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Approximate Output Regulation for a Spherical Inverted Pendulum

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Abstract— **The present study applies an approximate output regulator for controlling a MIMO nonlinear non-minimum phase system, considering the example of a four degree of freedom spherical inverted pendulum. The spherical pendulum consists of a slim cylinder attached to a universal joint upon which the planar control force acts. The aim in output tracking is to control the pendulum such that the base follows a desired reference trajectory as closely as possible while maintaining the upright position. The standard output regulator requires the solution to a mixed algebraic partial differential equation, which entails finding the manifold on which the system state trajectories result in exact tracking of the reference signal. This can be very difficult to solve in practice for the non-minimum phase case thus motivating the use of approximation methods. A local approximation method, based on a Taylor series expansion of the system dynamics is used, such that an output regulator may be applied to the spherical inverted pendulum. This gives the first application of output regulation for the output tracking of this system.**

I. Introduction

MUCH research has been carried out during the past few decades for stabilization and tracking of complex nonlinear systems considering the 2 degree of freedom (DoF) linear cart inverted pendulum as a benchmark (see [1] and the references therein). More recently, the spherical inverted pendulum, which is essentially a multi input multi output (MIMO) underactuated non-minimum phase nonlinear system, has become of interest as this provides a greater control challenge. The spherical inverted pendulum consists of a slim cylindrical rod attached to a universal joint at the base about which it can rotate with two degrees of freedom (Fig 1.). The universal joint is free to move in the horizontal plane with two translational degrees of freedom and is acted upon by a planar control force.

The spherical inverted pendulum control problem represents an increase in complexity from the standard 2-DoF linear cart pendulum due to the additional degrees of freedom and the coupled nature of the motion about the two rotational axes. To date very few results are available for either the stabilization or output tracking of the spherical inverted pendulum based on a non-simplified nonlinear model. Most of the results that have been found are pertaining to the stabilization of this system, e.g. [2], [3]. However, for nonlinear output tracking controllers these results are even more limited. The focus in output tracking is to control the pendulum such that the base follows a desired reference trajectory while maintaining the upright position of the pendulum. Some initial successful output tracking control results for the spherical inverted pendulum are presented in [4], [5], where non causal "exact" tracking methods were carried out using stable inversion in [4] and homotopy method in [5]. These methods were used to track specific individual trajectories which must be specified in advance due to their non-causal nature. In contrast to these methods, an output regulator can track a family of trajectories, where only the structure of the external dynamic system capable of generating these trajectories need to be known, not specific trajectories themselves. Also, unlike restrictions on the homotopy method used in [5], the desired reference trajectory generated by the exosystem is not required to be periodic.

The output regulation problem is the problem of finding a control law which allows the output of a system to asymptotically track a family of desired reference trajectories (and/or reject a family of known disturbances) that can be generated by an external dynamic system, referred to as the *exosystem*. The output regulator is casual as it is capable of tracking many possible reference trajectories that may be generated from the exosystem without explicit knowledge of which specific trajectory is to be tracked. In the seminal work on output regulation for nonlinear systems [6] it was established that the ability to design a controller to solve the nonlinear output regulation problem is dependent on the existence of a solution to a mixed algebraic partial differential equation. These equations, known as the regulator equations, are in practice very difficult to solve and in many cases it is often impossible to find a closed form solution. Much subsequent research in nonlinear output regulation control has been put into the advancement of output regulation theory ([7],[8],[9],[10] for example). These theoretical advancements have mostly been carried out without use of practical examples. This is generally because for most practical systems the solution to the regulator equations cannot readily be attained. Methods for finding an

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approximate solution to the regulator equations were also investigated in [11, 12], and are of more practical significance if the output regulation theory is to be applied to real systems. In [11] the approximation approach described was based on the Taylor series expansion of the regulator equations. In [12] the regulator equations were approximated by a neural network. Approximate output regulation, using the method of Taylor series expansion, has been applied to a linear cart inverted pendulum in [13] and a 2-dof rotary inverted pendulum in [14]. Research into methods for estimating the region of convergence for the local output regulation problem and local approximate output regulation problem are documented in [15]. The effectiveness of the approximation method based on the Taylor expansion for the output regulator as applied to the spherical inverted pendulum was investigated here. This work represents the first results on output regulation control of the spherical inverted pendulum and is based upon the theory of [16].

The paper is organised as follows. Section II briefly describes the output regulation problem for nonlinear systems and the approximation approach used here. Section III defines the control problem being solved. Section IV gives the control design and Section V gives the results of the simulation studies with conclusions in section-VI.

II. PRELIMINARIES

A. Standard output regulation problem

We briefly review the standard nonlinear output regulation problem considered herein.

Consider a general time-invariant nonlinear system,

$$\begin{aligned} \dot{x}(t) &= f(x(t),u(t),v(t)), \qquad x(0)=x_0 \\ y(t) &= h(x(t),u(t),v(t)) \end{aligned} \tag{2.1}$$

Where $x \in R^n$ is the system state, $u \in R^m$ is the control input, $y \in R^p$ is the system output and $v \in R^q$ is the *exosystem* signal generated by the *exosystem*

$$\dot{v} = a(v) \tag{2.2}$$

The control error associated with this system is

$$e(t) = y(t) - r(v(t)),$$

where *r(v(t))* gives the reference trajectories to be tracked and/or known disturbance signals to be rejected.

For the case of state feedback, a control law of the form $u(t) = k(x(t),v(t))$ is sought. This control law comprises some feedforward component based on the reference signal from the *exosystem* and feedback of the system states. This control law solves the output regulation problem if it renders the system (2.1) closed loop stable under the influence of the *exosystem* (2.2), such that the tracking error asymptotically approaches zero.

In [6] the most notable condition for the existence of the controller solving the output regulation problem was established. That is, there exists an invariant zero-error manifold for the closed loop system. The existence of such is posed by the task: find the smooth functions *x(v), u(v)* which solve the non trivial set of mixed algebraic partial differential equations, called the regulator equations

$$\begin{aligned} \frac{\partial x}{\partial v} a(v) &= f\big(x(v),u(v),v\big) \\ 0 &= h\big(x(v),u(v),v\big) - r(v) \end{aligned} \tag{2.3}$$

While the existence of such functions that solve the above problem can be proven to exist, finding the exact closed form solution has been found to be a very difficult, if not impossible task for many systems.

B. Local approximate output regulator

Due to the difficulty in finding a closed form solution to (2.3), the regulator equations were modified in [11] to allow for an approximate solution to the zero-error manifold to be found such that the output regulator could actually be applied in practice. The statement of the local approximate solution to the regulator equations is given next.

Definition 2.1 [11]: Let there exist an open neighbourhood of the origin in R^q, denoted as Γ, then there exists sufficiently smooth functions $x_{(k)}(v): \Gamma \to R^n$, $u_{(k)}(v): \Gamma \to R^m$ with $x_{(k)}(0)=0, u_{(k)}(0)=0$, such that for $v \in \Gamma$ the following modified regulator equations hold

$$\begin{aligned} \frac{\partial x_{(k)}}{\partial v} a(v) &= f\Big(x_{(k)}(v),u_{(k)}(v),v\Big) + O(v^{(k+1)}) \\ 0 &= h\Big(x_{(k)}(v),u_{(k)}(v),v\Big) - (r(v) + O(v^{(k+1)})) \end{aligned} \tag{2.4}$$

where $O(v^{k+1})$ is such that $\lim_{v \to 0} \left\| O(v^{(k+1)}) \right\| \big/ \left\| v^{(k+1)} \right\|$ is a finite constant. The functions $x_{(k)}(v), u_{(k)}(v)$ thus give the approximate solution to the regulator equations (2.3) of the kth order. The term $O(v^{(k+1)})$ essentially represents extra information about the zero-error manifold that is lost through approximation.

In [16] it was shown that through Taylor series expansion of the equations governing the system dynamics (2.1) the partial differential equation of (2.4) reduces to a set of linear matrix equations which are solvable provided the solution to (2.4) is proven to exist. A sufficient condition for the existence of the solution to (2.4) i.e. the k-th order approximate solution to (2.3) was given in [11] (Theorem 1.2).

III. APPROXIMATE OUTPUT TRACKING OF THE SPHERICAL INVERTED PENDULUM

A. Spherical inverted pendulum

The spherical inverted pendulum considered in the present study is shown in Fig. 1. The generalized coordinates shown are $x, y \in R$ which represent the position of the base of the pendulum in the horizontal plane, and $X, Y \in R$ represent the x and y positions of the of the vertical projection of the centre of the pendulum onto the horizontal plane. $F_x, F_y \in R$ are the control forces being applied to the base of the pendulum, m is the mass of the uniform rod, L is the distance from the base of the pendulum to the centre of mass, and g is the gravitational constant.

The full dynamics governing the motion of the spherical inverted pendulum based on the generalized coordinates shown in Fig 1 is given in [4]. Defining new state variables as: $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$, $z_1 = X$, $z_2 = \dot{X}$, $z_3 = Y$, $z_4 = \dot{Y}$ and $x = [x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; z_1 \;\; z_2 \;\; z_3 \;\; z_4]^T$, the ODEs defining the dynamics of the pendulum can be put into state-space form:

$$\begin{aligned} \dot{x} &= f(x) + g(x).u \\ y &= h(x) \end{aligned} \tag{3.1}$$

where

$$f(x) = \begin{bmatrix} x_2 \\ \left(12 z_1 z_3^2 - 3 z_1 (L^2 - z_1{}^2 + 3 z_3{}^2)\right)(C - B)/A \\ x_4 \\ \left(12 z_1^2 z_3 - 3 z_3 (L^2 + 3 z_1{}^2 - z_3{}^2)\right)(C - B)/A \\ z_2 \\ \left(3 z_1 (L^2 - z_1{}^2 + 3 z_3{}^2) - 12 z_1 z_3^2\right)(C - B)/A \\ z_4 \\ \left(3 z_3 (L^2 + 3 z_1{}^2 - z_3{}^2) - 12 z_1^2 z_3\right)(C - B)/A \end{bmatrix}$$

$$g(x) = \begin{bmatrix} 0 & 0 \\ 4(L^2 + 3z_3{}^2)/A & -12 z_1 z_3/A \\ 0 & 0 \\ -12 z_1 z_3/A & 4(L^2 + 3z_1{}^2)/A \\ 0 & 0 \\ -3(L^2 - z_1{}^2 + 3z_3{}^2)/A & 12 z_1 z_3/A \\ 0 & 0 \\ 12 z_1 z_3/A & -3(L^2 + 3z_1{}^2 - z_3{}^2)/A \end{bmatrix}$$

(2.2). Find a control law of the form $u(t) = k(x(t), v(t))$ such that the base of the pendulum given by ($x(t)$, $y(t)$) asymptotically converges to within a small bound of ($r_1(t)$, $r_2(t)$) i.e. $x(t) - r_1(t) \to \delta_1,\ y(t) - r_2(t) \to \delta_2,\ as\ t \to \infty$.

IV. Control design for the spherical inverted pendulum

The approximate output regulation method seeks to find an approximate solution to the regulator equations by using a truncated Taylor series expansion of the system dynamics.

First consider the input-output linearization of the system (3.1) [17]. It is immediately apparent that the system has relative degree $\{r_1, r_2\} = \{2,2\}$, and that the dynamics of the system can easily be separated into those directly describing the output dynamics and those describing the internal dynamics as in (3.1) without using any coordinate transformation. Taking the r[th] derivative of the output gives

$$\begin{bmatrix} d^{r_1} y / dt^{r_1} \\ d^{r_2} y / dt^{r_2} \end{bmatrix} = \begin{bmatrix} L_f{}^{r_1}(h(x)) \\ L_f{}^{r_2}(h(x)) \end{bmatrix} = \begin{bmatrix} \dot{x}_2 \\ \dot{x}_4 \end{bmatrix} = \alpha(x) + \beta(x) u \tag{4.1}$$

Using the following input transformation

$$u = \beta(x)^{-1}\left(\bar{u} - \alpha(x)\right) \tag{4.2}$$

where $\beta(x)$ is invertible, the output dynamics take the form

$$d^{r_i} y / dt^{r_i} = \bar{u}_i \tag{4.3}$$

for i = 1,2. For exact tracking, where $y = r(v)$, the new control input can be chosen as

$$\bar{u}_i = d^{r_i} r(v) / dt^{r_i} \tag{4.4}$$

Transforming $\bar{u}_i$ in (4.4) back to the original control input by (4.2) gives the following ideal control signal for exact tracking

$$u^* = \begin{bmatrix} F_x^* \\ F_y^* \end{bmatrix} = \begin{bmatrix} \dfrac{\left\{ \begin{aligned} & - z_1 (L^2 + 3z_1{}^2 + 3z_3{}^2)\left[3C(z_1{}^4 + z_3{}^4) - 3B(z_1{}^4 + z_3{}^4) - 6 z_1{}^2 z_3{}^2 (B - C) + A z_1 v_1 \omega_1^2 + A z_3 v_3 \omega_2^2\right] \\ & + 3(C z_1 - B z_1)(z_1{}^2 + z_3{}^2)(L^2 + 3z_1{}^2 + 3z_3{}^2) + A z_3 L^2 (z_1 v_3 \omega_2^2 - z_3 v_1 \omega_1^2) \end{aligned} \right\}}{4L^2 (z_1{}^2 + z_3{}^2)(L^2 + 3z_1{}^2 + 3z_3{}^2)} \\[2ex] \dfrac{\left\{ \begin{aligned} & - z_3 (L^2 + 3z_1{}^2 + 3z_3{}^2)\left[3C(z_1{}^4 + z_3{}^4) - 3B(z_1{}^4 + z_3{}^4) - 6 z_1{}^2 z_3{}^2 (B - C) + A z_1 v_1 \omega_1^2 + A z_3 v_3 \omega_2^2\right] \\ & + 3(C z_3 - B z_3)(z_1{}^2 + z_3{}^2)(L^2 + 3z_1{}^2 + 3z_3{}^2) + A z_1 L^2 (z_1 v_3 \omega_2^2 - z_3 v_1 \omega_1^2) \end{aligned} \right\}}{4L^2 (z_1{}^2 + z_3{}^2)(L^2 + 3z_1{}^2 + 3z_3{}^2)} \end{bmatrix} \tag{4.5}$$

$h(x) = [x_1 \quad x_3]^{\mathrm{T}}, \quad A = m(L^2 + 3(z_1{}^2 + z_3{}^2)),$

$B = 4m\left(L^2 (z_2{}^2 + z_4{}^2) - (z_2 z_3 - z_4 z_1)^2\right) / 3(L^2 - z_1{}^2 - z_3{}^2)^2,$

$C = mg \Big/ \sqrt{L^2 - z_1{}^2 - z_3{}^2}$

B. *Problem formulation*

The problem of the approximate output regulation of the spherical inverted pendulum can be formulated as follows.

Consider the equations governing the dynamics of the spherical inverted pendulum (3.1). Let ($r_1(t)$, $r_2(t)$) be the piece-wise reference trajectory generated by the exosystem

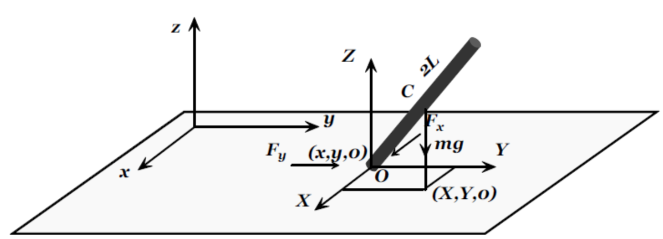

Fig. 1. Visualization of the spherical inverted pendulum with (x,y,X,Y) coordinates (reproduced from [4])

This control input cannot yet be implemented as the mappings $z_1(v), z_2(v), z_3(v), z_4(v)$, under the condition $e = 0$ are not yet known. These mappings under the zero tracking error condition represent the zero dynamics. By defining z_1^*, z_2^*, z_3^*, z_4^*, as the zero dynamic states, the zero dynamics of (3.1) become

$$\dot{z} = \gamma(z,v) = \begin{bmatrix} z_2^* \\ \frac{3(L^2 - z_1^{*2} + 3z_3^{*2})(-Bz_1^* + Cz_1^*) - 12z_1^* z_3^*(-Bz_3^* + Cz_3^*)}{A} - \frac{3(L^2 - z_1^{*2} + 3z_3^{*2})}{A} F_x^* + \frac{12 z_1^* z_3^*}{A} F_y^* \\ z_4^* \\ -\frac{12z_1^* z_3^*(-Bz_1^* + Cz_1^*) + 3(L^2 + 3z_1^{*2} - z_3^{*2})(-Bz_3^* + Cz_3^*)}{A} + \frac{12 z_1^* z_3^*}{A} F_x^* - \frac{3(L^2 + 3z_1^{*2} - z_3^{*2})}{A} F_y^* \end{bmatrix} \quad (4.6)$$

The zero dynamics given in (4.6) are unstable (the eigenvalues of $d\gamma/dz$ at v=0 are hyperbolic), implying that (3.1) describes a nonminimum phase system.

Solving the zero dynamics under the flow of the *exosystem* requires solving the following PDE, derived from the regulator equations (2.3), for $z_1(v)$, $z_2(v)$, $z_3(v)$, $z_4(v)$

$$\frac{\partial z(v)}{\partial v} a(v) = \gamma(z(v), v) \quad (4.7)$$

A closed form solution to (4.7) is not easily attainable and may even be nonexistent. Therefore, in the present study an approximate solution is sought based on Definition 2.1. The existence of the solution was affirmed by the satisfaction of the appropriate condition given in [18] (Lemma 4.13), which is equivalent to the condition referenced in Section II.

Following the approach described in [16, 18] to carry out the approximation approach through the use of Taylor series expansion, the zero dynamics (4.7) are rewritten as a power series

$$\gamma(z,v) = \sum_{l\geq 1} \sum_{\substack{i+k+l \\ i,k\geq 0}} \gamma_{ik} z^{(i)} \otimes v^{(k)}$$
$$a(v) = \sum_{l\geq 1} A_l v^{(l)} \quad (4.8)$$

where l denotes the order of the approximation used, and the superscript '(i)' denotes i successive uses of the Kroneckor product, e.g. $z^{(0)} = 1,\ z^{(1)} = z, z^{(2)} = z \otimes z$ etc.

An approximate solution to (3.5) is sought of the form

$$z^*(v) = \sum_{l\geq 1} Z_l v^{[l]} \quad (4.9)$$

where $v^{[l]}$ contains only the unique components of $v^{(l)}$.

Remark 4.1: The accuracy of an approximate solution to (4.7) depends on the bounds on the exact solution to (4.7). In fact, the closer the exact solutions for $z_1(v)$, $z_2(v)$, $z_3(v)$, $z_4(v)$ are to zero the more accurate the approximate solution will be, as the higher order terms which are truncated by the approximation have less effect on the solution. The bounds on the exact solutions will differ from case to case, depending on the dynamic system being considered as well as the *exosystem* of interest. Also, the order of the approximation used is obviously a factor in the accuracy as well. Increasing the order of the approximation can be done to increase the accuracy of the solution. However, increasing the order of approximation increases the dimensions of the linear matrix equations associated with the approximate solution thereby increasing the computational complexity at the initial stage of controller implementation.

The approximate solution to (4.7) based on the truncated power series expansion (4.8) was shown in [18]. This is achieved by the solving a number of linear matrix equations which result from substituting (4.8) & (4.9) into (4.7). These linear matrix equations are not shown here due to limitations of space and can be found in [16],[18]. A third order approximation was sought and was later seen to be sufficient to provide good tracking performance.

Remark 4.2: The fact that the applicability of this approximate output regulation problem is dependent on finding the solution to some linear matrix equations, as opposed to requiring the solution to the nonlinear APDEs of (2.3), is the key reason why this approach is more practical than the standard output regulation problem. If the existence of a solution to the regulator equations is proven, the solution to the linear matrix equations can always be found; however, the same cannot be said for exact solution for (2.3).

The dynamics of the *exosystem* which has been used to generate a family of trajectories to be tracked is chosen as

$$\dot{v} = a(v) = A_1 v = diag\left(\begin{bmatrix} 0 & \omega_1 \\ -\omega_1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \omega_2 \\ -\omega_2 & 0 \end{bmatrix}\right) v \quad (4.10)$$

with initial conditions $v(0)=[v_{1,0}, v_{2,0}, v_{3,0}, v_{4,0}]^T$. The desired output as $y_d=r(v)=[v_1,v_3]^T$, gives the desired output trajectories as a two periodic functions,
$y_d = \begin{bmatrix} v_1(t) \\ v_3(t) \end{bmatrix} = \begin{bmatrix} v_{1,0}\cos(\omega_1 t) + v_{2,0}\sin(\omega_1 t) \\ v_{3,0}\cos(\omega_2 t) + v_{4,0}\sin(\omega_2 t) \end{bmatrix}$. This exosystem is chosen for its ability to generate periodic trajectories which are generally of practical significance. This design procedure can easily be applied to track complex trajectories which can be generated by different exosystems. However, the use of a more complicated exosystem would increase the complexity of the linear matrix equations to be solved.

Whilst solvable, the process of solving the required linear matrix equations is non-trivial, especially as the plant and exosystem increase in order and complexity. Finding the approximate solution to the zero-dynamics for the current system (3.1) under the exosystem given (4.10) required the solution of over 80 simultaneous equations. Unlike the approximate solution for the SISO linear cart pendulum found in [14] which could be realistically solved by hand, the solution found here (as well as the construction of the matrices to be solved) required computational aide.

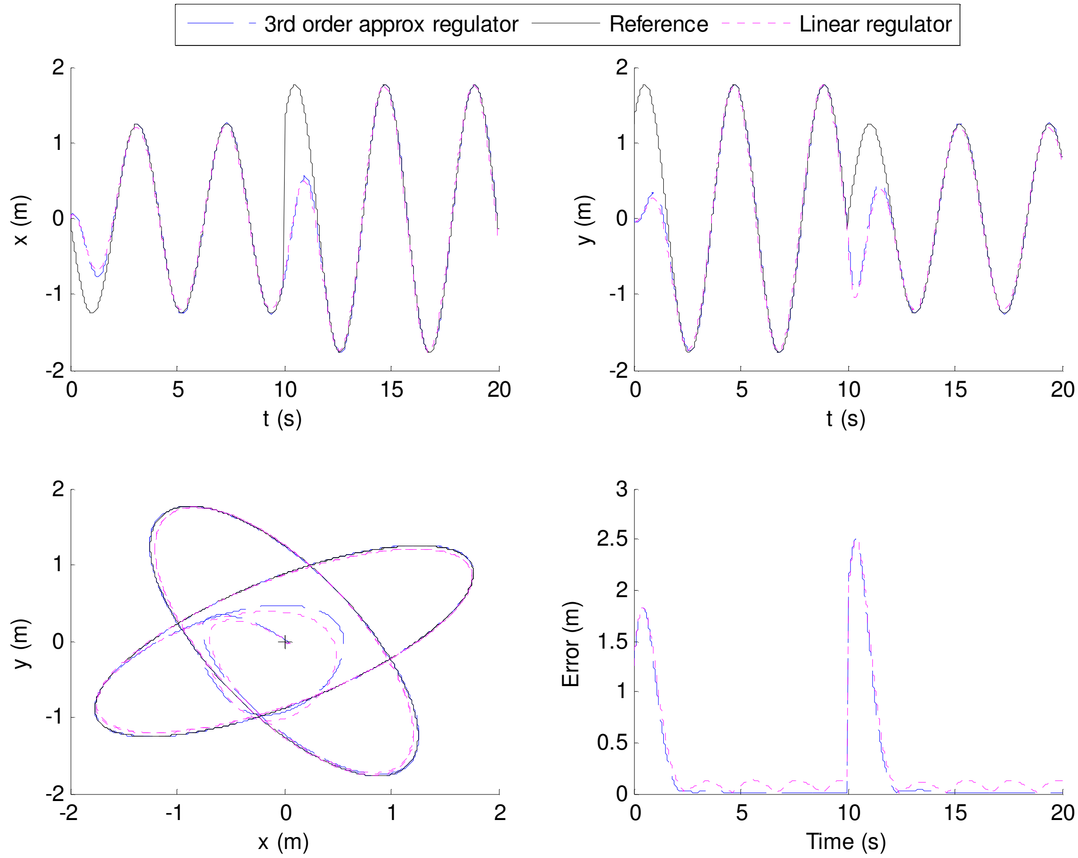

Figure 2 - Output state trajectories, pendulum base path transcribed in x-y plane, and tracking control error for the nonlinear 3rd order approximate regulator and linear regulator

Once the solution to the zero dynamics is resolved as a function of *exosystem* state, the feed forward control , $u^*(v)$, can be implemented. This control law would provide close approximate tracking of the desired output trajectory generated by the exosystem provided the system output lies on the desired trajectory and with the correct initial conditions. A stabilizing feedback control law can be additionally implemented to account for incorrect initial conditions or unknown disturbances by stabilizing the system states about state trajectories associated with exact tracking, $z^*(v)$.

Controllability of the system in the first approximation was verified, so an LQR stabilizing law was used here as a simple means of providing feedback control. Based on the linearization of the original system (3.1) about the point $(x,z)=0$ the linear feedback law, $u_{fb} = -Ke_x$, is used, where $e_x = (\underline{x} - \underline{x}^*)$ and feedback gain matrix K is designed by solving the LQR optimal control problem.

The complete controller used here is then given as

$$u(x,v) = u^*(v) + u_{fb}(x,v) \qquad (4.11)$$

V. Simulation Studies

Simulation of the controller was carried out in MATLAB/SIMULINK using system parameters, m=0.050 kg, L=0.650m, $\omega_{1,2}$=1.5 rad/s, and initial conditions (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)=(0,0,0,0,0,0,0,0) for the pendulum system and $(v_1(0),v_2(0),v_3(0),v_4(0)) = (0,-1.25,1.25,1.25)$ for the *exosystem*. During the simulation (at $t = 10s$) the exosystem is reset such that a new desired trajectory of similar magnitude is generated, achieved by the conditions, $(v_1(10),v_2(10),v_3(10),v_4(10)) = (1.25,1.25,0,1.25)$. This is done to emphasize the causal nature of the output regulator in contrast to non-causal inversion techniques which have so far been applied to the spherical inverted pendulum i.e. that a piece-wise trajectory that can be generated by the exosystem can be the tracked without prior knowledge of the specific trajectory. These initial conditions give the reference signals as two elliptical trajectories.

For the feedback part of the output regulator, the weighting matrices that were found to perform well were $Q=diag(70,10,70,10,50,750,50,750)$, $R=diag(1,1)$.

It can be seen in Fig. 2 that the implementation of the approximate output regulator results in the pendulum following the elliptical paths. A third order approximation was made, i.e. k = 3. Despite the approximation being used in designing the controller, the tracking error after convergence remains within a very small bound (maximum steady state error of 1.57×10^{-2} m) for the given scenario.

Remark 5.1: Using the approximation order k = 1 gives a linear controller. It is interesting to note that in this case the matrix equations to be solved are equivalent to the standard linear regulator equations (which can be found in [18]).

For purpose of comparison, the performance of a linear regulator was also studied (taking k = 1). The results, also seen in Fig 2 show that the linear regulator is also capable of stable regulation of the system, however the steady state errors are more significant than when the 3rd order approximation is used (the maximum steady state error is 8.6 times greater in magnitude).

Remark 5.2: This difference in performance will vary in different cases; the extent of this difference is dependent on the system under consideration, as well as the specific trajectories to be tracked. Regardless, the 3rd (or higher) order approximate regulator is expected to always provide better steady state performance than the standard linear regulator, as the nonlinear controller uses more information about the system. This was witnessed in the simulation studies carried out. Thus, it is desirable to use the nonlinear approximate regulator over the standard linear regulator where tracking performance is concerned.

The local nature of the approximate output regulator used here means that the performance decreases as reference trajectories require the zero-error manifold to deviate further from the origin. This was illustrated for the spherical inverted pendulum by the results in Table 1 which show the maximum steady state error when the pendulum is made to track ellipses of fixed size (the same size as in Fig. 1) but with different speeds. Variation of the reference signal frequency alone was done to emphasize that the zero dynamics manifold is what directly affects the accuracy of tracking, rather than the deviation of the output itself from the origin. As expected the steady state tracking error increases with increase in the speed at which the pendulum is required to traverse the path, even up to five orders of magnitude comparing results of $\omega_{1,2}=0.5$ *rad/s* and $\omega_{1,2} = 2$ *rad/s*. However, for the first three cases at least (in the case of the nonlinear controller), the tracking errors are still very small in relation to trajectories being tracked. Again, the improved tracking performance of the nonlinear approximate regulator over the standard linear regulator can be seen.

Table 1 - Maximum steady state tracking error

Frequency (rad/s)	Max steady state error (m)	
	Nonlinear, k=3	Linear, k=1
0.5	0.00000107	0.000351
1.0	0.000449	0.0175
1.5	0.0157	0.135
2.0	0.139	0.471

VI. CONCLUSION

In this paper an output regulator was applied to the spherical inverted pendulum for the tracking of a series of elliptical paths. This is the first application of output regulation to this system. The solution to the regulator equations for the spherical inverted pendulum is more complex compared to the linear cart and rotary pendulums for which this approach has been applied to before. The approximation approach based on a Taylor series expansion of the system dynamics had to be adapted to this more complex system. Simulations were conducted to evaluate the effectiveness of this approach and showed that tracking of a series of elliptical trajectories was achieved with very small steady state error. Comparison of a third order nonlinear approximate regulator to the standard linear regulator showed the benefit of the nonlinear approximate regulator as witnessed by greater steady state tracking performance. The application of output regulation allowed specific trajectories to be tracked which were unknown at the control design stage, giving an advantage over the non-causal tracking controllers that have recently been applied to the spherical inverted pendulum.

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