2011 American Control Conference
on O'Farrell Street, San Francisco, CA, USA
June 29 - July 01, 2011

$\mathcal{L}_1$ Adaptive Output Feedback Controller for Minimum Phase Systems

Evgeny Kharisov and Naira Hovakimyan

***Abstract*— This paper presents an $\mathcal{L}_1$ adaptive output feedback controller for a class of uncertain nonlinear systems in the presence of time and output dependent unknown nonlinearities. As compared to earlier introduced $\mathcal{L}_1$ adaptive output feedback control architectures, the architecture in this paper relies on system inversion, and is therefore limited to minimum phase systems. Similar to prior solutions in $\mathcal{L}_1$ adaptive control theory, the feedback structure is comprised of the three main elements, involving predictor, adaptation laws and low–pass filter, with the only difference that the predictor here is an input predictor and not a state predictor. Whereas in prior architectures of $\mathcal{L}_1$ adaptive output feedback control the verification of the sufficient condition for stability, written in terms of $\mathcal{L}_1$ norm of cascaded systems, was not straightforward, the solution proposed in this paper, under mild assumptions on system dynamics, provides a complete parametrization of the low–pass filters for the design purposes. The closed–loop system achieves arbitrarily close tracking of the input and the output signals of the reference system. Simulations verify the theoretical findings.**

I. Introduction

This paper presents an $\mathcal{L}_1$ adaptive output feedback control architecture for minimum phase systems in the presence of time and output dependent nonlinearities. The $\mathcal{L}_1$ adaptive control architecture in this paper is a modification of the architecture of Monopoli from [1], which includes the *augmented error signal*. In this paper the augmented error avoids the use of pure differentiators in the control law. Several adaptive control laws using the augmented error were later proposed in [2], [3]. In [2], the use of the augmented error helped to relax the constraints on the relative degree of the system and allowed for application of the adaptive controllers to systems with arbitrary relative degree.

The $\mathcal{L}_1$ adaptive control architecture in this paper achieves uniform transient and steady–state performance bounds with respect to the signals of a bounded reference system, which assumes partial cancelation of uncertainties. We notice that adaptive algorithms achieving arbitrarily improved transient performance for system's output were reported in [4]–[10]. The $\mathcal{L}_1$ adaptive control architecture gives the opportunity to regulate also the performance bound for system's input signal, including its frequency spectrum, by rendering it arbitrarily close to the corresponding signal of a bounded linear reference system. $\mathcal{L}_1$ adaptive controller for non–SPR reference systems was first presented in [11] without using system inversion, and therefore it could be applied for control of systems with non–minimum phase zeros as well. However, the verification of the sufficient condition for stability is not straightforward, and for certain classes of systems may be impossible, as observed in [12]. As compared to that solution, the architecture in this paper relies on system inversion, and hence cannot be used in the presence of non–minimum phase zeros. However, it contains more parameters for tuning than the $\mathcal{L}_1$ controller in [11]. In particular it gives an opportunity to set the estimation dynamics by tuning the poles of the observer polynomial. We note that the problem of the filter design for the $\mathcal{L}_1$ adaptive controller in general setting is considered in [13], where results from disturbance observer (DOB) literature are used along with μ–synthesis to address the problem for both minimum phase and non–minimum phase systems.

An $\mathcal{L}_1$ like control architecture was also proposed in [14]. However, the performance bounds for the architecture in [14] cannot be arbitrarily reduced in the presence of bounded disturbances due to the lack of estimation loop for the disturbances.

This paper is organized as follows: Section II gives the problem formulation. Section III derives an equivalent system representation, which is used in the analysis of the control system. Section IV presents the adaptive control architecture. Stability and performance bounds are derived in Section V. Section VI presents simulation results. Section VII concludes the paper. The proofs of the results are collected in Appendix.

II. Problem Formulation

Consider the system given by

$$y(s) = W(s)(u(s) + \sigma(s)), \quad W(s) \triangleq k_0 \frac{B(s)}{A(s)}, \qquad (1)$$

where $A(s)$ and $B(s)$ are relatively prime unknown monic polynomials with $B(s)$ being Hurwitz; $k_0 \in \mathbb{R}$ is the unknown high frequency gain of the system with known sign; $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ are the system input and the output respectively; $\sigma(s)$ is the Laplace transform of $\sigma(t) \triangleq f(t, y(t))$, and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the unknown nonlinearity. Let $n_A \triangleq \deg(A(s))$ and $n_B \triangleq \deg(B(s))$ be the unknown degrees of the polynomials $A(s)$ and $B(s)$ with $n_A > n_B$. Assume that the upper bound $n > n_A$ and the relative degree $n_r \triangleq n_A - n_B$ are known. Further, let the following assumptions hold.

Assumption 1. Let $0 < k_m < k_0 < k_M$, where $k_m, k_M \in \mathbb{R}^+$ are known conservative bounds.

Assumption 2. Assume that for arbitrary $y_1, y_2 \in \mathbb{R}$ there exist known constants $L \in \mathbb{R}^+$ and $L_0 \in \mathbb{R}^+$, verifying

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|, \quad \forall\, t \ge 0. \qquad (2)$$

$$|f(t, 0)| \le L_0, \quad \forall\, t \ge 0. \qquad (3)$$

The control objective is to design a control law $u(t)$, which ensures that the system output $y(t)$ tracks a bounded reference signal $r(t) \in \mathbb{R}$, $\|r\|_{\mathcal{L}_\infty} \leq \bar{r}$, with performance specifications given by the following *desired (ideal) system*:

$$y(s) = k_g M(s) r(s)\,, \quad M(s) \triangleq N(s)/D(s)\,, \tag{4}$$

where $N(s)$ and $D(s)$ are arbitrary known monic Hurwitz polynomials with $n_D \triangleq \deg(D(s))$ and $n_N \triangleq \deg(N(s))$, such that $n < n_N$, and the relative degree of $M(s)$ is equal to n_r, i.e. $n_D - n_N = n_r$; $k_g \in \mathbb{R}^+$ is the high–frequency gain of the system, given by $k_g \triangleq 1/M(0)$, which ensures that $y(t)$ tracks constant $r(t)$ with zero steady state error.

III. System Parametrization

To proceed with the developments in this paper, we refer to a lemma from [15] on system parametrization.

Lemma 1. [15] The system in (1) can be rewritten as follows:

$$y(s) = M(s)\big[k_0 u(s) + h^\top \phi_u(s) + k^\top \phi_y(s) + w(s)\big]\,, \tag{5}$$

where $h,\, k \in \mathbb{R}^n$ are unknown system parameters; $\phi_u(s) \triangleq P(s)u(s)$, $\phi_y(s) \triangleq P(s)y(s)$ are computable signals with $P(s) \triangleq \frac{\lambda(s)}{p(s)}$, $p(s) = N(s)p^*(s)$, where $\lambda(s) \triangleq \begin{bmatrix}1 & s & s^2 & \cdots & s^{n-1}\end{bmatrix}^\top$, and $p^*(s)$ is an arbitrary monic Hurwitz polynomial of degree $n_{p^*} \triangleq n - n_N$; $w(s) \triangleq (k_0 + h^\top P(s))\sigma(s)$.

The system in (5) can be further rewritten as:

$$y(s) = M(s)\left(k_m u(s) + \theta^\top \phi(s) + w(s)\right)\,, \tag{6}$$

$$\theta \triangleq \begin{bmatrix} k_0 - k_m & h^\top & k^\top \end{bmatrix}^\top \in \mathbb{R}^{2n+1}\,,$$

$$\phi(s) \triangleq \begin{bmatrix} u(s) & \phi_u^\top(s) & \phi_y^\top(s) \end{bmatrix}^\top \in \mathbb{R}^{2n+1}\,. \tag{7}$$

Using the conservative information about the location of the coefficients of $A(s)$ and $B(s)$, one can obtain the conservative set Θ, to which the parameter θ belongs.

IV. $\mathcal{L}_1$ Adaptive Control Architecture

The $\mathcal{L}_1$ adaptive output feedback controller in this paper is based on *system inversion*. A low–pass filter at the system output is used to render the transfer function of the inverted system proper. The adaptation law uses the prediction error augmented by an *auxiliary error*. The control law generates the control signal via the output of a low–pass filter.

A. Definitions and $\mathcal{L}_1$ Stability Condition

Consider the following three filters

$$C_G(s) \triangleq \frac{\omega_G}{s + \omega_G}\,, \quad C_E(s) \triangleq \frac{(\omega_E)^l}{(s + \omega_E)^l}\,, \quad C_0(s)\,, \tag{8}$$

where $C_0(s)$ is a stable strictly proper transfer function with unit DC gain $C_0(0) = 1$; $l \geq n_r$ is the order of the low pass filter; and $\omega_E \in \mathbb{R}^+$, $\omega_G \in \mathbb{R}^+$ are the parameters of the filters. Next, define

$$C_H(s) \triangleq C_G(s) C_E(s)\,, \quad C_F(s) \triangleq C_H(s) C_0(s)\,. \tag{9}$$

Let the choice of the filters in (8) satisfy the following $\mathcal{L}_1$ *norm condition*:

$$\|H_{yy}(s)\|_{\mathcal{L}_1} + \|H_{yw}(s)\|_{\mathcal{L}_1} L < 1\,, \tag{10}$$

where the constant L is defined in (2), and

$$H_{yy}(s) \triangleq \frac{k_m(1 - C_F(s))}{k_m + C_F(s)(k_0 - k_m)}\left[1 - \frac{A(s)N(s)}{B(s)D(s)}\right]\,, \tag{11}$$

$$H_{yw}(s) \triangleq \frac{k_0 k_m (1 - C_F(s))}{k_m + C_F(s)(k_0 - k_m)} \frac{N(s)}{D(s)}\,. \tag{12}$$

Next, define

$$\rho_{y_{\mathrm{rf}}} \triangleq \frac{\|H_{yw}(s)\|_{\mathcal{L}_1} L_0 + \|H_{yr}(s)\|_{\mathcal{L}_1} \bar{r}}{1 - \|H_{yy}(s)\|_{\mathcal{L}_1} + \|H_{yw}(s)\|_{\mathcal{L}_1} L}\,, \tag{13}$$

where L_0 is defined in (3), and

$$H_{yr}(s) \triangleq \frac{N(s)}{D(s)} \frac{k_0 k_g}{k_m + C_F(s)(k_0 - k_m)}\,. \tag{14}$$

Further, let $\gamma_y' \in \mathbb{R}$ be an arbitrary (small) constant, and let $\rho_y \triangleq \rho_{y_{\mathrm{rf}}} + \gamma_y'$. Finally, let

$$\bar{w}_0 \triangleq L\rho_y + L_0\,, \quad \bar{w} \triangleq \left\|k_0 + h^\top P(s)\right\|_{\mathcal{L}_1} \bar{w}_0\,. \tag{15}$$

B. Filtered Inversion of the System

Define the filtered inverse of the system in (6) as

$$\nu(s) = \frac{C_H(s)}{M(s)} y(s)\,. \tag{16}$$

Notice that this corresponds to the filtered total system input:

$$\nu(s) = C_H(s)\left(k_m u(s) + \theta^\top \phi(s) + w(s)\right)\,.$$

C. Input Predictor

Consider the following input predictor:

$$\hat{\nu}(s) = C_H(s)\left(k_m u(s) + \hat{\mu}_\phi(s)\right) + C_G(s)\hat{w}(s)\,, \tag{17}$$

where $\hat{\nu}(t) \in \mathbb{R}$ is the estimate of the total system input, $\hat{\mu}_\phi(s)$ is the Laplace transform of $\hat{\mu}_\phi(t) \triangleq \hat{\theta}^\top(t)\phi(t)$, and $\hat{\theta}(t) \in \mathbb{R}^{2n+1}$, $\hat{w}(t) \in \mathbb{R}$ are the adaptive estimates.

D. Augmented Error

Let $\tilde{\nu}(t) \triangleq \nu(t) - \hat{\nu}(t)$ be the prediction error. We consider the following auxiliary error:

$$\eta(s) \triangleq C_H(s)\hat{\mu}_\phi(s) - C_G(s)\hat{\mu}_X(s)\,, \tag{18}$$

where $\hat{\mu}_X(t) \triangleq \hat{\theta}^\top(t)X(t)$, and $X(t)$ is the filtered version of $\phi(t)$ given by $X(s) \triangleq C_E(s)\phi(s)$. Using this auxiliary error, we define the augmented error:

$$\varepsilon(t) = \tilde{\nu}(t) + \eta(t)\,. \tag{19}$$

E. Adaptation Laws

The adaptation laws are given by:

$$\begin{aligned} \dot{\hat{\theta}}(t) &= \Gamma \mathrm{Proj}\left(\hat{\theta}(t),\, \omega_G \varepsilon(t) X(t)\right)\,, & \hat{\theta}(0) &= \hat{\theta}_0\,, \\ \dot{\hat{w}}(t) &= \Gamma \mathrm{Proj}\left(\hat{w}(t),\, \omega_G \varepsilon(t)\right)\,, & \hat{w}(0) &= \hat{w}_0\,, \end{aligned} \tag{20}$$

where $\Gamma \in \mathbb{R}^+$ is the adaptation gain, and the projection bounds are set to ensure $\hat{\theta}(t) \in \Theta$, and $\hat{w}(t) \in \Delta \triangleq [-\bar{w},\, \bar{w}]$ for all $t \geq 0$, where $\bar{w}$ was defined in (15).

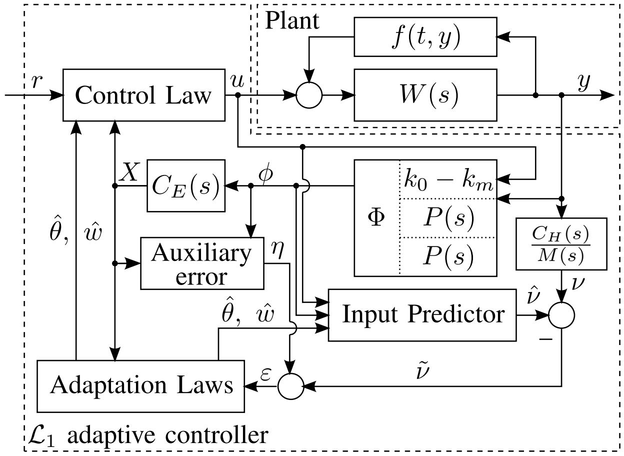


Fig. 1: Closed–loop adaptive system

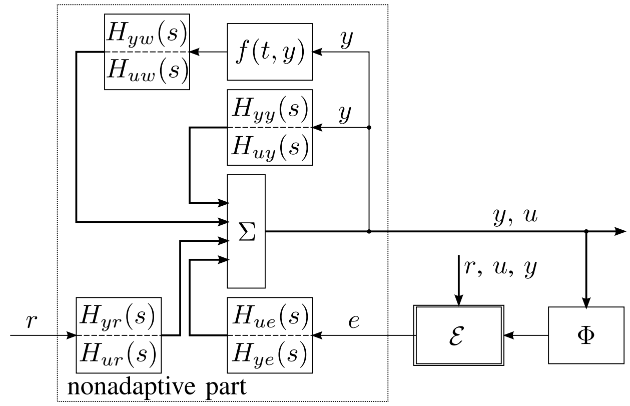


Fig. 2: Closed–loop adaptive system in (22)–(23)

F. Control Law

The $\mathcal{L}_1$ adaptive control law is given by

$$u(s) = \frac{1}{k_m}\left(k_g r(s) - C_F(s)\left(\hat{\mu}_\phi(s) + \hat{w}(s)\right)\right), \tag{21}$$

where the filter $C_F(s)$ is defined in (9).

G. Control System Architecture

The $\mathcal{L}_1$ controller consists of the system given by (1), the inversion law in (16), the input predictor in (17), the adaptation laws in (20) along with (18)–(19), and the control law given by (21). The block diagram of the closed–loop system is given in Figure 1. In this block diagram the block with $\Phi : (u(t), y(t)) \to \phi(t)$ is defined according to (7).

V. Analysis of the $\mathcal{L}_1$ Adaptive Controller

A. Closed–loop System Representation

We start the analysis of the closed–loop adaptive system by rewriting it in a more convenient form.

Lemma 2. The closed–loop adaptive system shown in Figure 1 can be equivalently represented as

$$\begin{aligned} y(s) &= H_{yy}(s)y(s) + H_{yw}(s)\sigma(s) \\ &\quad + H_{yr}(s)r(s) + H_{ye}(s)e(s)\,, \end{aligned} \tag{22}$$

$$\begin{aligned} u(s) &= H_{uy}(s)y(s) + H_{uw}(s)\sigma(s) \\ &\quad + H_{ur}(s)r(s) + H_{ue}(s)e(s)\,, \end{aligned} \tag{23}$$

where

$$H_{ye}(s) \triangleq \frac{N(s)}{D(s)} \frac{k_0 C_0(s)}{k_m + C_F(s)(k_0 - k_m)}\,, \tag{24}$$

$$\begin{aligned} H_{uy}(s) &\triangleq \frac{C_F(s)}{k_m + C_F(s)(k_0 - k_m)} \frac{D(s)}{N(s)} \left[\frac{A(s)N(s)}{B(s)D(s)} - 1\right], \\ H_{uw}(s) &\triangleq -\frac{k_0 C_F(s)}{k_m + C_F(s)(k_0 - k_m)}\,, \\ H_{ur}(s) &\triangleq \frac{k_g}{k_m + C_F(s)(k_0 - k_m)}\,, \\ H_{ue}(s) &\triangleq \frac{C_0(s)}{k_m + C_F(s)(k_0 - k_m)}\,, \end{aligned} \tag{25}$$

the transfer functions $H_{yy}(s)$, $H_{yw}(s)$, $H_{yr}(s)$ are defined in (11), (12), (14) respectively, and

$$\begin{aligned} e(s) &\triangleq C_H(s)\left(\tilde{\mu}_\phi(s) + \tilde{w}(s)\right)\,, \\ \tilde{\mu}_\phi(t) &\triangleq \tilde{\theta}^\top(t)\phi(t),\ \ \tilde{\theta}(t) \triangleq \theta - \hat{\theta}(t),\ \ \tilde{w}(t) \triangleq w(t) - \hat{w}(t). \end{aligned} \tag{26}$$

Let $\mathcal{E}$ be the map, generating the error $e(t)$:

$$\mathcal{E}\ :\ (t, r(t), u(t), y(t), \phi(t)) \to e(t)\,. \tag{27}$$

Notice that $\mathcal{E}$ denotes the adaptive part of the system. Figure 2 shows a block diagram of the system in (22)–(23).

B. Reference System

Consider the *closed–loop reference system*:

$$\begin{aligned} y_{\mathrm{rf}}(s) &= H_{yy}(s)y_{\mathrm{rf}}(s) + H_{yw}(s)\sigma_{\mathrm{rf}}(s) + H_{yr}(s)r(s), \\ u_{\mathrm{rf}}(s) &= H_{uy}(s)y_{\mathrm{rf}}(s) + H_{uw}(s)\sigma_{\mathrm{rf}}(s) + H_{uer}(s)r(s), \end{aligned} \tag{28}$$

where $\sigma_{\mathrm{rf}}(t) \triangleq f(t, y_{\mathrm{rf}}(t))$.

Lemma 3. If the $\mathcal{L}_1$ norm condition in (10) is satisfied, then the closed–loop reference system in (28) is BIBO stable, and:

$$\|y_{\mathrm{rf}}\|_{\mathcal{L}_\infty} \le \rho_{y_{\mathrm{rf}}}\,, \quad \|u_{\mathrm{rf}}\|_{\mathcal{L}_\infty} \le \rho_{u_{\mathrm{rf}}}\,, \tag{29}$$

where $\rho_{y_{\mathrm{rf}}}$ is defined in (13), and $\rho_{u_{\mathrm{rf}}} \triangleq \|H_{uy}(s)\|_{\mathcal{L}_1}\rho_{y_{\mathrm{rf}}} + \|H_{uw}(s)\|_{\mathcal{L}_1}(L\rho_{y_{\mathrm{rf}}} + L_0) + \|H_{uer}(s)\|_{\mathcal{L}_1}\bar{r}$.

Remark 1. Notice that if the system in (1) is linear, i.e. $f(t, y(t)) \equiv w(t)$, which is uniformly bounded, then the stability condition in (10) is reduced to the requirement of stability of $H_{yy}(s)$ closed with unit feedback. In this case it can be shown that the stability of the reference system can be always achieved by increasing the bandwidth of $C_F(s)$.

Remark 2 (*Reference system vs. Ideal system*). To see the connection between the reference system in (28) and the ideal system defined in (4), we consider the limiting case of filters with $\omega_G, \omega_E \to \infty$ and $C_0(s) \to 1$. Further, the transfer functions in (24)–(25), (11), (12), and (14) reduce to:

$$\begin{aligned} H_{yy}(s) &= 0\,, \qquad H_{uy}(s) = \frac{1}{k_0}\frac{D(s)}{N(s)}\left(\frac{A(s)N(s)}{B(s)D(s)} - 1\right), \\ H_{yw}(s) &= 0\,, \qquad H_{uw}(s) = -1\,, \end{aligned}$$

$$H_{yr}(s) = k_g \frac{N(s)}{D(s)}\,, H_{ur}(s) = \frac{k_g}{k_0}\,.$$

We see that the output of the reference system is identical to the output of the ideal system, when $C_F(s) \to 1$. However, the control input of the reference system is not implementable, as $H_{uy}(s)$ is now improper.

C. Error Dynamics

From the system in (6), we obtain $\frac{C_H(s)}{M(s)} y(s) = C_H(s)\left(k_m u(s) + \theta^\top \phi(s) + w(s)\right)$. Using this, we can rewrite (19) as

$$\varepsilon(s) = C_G(s)\left(\tilde{\mu}_X(s) + \tilde{w}_E(s)\right)\,, \tag{30}$$

where $\tilde{\mu}_X(t) \triangleq \tilde{\theta}(t)X(t)$, $\tilde{w}_E(t) \triangleq w_E(t) - \hat{w}(t)$, and $w_E(s) \triangleq C_E(s)w(s)$. Using the definition of $C_G(s)$ we can write the error dynamics in the state space form as follows:

$$\dot{\varepsilon}(t) = -\omega_G \varepsilon(t) + \omega_G\Big[\tilde{\theta}^\top(t)X(t) + \tilde{w}_E(t)\Big],\ \varepsilon(0) = 0. \tag{31}$$

Lemma 4. Consider the error dynamics given by (31). If for some $\tau \geq 0$ we have $\|w_\tau\|_{\mathcal{L}_\infty} \leq \bar{w}$, then:

$$\|\varepsilon_\tau\|_{\mathcal{L}_\infty} \leq \frac{\bar{\varepsilon}}{\sqrt{\Gamma}}\,, \tag{32}$$

where $\bar{\varepsilon} \triangleq \sqrt{2\bar{w}\bar{w}_{Ed}/\omega_G + 4\left(\max_{\theta\in\Theta}\|\theta\|^2 + \bar{w}\right)}$, and $\bar{w}_{Ed} \triangleq \|sC_E(s)\|_{\mathcal{L}_1}\bar{w}$. Also we have:

$$\left\|\dot{\tilde{\theta}}(t)\right\|_\infty \leq \sqrt{\Gamma}\omega_G\bar{\varepsilon}\|X(t)\|_\infty\,, \quad \forall\ t \in [0,\ \tau]\,. \tag{33}$$

The proof of the lemma is done using the standard Lyapunov argument and is omitted due to space limitations.

D. Boundedness of the Adaptation Error

In this section we study the properties of $\mathcal{E}$ defined in (27).

Lemma 5. For the closed–loop adaptive system shown in Figure 1, if for some $\tau \geq 0$ we have $\|w_\tau\|_{\mathcal{L}_\infty} \leq \bar{w}$, then

$$\|e_\tau\|_{\mathcal{L}_\infty} \leq \bar{e}_0(\Gamma) + \bar{e}_\phi(\Gamma,\,\omega_E)\|\phi_\tau\|^2_{\mathcal{L}_\infty}\,, \tag{34}$$

where

$$\bar{e}_0(\Gamma) \triangleq \frac{\bar{\varepsilon}}{\sqrt{\Gamma}} + \|1 - C_E(s)\|_{\mathcal{L}_1}\bar{w}\,, \tag{35}$$

$$\bar{e}_\phi(\Gamma,\,\omega_E) \triangleq \frac{\sqrt{2n+1}\sqrt{\Gamma}\omega_G}{\omega_E}\sum_{k=0}^{l-1}\frac{\bar{\varepsilon}}{k!}\,. \tag{36}$$

E. Stability and Performance Bounds of the Closed–loop Adaptive System

Consider the closed–loop system in (22), (23). The next theorem proves the main result of the paper.

Theorem 1. Let the filter $C_F(s)$ satisfy the $\mathcal{L}_1$ norm stability condition in (10). For any fixed $\omega_G > 0$ and arbitrary (small) constant $\epsilon \in \mathbb{R}^+$, if we set $\omega_E \geq \Gamma$ and set the adaptive gain large enough to satisfy the following conditions

$$\|1 - C_E(s)\|_{\mathcal{L}_1} < \frac{\epsilon}{2\bar{\phi}_e\bar{w}}\,, \quad \gamma_y < \gamma'_y\,, \tag{37}$$

and

$$\Gamma > \max\left\{(2n+1)\left[\frac{\bar{\varepsilon}\omega_G(\bar{\phi}_0+2\epsilon)^2\bar{\phi}_e}{\epsilon}\sum_{k=0}^{l-1}\frac{1}{k!}\right]^2;\ \left[\frac{2\bar{\phi}_e\bar{\varepsilon}}{\epsilon}\right]^2\right\}, \tag{38}$$

then the closed–loop system is stable, and the following uniform performance bounds hold

$$\|y_{\rm rf} - y\|_{\mathcal{L}_\infty} < \gamma_y\,, \quad \|u_{\rm rf} - u\|_{\mathcal{L}_\infty} < \gamma_u\,, \tag{39}$$

where

$$\begin{aligned}
\gamma_e &\triangleq \bar{e}_0(\Gamma) + \bar{e}_\phi(\Gamma,\,\omega_E)(\bar{\phi}_0 + 2\epsilon)^2\,,\\
\gamma_y &\triangleq \frac{\|H_{ye}(s)\|_{\mathcal{L}_1}}{1 - \|H_{yy}(s)\|_{\mathcal{L}_1} - \|H_{yw}(s)\|_{\mathcal{L}_1}L}\gamma_e\,,\\
\gamma_u &\triangleq \left(\|H_{uy}(s)\|_{\mathcal{L}_1} + \|H_{uw}(s)\|_{\mathcal{L}_1}L\right)\gamma_y + \|H_{ue}(s)\|_{\mathcal{L}_1}\gamma_e\,,\\
\bar{\phi}_0 &\triangleq \max\left\{\max\{1,\,\|P(s)\|_{\mathcal{L}_1}\}\bar{u}_0,\,\bar{y}_0\right\}\,,\\
\bar{\phi}_e &\triangleq \max\left\{\max\{1,\,\|P(s)\|_{\mathcal{L}_1}\}\bar{u}_e,\,\bar{y}_e\right\}\,,\\
\bar{u}_0 &\triangleq \left\|\frac{H_{uy}(s)H_{yw}(s)}{1 - H_{yy}(s)} + H_{uw}(s)\right\|_{\mathcal{L}_1}\bar{w}_0\\
&\quad + \left\|\frac{H_{uy}(s)H_{yr}(s)}{1 - H_{yy}(s)} + H_{ur}(s)\right\|_{\mathcal{L}_1}\bar{r}\,,\\
\bar{u}_e &\triangleq \left\|\frac{H_{uy}(s)H_{ye}(s)}{1 - H_{yy}(s)} + H_{ue}(s)\right\|_{\mathcal{L}_1}\,,\\
\bar{y}_0 &\triangleq \left\|\frac{P(s)H_{yw}(s)}{1 - H_{yy}(s)}\right\|_{\mathcal{L}_1}\bar{w}_0 + \left\|\frac{P(s)H_{yr}(s)}{1 - H_{yy}(s)}\right\|_{\mathcal{L}_1}\bar{r}\,,\\
\bar{y}_e &\triangleq \left\|\frac{P(s)H_{ye}(s)}{1 - H_{yy}(s)}\right\|_{\mathcal{L}_1}\,.
\end{aligned}$$

Remark 3. For the bound in (34), notice that for any fixed value of ω_G, if we set $\omega_E = \Gamma$, then $\lim_{\Gamma\to\infty}\bar{e}_0(\Gamma) = 0\,$, $\lim_{\Gamma\to\infty}\bar{e}_\phi(\Gamma,\,\omega_E = \Gamma) = 0\,$. Therefore it follows that

$$\lim_{\Gamma\to\infty}\gamma_e = \lim_{\Gamma\to\infty}\bar{e}_0(\Gamma) + (\bar{\phi}_0 + 2\epsilon)^2\lim_{\Gamma\to\infty}\bar{e}_\phi(\Gamma,\,\Gamma) = 0\,.$$

This implies

$$\lim_{\Gamma\to\infty}\gamma_y = 0\,, \quad \lim_{\Gamma\to\infty}\gamma_u = 0\,.$$

VI. SIMULATIONS

Consider $W_1(s) \triangleq 5\frac{s+0.2}{s^2+s+2}\,$, $M(s) \triangleq \frac{s+20}{s^2+12s+20}\,$. Assume that our conservative knowledge of the unknown plant $W_1(s)$ leads to the following minimum value of the plant high frequency gain $k_m = 1$, and the parameter sets $\Theta = \left\{\theta \in \mathbb{R}^{2n+1} : \|\theta\| \leq 1000\right\}$, $\Delta = [-1000\ \ 1000]$. Set $\Gamma = 1000$, and let $\omega_G = 100$, $\omega_E = 100\,000$, and $C_0(s) = 1/(0.15s+1)\,$, leading to $C_F(s) = \frac{1}{0.15s+1}\frac{100\,000}{s+100\,000}\frac{100}{s+100}\,$. Notice that according to the theory we need to keep ω_E high enough to ensure that the closed–loop adaptive system is close to the reference system. Further, let $p = N(s)(s+5) = s^2 + 25s + 100\,$.

Figure 3 shows the simulation results in the presence of $f_1(t,y) = \sin y - 0.5 + \sin(0.3t)\,$, $f_2(t,y) = |y| + 0.5 + 0.1\sin(2t)\,$, $f_3(t,y) = -\cos y + 0.5(|y| + y) + 1 + 0.5\sin(0.5t) + 0.2\sin t\,$. We see that the transient of the closed–loop adaptive system almost coincides with the transient of the reference system. All simulations are done without any retuning of the controller parameters.

Figure 4 shows the system response to the step reference commands of different amplitudes in the presence of input disturbance. The system response scales with the reference commands, implying that the $\mathcal{L}_1$ controller leads to prdictable response.

VII. Conclusions

An $\mathcal{L}_1$ adaptive output feedback control architecture for minimum phase systems is presented, which leads to uniform performance bounds for system's both signals in transient and steady–state in the presence of fast adaptation.

Acknowledgments

The authors would like to thank Roberto Cristi for thoughtful discussions and helpful comments.

References

[1] R. V. Monopoli, "Model reference adaptive control with an augmentation error signal," *IEEE Transactions on Automatic Control*, vol. 19, no. 5, October 1974.

[2] A. Feuer and A. S. Morse, "Adaptive control of single-input, single-output linear systems," vol. 16, December 1977, pp. 1030–1035.

[3] K. S. Narendra and L. S. Valavani, "Stable adaptive controller design–direct control," *IEEE Transactions on Automatic Control*, vol. 23, no. 4, pp. 570–583, August 1978.

[4] G. Bartolini, A. Ferrara, and A. A. Stotsky, "Robustness and performance of an indirect adaptive control scheme in the presence of bounded disturbances," *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 789–793, April 1999.

[5] J. Sun, "A modified model reference adaptive control scheme for improved transient performance," *IEEE Transactions on Automatic Control*, vol. 38, no. 8, pp. 1255–1259, August 1993.

[6] M. Krstić, P. V. Kokotović, and I. Kanellakopoulos, "Transient-performance improvement with a new class of adaptive controllers," *Systems & Control Letters*, vol. 21, no. 6, pp. 451–461, December 1993.

[7] Z. Zang and R. R. Bitmead, "Transient bounds for adaptive control systems," in *IEEE Conference on Decision and Control*, vol. 5, Honolulu, HI, December 1990, pp. 2724–2729.

[8] A. Datta and P. A. Ioannou, "Performance analysis and improvement in model reference adaptive control," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2370–2387, December 1994.

[9] R. Marino and P. Tomei, "An adaptive output feedback control for a class of nonlinear systems with time-varying parameters," *IEEE Transactions on Automatic Control*, vol. 44, no. 11, pp. 2190–2194, November 1999.

[10] M. A. Arteaga and Y. Tang, "Adaptive control of robots with an improved transient performance," *IEEE Transactions on Automatic Control*, vol. 47, no. 7, pp. 1198–1202, July 2002.

[11] C. Cao and N. Hovakimyan, "$\mathcal{L}_1$ adaptive output-feedback controller for non-stricly-positive-real reference systems: Missile longitudinal autopilot design," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 32, no. 3, pp. 717–726, May-June 2009.

[12] J. Wang, C. Cao, N. Hovakimyan, R. Hindman, and D. B. Ridgely, "$\mathcal{L}_1$ adaptive controller for a missile longitudinal autopilot design," in *AIAA Guidance, Navigation and Control Conference*, Honolulu, HI, August 2008, AIAA-2008-6282.

[13] K. K. K. Kim, E. Kharisov, and N. Hovakimyan, "$\mathcal{L}_1$ adaptive controller and disturbance observer. part II: Filter design for $\mathcal{L}_1$ adaptive output–feedback controller," in *IEEE Conference on Decision and Control*, Orlando, FL, December 2011, submitted.

[14] R. Cristi, J. J. Kim, and J. H. Yang, "Robust $\mathcal{L}_1$ adaptive control," July 2010, submitted.

[15] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*, ser. Information and System Sciences. Englewood Cliffs, NJ: Prentice Hall, 1989.

Appendix

Proof of Lemma 2: Substituting $k_0u(s)$ from (1) into (6), gives $y(s) = \frac{N(s)}{D(s)}\Big(\frac{A(s)}{B(s)}y(s) - k_0\sigma(s) - (k_0 - k_m)u(s) + \theta^\top\phi(s) + w(s)\Big)$. This leads to

$$\begin{aligned}\theta^\top\phi(s) + w(s) = \frac{D(s)}{N(s)}\left(1 - \frac{N(s)A(s)}{D(s)B(s)}\right)y(s) \\ + k_0\sigma(s) + (k_0 - k_m)u(s)\,.\end{aligned} \tag{40}$$

Consider the control law given in (21). Using (40), it can be rewritten as follows: $k_mu(s) = k_gr(s) + C_0(s)e(s) - C_F(s)(k_0 - k_m)u(s) - C_F(s)\left[\frac{D(s)}{N(s)}\left[1 - \frac{N(s)A(s)}{D(s)B(s)}\right]y(s) + k_0\sigma(s)\right]$. Isolating $u(s)$, we obtain (23). To prove the equality for $y(s)$ we substitute the control law given by (21) into (6): $y(s) = \frac{N(s)}{D(s)}\big(k_gr(s) + C_0(s)e(s) + (1 - C_F(s))(\theta^\top\phi(s) + w(s))\big)$. Finally, substituting (40) into this equation, and then substituting (23) into the resulting equation, we obtain (22). □

Proof of Lemma 3: The equation in (28) leads to the following upper bound, valid for arbitrary $\tau \geq 0$: $\|y_{\mathrm{rf}_\tau}\|_{\mathcal{L}_\infty} \leq \|H_{yy}(s)\|_{\mathcal{L}_1}\|y_{\mathrm{rf}_\tau}\|_{\mathcal{L}_\infty} + \|H_{yw}(s)\|_{\mathcal{L}_1}\|\sigma_\tau\|_{\mathcal{L}_\infty} + \|H_{yr}(s)\|_{\mathcal{L}_1}\bar{r}$. Notice that from (2) and (3) it follows that $|f(t, y_{\mathrm{rf}}(t))| \leq L|y_{\mathrm{rf}}(t)| + L_0\,,\quad \forall\ t \geq 0$, which further leads to $\|y_{\mathrm{rf}_\tau}\|_{\mathcal{L}_\infty} \leq \frac{\|H_{yw}(s)\|_{\mathcal{L}_1}L_0 + \|H_{yr}(s)\|_{\mathcal{L}_1}\bar{r}}{1 - \|H_{yy}(s)\|_{\mathcal{L}_1} - \|H_{yw}(s)\|_{\mathcal{L}_1}L}$. The $\mathcal{L}_1$–norm condition in (10) ensures that the RHS of this bound is positive. The fact that the RHS is independent of τ yields the uniform bound in (29). The uniform boundedness of $y_{\mathrm{rf}}(t)$, and hence $\sigma_{\mathrm{rf}}(t)$, lead to the following upper bound for the control signal of the reference system, given by (28): $\|u_{\mathrm{rf}}\|_{\mathcal{L}_\infty} \leq \|H_{uy}(s)\|_{\mathcal{L}_1}\|y_{\mathrm{rf}}\|_{\mathcal{L}_\infty} + \|H_{uw}(s)\|_{\mathcal{L}_1}\|\sigma_{\mathrm{rf}}\|_{\mathcal{L}_\infty} + \|H_{uer}(s)\|_{\mathcal{L}_1}\bar{r} \leq \rho_{u_{\mathrm{rf}}}$. □

Proof of Lemma 5: The error in (26) can be rewritten as

$$e(s) = \varepsilon(s) + \eta(s) + \eta_w(s)\,, \tag{41}$$

where $\varepsilon(s)$ is the augmented error from (30), $\eta(s)$ is the auxiliary error defined in (18), and $\eta_w(s) \triangleq C_G(s)\left(\hat{w}(s) - C_E(s)\hat{w}(s)\right)$. Next we prove boundedness of each of the signals $\varepsilon(t)$, $\eta(t)$, and $\eta_w(t)$ on $t \in [0,\ \tau]$.

Boundedness of $\varepsilon(t)$. Application of Lemma 4 leads to the result in (32).

Boundedness of $\eta(t)$. Let

$$\eta(s) = C_G(s)\eta_E(s),\ \eta_E(s) \triangleq C_E(s)\hat{\mu}_\phi(s) - \hat{\mu}_X(s). \tag{42}$$

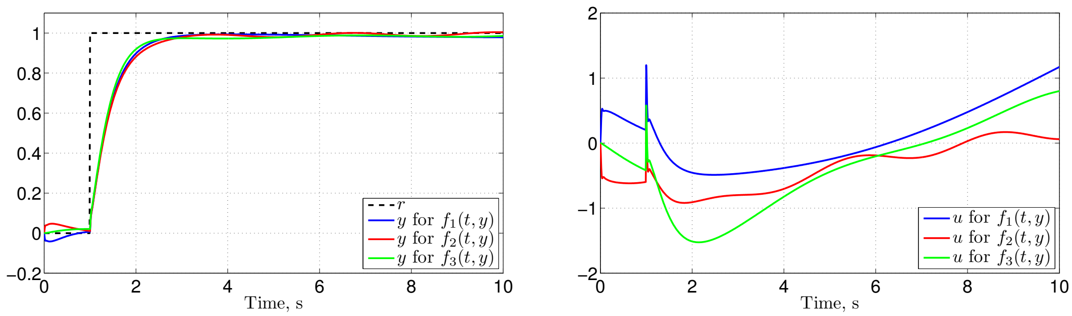
Notice that $\eta_E(s) = C_E(s)\tilde{\mu}_\phi(s) - \tilde{\mu}_X(s)$. Let $c_E(t)$ be the impulse response for $C_E(s)$. Then $\eta_E(t) = -\int_0^t \dot{\tilde{\theta}}^\top(\lambda)\int_{t-\lambda}^t c_E(\xi)\phi(t - \xi)d\xi d\lambda$. From the definition of $C_E(s)$ it follows that $\|C_E(s)\|_{\mathcal{L}_1} = 1$. Therefore on $t \in [0,\ \tau]$ we have $\|X(t)\|_\infty \leq \|\phi_\tau\|_{\mathcal{L}_\infty}$, and the upper bound in (33) can be rewritten as $\left\|\dot{\hat{\theta}}(t)\right\|_\infty \leq \sqrt{\Gamma}\omega_G\bar{\varepsilon}\|\phi_\tau\|_{\mathcal{L}_\infty}$, which yields

$$|\eta_E(t)| \leq \sqrt{(2n+1)\Gamma}\omega_G\bar{\varepsilon}\int_0^t\int_{t-\lambda}^\infty |c_E(\xi)|d\xi d\lambda\|\phi_\tau\|^2_{\mathcal{L}_\infty} \tag{43}$$

Consider the inner integral. Since $C_E(s) = (\omega_E)^l/(s + \omega_E)^l$, then $\int_{t-\lambda}^\infty |c_E(\xi)|d\xi = \sum_{k=0}^{l-1}\frac{(\omega_E(t-\lambda))^k}{k!}\mathrm{e}^{-\omega_E(t-\lambda)}$. Using this equation, we obtain the following upperbound $\int_0^t\int_{t-\lambda}^\infty |c_E(\xi)|d\xi d\lambda \leq \frac{1}{\omega_E}\sum_{k=0}^{l-1}\frac{1}{k!}$. Substituting this into (43), we obtain

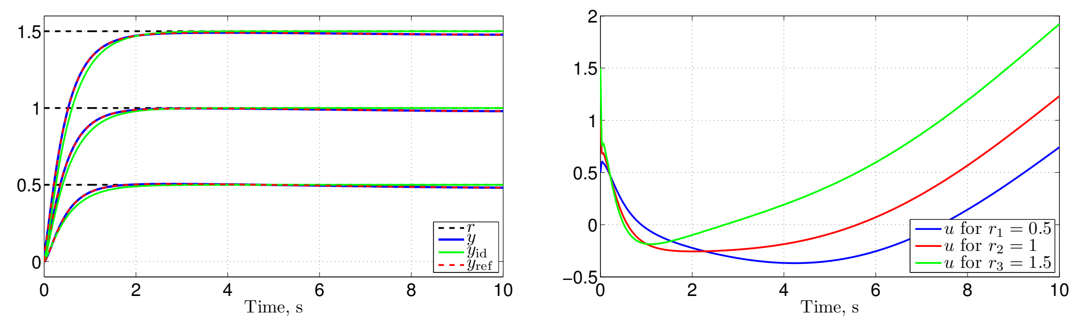
$$|\eta_E(t)| \leq \left(\frac{\sqrt{2n+1}\sqrt{\Gamma}\omega_G}{\omega_E}\sum_{k=0}^{l-1}\frac{\bar{\varepsilon}}{k!}\right)\|\phi_\tau\|^2_{\mathcal{L}_\infty}\,. \tag{44}$$

From (42) and the fact that $\|C_G(s)\|_{\mathcal{L}_1} = 1$, it follows that $\|\eta_\tau\|_{\mathcal{L}_\infty} \leq \|\eta_{E_\tau}\|_{\mathcal{L}_\infty}$.



(a) Transient performance (b) Control history

Fig. 3: Closed–loop system response for the disturbances $f_1(t)$, $f_2(t)$, $f_3(t)$



(a) Transient performance (b) Control history

Fig. 4: Closed–loop system response to the step signals of different size

Boundedness of $\eta_w(t)$. From the definition of $\eta_w(t)$ it immediately follows that

$$\|\eta_{w_\tau}\|_{\mathcal{L}_\infty} \le \|1 - C_E(s)\|_{\mathcal{L}_1} \bar{w}\,. \tag{45}$$

Combining the bounds in (32), (44) and (45) according to (41), we obtain the upper bound in (34). □

Proof of Theorem 1: We prove the bounds in (39) using a contradiction argument. Assume that either one or both of (39) does not hold. Then continuity of $y(t)$, $y_{\rm rf}(t)$, $u(t)$, and $u_{\rm rf}(t)$ along with the fact that $y(0) = y_{\rm rf}(0) = u(0) = u_{\rm rf}(0) = 0$ implies that there exists time $\tau > 0$, such that either or both of the following conditions hold

$$|y_{\rm rf}(t) - y(t)| < \gamma_y\,, \quad |y_{\rm rf}(\tau) - y(\tau)| = \gamma_y\,, \tag{46}$$

$$|u_{\rm rf}(t) - u(t)| < \gamma_y\,, \quad |u_{\rm rf}(\tau) - u(\tau)| = \gamma_u\,. \tag{47}$$

Consider the case when (46) holds. Using the upper bound in (29) from Lemma 3 we obtain $\|y_\tau\|_{\mathcal{L}_\infty} \le \rho_{y_{\rm rf}} + \gamma_y < \rho_{y_{\rm rf}} + \gamma_y' = \rho_y$. Notice that from (2) and (3) it follows that $\|f_\tau\|_{\mathcal{L}_\infty} \le L\rho_y + L_0 = \bar{w}_0$, where f stands for $f(t, y(t))$, and $\bar{w}_0$ was defined in (15). Using the definition of $w(t)$ in Lemma 1, we obtain $\|w_\tau\|_{\mathcal{L}_\infty} \le \|k_0 + h^\top P(s)\|_{\mathcal{L}_1}(L\rho_y + L_0) = \bar{w}$. The closed–loop system in (22) can be written as $y(s) = \frac{H_{yw}(s)\sigma(s) + H_{yr}(s)r(s) + H_{ye}(s)e(s)}{1 - H_{yy}(s)}$. Substituting it into (23), we obtain the following upperbound for the signals $u(t)$ and $\phi_y(t)$: $\|u_\tau\|_{\mathcal{L}_\infty} \le \bar{u}_0 + \bar{u}_e\|e_\tau\|_{\mathcal{L}_\infty}$, $\|(\phi_y)_\tau\|_{\mathcal{L}_\infty} \le \bar{y}_0 + \bar{y}_e\|e_\tau\|_{\mathcal{L}_\infty}$. Next, using (7) we obtain the following upper bound

$$\|\phi_\tau\|_{\mathcal{L}_\infty} \le \bar{\phi}_0 + \bar{\phi}_e\|e_\tau\|_{\mathcal{L}_\infty}\,. \tag{48}$$

Let $\bar{\phi} \triangleq \bar{\phi}_0 + 2\epsilon$. Next we show that

$$\|\phi_\tau\|_{\mathcal{L}_\infty} < \bar{\phi}\,. \tag{49}$$

We use a contradiction to prove this upper bound. Notice that $\phi(t)$ is a continuous function, and that $u(0) = 0$, $\phi(0) = 0$. Thus, if (49) is not true, then there exists some time $\tau_1 \in [0,\ \tau]$, such that

$$\|\phi(\tau_1)\|_\infty = \bar{\phi}\,, \quad \|\phi(t)\|_\infty < \bar{\phi}\,, \quad t < \tau_1\,. \tag{50}$$

Substituting (38) into (36) and setting $\omega_E \ge \Gamma$, we obtain

$$\bar{e}_\phi \le \frac{\sqrt{2n+1}\omega_G}{\sqrt{\Gamma}} \sum_{k=0}^{l-1} \frac{\bar{\varepsilon}}{k!} < \frac{\epsilon}{(\bar{\phi}_0 + 2\epsilon)^2 \bar{\phi}_e} < \frac{\epsilon}{\bar{\phi}^2 \bar{\phi}_e}\,,$$

which can be equivalently represented as $\bar{\phi}^2\bar{\phi}_e\bar{e}_\phi < \epsilon$. Next, consider (35). Substituting (37) and (38) leads to $\bar{e}_0 = \frac{\bar{\varepsilon}}{\sqrt{\Gamma}} + \|1 - C_E(s)\|_{\mathcal{L}_1}\bar{w} < \frac{\epsilon}{\bar{\phi}_e}$, which further yields $\bar{\phi}_e\bar{e}_0 < \epsilon$. Substituting (34) into (48) leads to $\|\phi_{\tau_1}\|_{\mathcal{L}_\infty} \le \bar{\phi}_0 + \bar{\phi}_e\bar{e}_0 + \bar{\phi}_e\bar{e}_\phi\bar{\phi}^2 < \bar{\phi}_0 + 2\epsilon = \bar{\phi}$, which gives a contradiction to (50). Therefore, (49) holds. Now we proceed with construction of the contradiction to the claim in (46). Substituting (49) into (34), we obtain

$$\|e_\tau\|_{\mathcal{L}_\infty} < \bar{e}_0 + \bar{e}_\phi\bar{\phi}^2 = \bar{e}_0 + \bar{e}_\phi(\bar{\phi}_0 + 2\epsilon)^2 = \gamma_e\,. \tag{51}$$

Subtracting $y_{\rm rf}(t)$, given by (28), from $y(t)$, given by (22) and (23), respectively, and taking into account (2), we obtain $\|(y - y_{\rm rf})_\tau\|_{\mathcal{L}_\infty} \le \|H_{yy}(s)\|_{\mathcal{L}_1}\|(y - y_{\rm rf})_\tau\|_{\mathcal{L}_\infty} + \|H_{yw}(s)\|_{\mathcal{L}_1} L\|(y - y_{\rm rf})_\tau\|_{\mathcal{L}_\infty} + \|H_{ye}(s)\|_{\mathcal{L}_1}\|e_\tau\|_{\mathcal{L}_\infty}$. Using the upper bound in (51), this can be written as $\|(y - y_{\rm rf})_\tau\|_{\mathcal{L}_\infty} < \gamma_y$, which contradicts (46), and hence proves the performance bound in (39). Next, subtracting $u_{\rm rf}(t)$, given by (28), from $u(t)$, given by (23), we obtain $u(s) - u_{\rm rf}(s) = H_{uy}(s)(y(s) - y_{\rm rf}(s)) + H_{uw}(s)(\sigma(s) - \sigma_{\rm rf}(s)) + H_{ue}(s)e(s)$. This leads to the following bound $\|(u - u_{\rm rf})_\tau\|_{\mathcal{L}_\infty} < \gamma_u$, which contradicts (47) and completes the proof. □