308 \\ 0.1308 & 0.8257 \end{bmatrix}. \tag{19}$$

Consider a set $\mathcal{X}_0$ which is the 1-norm unit circle in $\mathbb{R}^2$. By applying the Algorithm IV.7 the inner-outer approximation of the limit $\mathcal{D}$-invariant set is obtained iteratively. Figure 1 presents the inner-outer approximation for each iteration (indexed on the z axis).

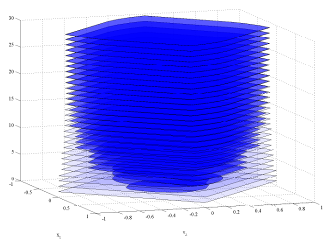

Fig. 1. Sequences of inner and outer approximations $(\mathcal{R}_i, \mathcal{P}_i)$ indexed on the z axis according to the number of iterations in the numerical procedure.

Figure 2 presents the evolution of the Hausdorff distance between the inner and the outer approximation as a function of the number of iterations.

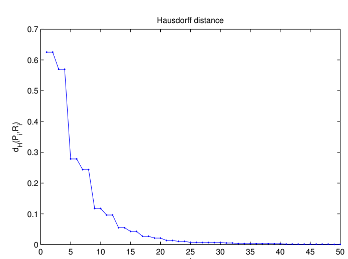

Fig. 2. $d_H(\mathcal{R}_i, \mathcal{P}_i)$ vs. the iteration index i.

The test of $\mathcal{D}$-invariance is depicted in Figure V.

VI. CONCLUSION

The paper introduced a new concept of set invariance ($\mathcal{D}$-invariance) for a class of discrete-time dynamical systems affected by delay. The main contribution is an algorithm for the effective construction of a $\mathcal{D}$-invariant sets via set-iterates. It is shown that a non-decreasing sequence of $\mathcal{D}$-invariant sets can be obtained in a predefined region of the state space.

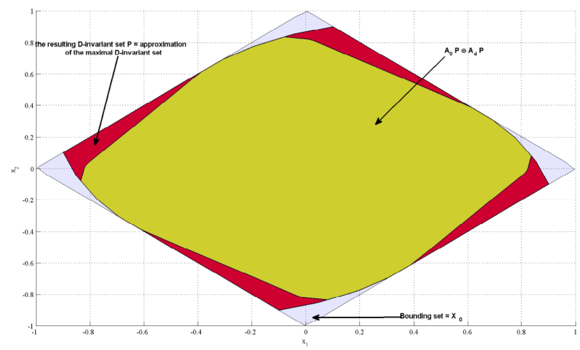

Fig. 3. Graphical illustration of $\mathcal{D}$-invariance $A_0\mathcal{P} \oplus A_d\mathcal{P} \subseteq \mathcal{P}$. In blue, the bounding box - $\mathcal{X}_0$; in red, the $\mathcal{D}$-invariant set; in yellow, the set $A_0\mathcal{P} \oplus A_d\mathcal{P}$.

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